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A New Method for the Analysis of Unitary Representations of SL(n, C)

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Abstract. It is shown that homogeneous variables may be used in a covariant fashion to analyze unitary as well as finite dimensional representations of the group SL(n, C). In particular the reduction of these representations with respect to the maximal compact subgroup SU(n) is carried out in an explicitly covariant manner. The method is particularly useful for degenerate representations.

1. Introduction

Recently, in connection with infinite sequences of resonances [1], with strong-coupling theory [2], with dynamical symmetry groups [3], with current algebras [4] and with Regge poles [4], unitary representations of noncompact groups have come to play an interesting role in elementary particle physics. In particular unitary representations of the group of complex unimodular $n \times n$ matrices, SL(n, C), have been used in a number of applications. In a remarkable early investigation on field equations with infinitely many components, MAJORANA [5] had made use of unitary representations of the covering group SL(2, C) of the homogeneous Lorentz group. More recent applications of unitary representations of the groups SL(2, C) and SL(6, C) can be found in the bibliography of Ref. [6].

The mathematical theory of unitary representations of the classical groups SL(n, C), SO(n, C) and Sp(n, C) has been developed by GELFAND and NAIMARK [7, 8]. We shall briefly review their theory for the case of SL(n, C) below. Unfortunately, their elegant method is not suitable for practical applications, because it does not display explicitly the covariance of the results. The purpose of this paper is to show that a modification of the technique of GELFAND and NAIMARK restores the full symmetry in cases of practical interest. In particular it is shown that the reduction of the representations with respect to the maximal compact subgroup, which constitutes an important step in the applications, can be handled in an explicitly covariant fashion.

A quite different technique, based on the analytic continuation of matrix elements of finite dimensional representations, has been proposed by FRONSDAL [10]. In the case of SL(2, C) still another method dealing directly with the elements of the canonical basis associated with the reduction SL(2, C) $\supset$ SU(2) has proved to be useful [11]. Each of these methods has its advantages and may be more convenient for a specific type of problems than the others.

To present the main features of our techniques in as simple a manner as possible we shall first deal with unitary representations of the group SL(2, C). We demonstrate the usefulness of the method by means of simple examples in Sections 4 and 5. The extension of the technique to SL(n, C) is given in Sections 6–8.

We shall not try to give a rigorous treatment of our method, but shall rather emphasize in a heuristic fashion those aspects which are useful in practical applications.

Rather elaborate applications of the method to SL(2, C) and to SL(6, C) have been given by ZULAUF [12] and by GORGÉ and LEUTWYLER [6].

2. Principal Series of Unitary Representations of the Group SL(2, C)

According to NAIMARK [13] there are two types of unitary representations of SL(2, C), the representations of the so-called principal series and those of the supplementary series. We shall in the following restrict ourselves to the principal series although the results can be extended to the supplementary series in a straightforward fashion. The representations of the principal series are constructed as follows. One considers the subgroups Z and K of SL(2, C) consisting of elements of the form

$$\boldsymbol{z} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \qquad \boldsymbol{k} = \begin{pmatrix} k^{-1} & l \\ 0 & k \end{pmatrix}. \tag{2.1}$$

These subgroups are complementary in the sense that almost all elements $\boldsymbol{s} \in$ SL(2, C) can be decomposed as

$$\boldsymbol{s} = \boldsymbol{k}\, \boldsymbol{z} \tag{2.2}$$

and this decomposition is unique[1]).

The representations of the principal series are then defined on a Hilbert space of complex functions $f(z)$ on the subgroup Z with the scalar product

$$(f, g) = \int dz \overset{*}{f}(z)\, g(z)\ . \tag{2.3}$$

The integration extends over real and imaginary parts of the complex number z

$$z = x + i\, y \qquad dz = dx\, dy\ .$$

The action of the group on these functions is defined as follows:

$$f \to f' = U(\boldsymbol{g})\, f$$

$$f'(z) = \alpha(\boldsymbol{k}')\, f(z') \tag{2.4}$$

where $\boldsymbol{k}'$ and $\boldsymbol{z}'$ are defined by

$$\boldsymbol{k}'\, \boldsymbol{z}' = \boldsymbol{z}\, \boldsymbol{g}\ . \tag{2.5}$$

If we write

$$\boldsymbol{g} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

we have explicitly

$$z' = \frac{\alpha\, z + \gamma}{\beta\, z + \delta}\ ; \quad k' = \beta\, z + \delta; \quad l' = \beta\ . \tag{2.6}$$

[1]) The decomposition fails if $s_2^2 = 0$.

Consistency of this definition of the representation $U(\boldsymbol{g})$ with the group structure requires

$$\alpha(\boldsymbol{k}_1)\,\alpha(\boldsymbol{k}_2) = \alpha(\boldsymbol{k}_1\,\boldsymbol{k}_2)\ .$$

This implies

$$\alpha(\boldsymbol{k}) = \alpha(k) = k^{a^+}\,k^{*\,a^-} \tag{2.7}$$

where k is the lower element on the diagonal of the matrix $\boldsymbol{k}$.

The requirement that the representation be unitary fixes the absolute value of $\alpha(\boldsymbol{k})$, i.e. the sum of the real parts of the exponents a^+ and a^-. On the other hand the phase of $\alpha(\boldsymbol{k})$ is only subject to the condition that $\alpha(\boldsymbol{k})$ be a one-valued function on the subgroup K as otherwise $U(\boldsymbol{g})$ is not a global but only a local representation of SL(2, C). These requirements lead to the following expressions for a^+ and a^-

$$a^{\pm} = \frac{1}{2}\,(\mp\, m + i\,\varrho) - 1 \tag{2.8}$$

where m is an integer, and ϱ is an arbitrary real parameter[2]). These two parameters are invariants of the representation; in fact they fix the representation defined above completely. Furthermore the following theorem is proved in Ref. [13]: Two representations characterized by (m_1, ϱ_1) and (m_2, ϱ_2) respectively are equivalent if and only if

$$m_1 = m_2;\quad \varrho_1 = \varrho_2 \quad \text{or} \quad m_1 = -\,m_2;\quad \varrho_1 = -\,\varrho_2\ .$$

The representations of the principal series may therefore be denoted by $U_{m,\varrho}$, labelled by an integer $m = 0, \pm 1, \pm 2, \ldots$ and an arbitrary real parameter ϱ. These representations are mutually inequivalent, except for $U_{m,\varrho} \simeq U_{-m,-\varrho}$.

3. Homogeneous Variables

It is evident that the explicit transformation law for the variable z given in (2.6) is rather clumsy for practical applications. In particular, if the element $\boldsymbol{g} \in$ SL(2, C) represents a Lorentz transformation, the transformation law for z contains a rather strange combination of the conventional parameters, i.e. angles of rotation and relative velocities of the reference frames connected by this transformation. A much more transparent form is obtained if one makes use of homogeneous variables. These variables may be introduced as follows. Associate with every complex number z a family of spinors ζ by means of the definition

$$\zeta = (\zeta_1, \zeta_2) = \lambda(z, 1) \tag{3.1}$$

where λ is an arbitrary complex parameter. Accordingly we define a function $F(\zeta)$ associated with the element $f(z)$ of the representation space by

$$F(\zeta) = \alpha(\lambda)\; f(z)\ . \tag{3.2}$$

Since $\alpha(\lambda)$ is a homogeneous function so is $F(\zeta)$:

$$F(\mu\,\zeta) = \alpha(\mu)\;F(\zeta) = \mu^{1/2\,(-m+i\varrho)-1}\;\mu^{*\,1/2\,(+m+i\varrho)-1}\;F(\zeta)\ . \tag{3.3}$$

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[2]) We adhere to the notation used in Ref. [7]. Note that the conventions used in Ref. [13] differ from those used here in the sign of m.

It is easy to verify that the transformation law for the elements $f(z)$ implies the following transformation law for the homogeneous functions $F(\zeta)$ [3])

$$F \to F' = U(\mathbf{g})\, F$$
$$F'(\zeta) = F(\zeta\, \mathbf{g}); \quad (\zeta\, \mathbf{g})_\alpha = \zeta_\beta\, g^\beta_\alpha\,. \tag{3.4}$$

In other words, the quantity ζ does indeed transform like a spinor under SL(2, C). The degree of homogeneity of $F(\zeta)$ characterizes the representation, whereas the transformation law is the same for all representations.

Generators

The infinitesimal elements of SL(2, C) may be parametrized by means of the conventional antisymmetric quantity $\omega^{\mu\nu}$

$$\mathbf{g}(\omega) = \mathbf{1} + \frac{1}{4}\, \omega^{\mu\nu}\, \boldsymbol{\sigma}_{\mu\nu} + \cdots \tag{3.5}$$

where the generators $\boldsymbol{\sigma}_{\mu\nu}$ are defined by [3])

$$\boldsymbol{\sigma}_{\mu\nu} = \frac{1}{2}\, (\tilde{\boldsymbol{\sigma}}_\mu\, \boldsymbol{\sigma}_\nu - \tilde{\boldsymbol{\sigma}}_\nu\, \boldsymbol{\sigma}_\mu)\,. \tag{3.6}$$

It is straightforward to verify that the corresponding hermitean generators $J_{\mu\nu}$ of the representation in terms of homogeneous functions are given by

$$U(\mathbf{g}) = 1 - \frac{i}{2}\, \omega^{\mu\nu}\, J_{\mu\nu} + \cdots$$
$$J_{\mu\nu} = \frac{i}{2} \left\{ \zeta\, \boldsymbol{\sigma}_{\mu\nu}\, \frac{\partial}{\partial \zeta} + \zeta^*\, \boldsymbol{\sigma}^*_{\mu\nu}\, \frac{\partial}{\partial \zeta^*} \right\}. \tag{3.7}$$

The Casimir operators

$$C_1 = \frac{1}{2}\, J^{\mu\nu}\, J_{\mu\nu}; \qquad C_2 = \frac{1}{4}\, \varepsilon^{\mu\nu\varrho\sigma}\, J_{\mu\nu}\, J_{\varrho\sigma} \tag{3.8}$$

can be rewritten in terms of the operators

$$H = \zeta_\alpha\, \frac{\partial}{\partial \zeta_\alpha}; \quad \tilde{H} = \zeta^*_{\dot\alpha}\, \frac{\partial}{\partial \zeta^*_{\dot\alpha}} \tag{3.9}$$

as follows

$$C_1 = \frac{1}{2}\, H\, (H + 2) + \frac{1}{2}\, \tilde{H}\, (\tilde{H} + 2)$$
$$C_2 = \frac{i}{2}\, H\, (H + 2) - \frac{i}{2}\, \tilde{H}\, (\tilde{H} + 2)\,. \tag{3.10}$$

The values of the operators H and $\tilde{H}$ in an irreducible representation are determined by the degree of homogeneity of $F(\zeta)$ given in (3.3). The resulting expressions for the Casimir operators are

$$C_1 = \frac{1}{4}\, (m^2 - \varrho^2) - 1 \qquad C_2 = + \frac{1}{2}\, m\, \varrho\,. \tag{3.11}$$

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[3]) Notation: $\alpha, \beta = 1, 2$; $\mu, \nu = 0, 1, 2, 3$. Metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Antisymmetric tensors in two dimensions $\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$: $\varepsilon_{12} = \varepsilon^{12} = 1$ and in four dimensions $\varepsilon_{\mu\nu\varrho\sigma}$, $\varepsilon^{\mu\nu\varrho\sigma}$: $\varepsilon_{0123} = -\varepsilon^{0123} = -1$. The generalized Pauli matrices are defined by $\tilde{\sigma}^\mu\, \sigma^\nu + \tilde{\sigma}^\nu\, \sigma^\mu = 2\, g^{\mu\nu}\, \mathbf{1}$.

Finite Dimensional Representations

The connection of these homogeneous functions with the theory of finite dimensional representations of SL(2, C) is obvious. The finite dimensional representation $D(j_1, j_2)$, $(j_1, j_2 = 0, \pm 1/2, \pm 1, \ldots)$, can be defined on tensors of the type

$$A^{\alpha_1 \ldots \alpha_{2j_1} \dot{\beta}_1 \ldots \dot{\beta}_{2j_2}} \tag{3.12}$$

symmetric in $\alpha_1 \ldots \alpha_{2j_1}$ and in $\dot{\beta}_1 \ldots \dot{\beta}_{2j_2}$. The conventional transformation law for these tensors reads

$$A \to A' = D(g)\, A$$

$$A'^{\alpha_1 \ldots \dot{\beta}_{2j_2}} = A^{\gamma_1 \ldots \dot{\delta}_{2j_2}} g^{\alpha_1}{}_{\gamma_1} \ldots g^{*\dot{\beta}_{2j_2}}{}_{\dot{\delta}_{2j_2}} \,. \tag{3.13}$$

With each tensor of this type we may associate a homogeneous function

$$F(\zeta) = A^{\alpha_1 \ldots \alpha_{2j_1} \dot{\beta}_1 \ldots \dot{\beta}_{2j_2}} \zeta_{\alpha_1} \ldots \zeta_{\alpha_{2j_1}} \zeta^*_{\dot{\beta}_1} \ldots \zeta^*_{\dot{\beta}_{2j_2}} \,. \tag{3.14}$$

The degree of homogeneity is in this case

$$\alpha(\mu) = \mu^{2j_1} \mu^{*\,2j_2} \,. \tag{3.15}$$

This is of the form (3.3) only if $j_1 = j_2 = 0$, reflecting the fact that the only finite dimensional unitary representation of SL(2, C) is the trivial one.

The finite dimensional representations and the unitary representations can thus be treated in the same framework. This observation is at the basis of Fronsdal's theory of representations 'with continuous number of indices' [10]. In this connection the reader is referred to the more detailed discussion at the end of Section 4.

Scalar Product

The scalar product (2.3) now takes the form

$$(F, G) = \int d\mu(\zeta)\; F^*(\zeta)\; G(\zeta) \,. \tag{3.16}$$

The measure $d\mu(\zeta)$ is given by

$$d\mu(\zeta) = d\zeta\; \delta\,(\zeta_2 - 1)$$

where $d\zeta$ stands for the product of the differentials of real and imaginary parts of ζ_1 and ζ_2. It is obvious that the measure $d\mu(\zeta)$ is not explicitly covariant. In fact the above expression for the scalar product (F, G) is not invariant with respect to transformations under SL(2, C) defined in (3.4) unless the product $F^* G$ is a homogeneous function of the type

$$F^* G(\mu\, \zeta) = |\mu|^{-4}\, F^* G(\zeta) \,.$$

This relation is satisfied if the functions F, G are elements of the representation space and therefore obey (3.3).

In the applications of the unitary representations of SL(2, C) one of the central problems concerns the reduction of a given representation with respect to the so-called canonical basis associated with the maximal compact subgroup SU(2). If use is made of homogeneous functions the elements of the canonical basis can be written down in a very simple fashion. However, the reduction requires not only knowledge of the elements of the canonical basis, but also the evaluation of scalar products of these elements with an arbitrary vector of the representation space, and the non-covariant form of the measure which defines this scalar product is a serious drawback. Fortunately, it turns out that the integral defining the scalar product of an arbitrary element with an element of the canonical basis can be rewritten in such a fashion as to make it explicitly covariant. The advantages of this rearrangement are quite remarkable. The situation somewhat resembles the relationship between the covariant form of perturbation theory and the old-fashioned, non-covariant method. In fact the integration technique we shall employ is a variant of the method used by Bogoliubov and Shirkov in their book on the theory of quantized fields [14].

4. The Reduction SL(2, C) ⊃ SU(2)

Let us define the subgroup $SU(2)_M$ by

$$\boldsymbol{g}\,\boldsymbol{M}\,\boldsymbol{g}^+ = \boldsymbol{M} \qquad \boldsymbol{g} \in SU(2)_M \tag{4.1}$$

where $\boldsymbol{M}$ is a positive definite, hermitean matrix, the metric of the group $SU(2)_M$. For convenience, let us normalize this metric by

$$\det \boldsymbol{M} = 1\,. \tag{4.2}$$

(Note that if instead a metric $\tilde{M}$ is chosen with $\det \tilde{M} = -1$ then the resulting subgroup is the noncompact group SU(1, 1), locally isomorphic to the Lorentz group SO(2, 1) in 3 dimensions [15].)

Canonical Basis

We first want to show that the representation $U_{m,\varrho}$ contains a subspace, the elements of which transform irreducibly according to the representation $D(|m|/2)$ of $SU(2)_M$. In fact consider the set of functions

$$\begin{aligned} F_{\dot\alpha_1\ldots\dot\alpha_m}(\zeta;\boldsymbol{M}) &= \zeta^*_{\dot\alpha_1}\ldots\zeta^*_{\dot\alpha_m}(\zeta\,\boldsymbol{M}\,\zeta^+)^{1/2(-m+i\varrho)-1} \quad \text{if} \quad m \geqslant 0 \\ F_{\alpha_1\ldots\alpha_{|m|}}(\zeta;\boldsymbol{M}) &= \zeta_{\alpha_1}\ldots\zeta_{\alpha_{|m|}}(\zeta\,\boldsymbol{M}\,\zeta^+)^{1/2(m+i\varrho-1} \quad \text{if} \quad m \leqslant 0\,. \end{aligned} \tag{4.3}$$

We shall refer to these quantities as generating functions.

Clearly these functions are homogeneous of the required degree. Furthermore, under the action of the elements of $SU(2)_M$ the quantity $(\zeta\,\boldsymbol{M}\,\zeta^+)$ is invariant, whereas the monomials $\zeta^*_{\dot\alpha_1}\ldots\zeta^*_{\dot\alpha_m}$ and $\zeta_{\alpha_1}\ldots\zeta_{\alpha_{|m|}}$ do indeed transform according to $D(|m|/2)$. What must be shown in addition is that these functions actually belong to the Hilbert space of $U_{m,\varrho}$, i.e. that their norm, defined by the scalar product (3.16) is finite. We shall compute this norm explicitly as a byproduct of the more general calculation carried out below. Before we turn to this calculation let us note the transformation properties of the generating functions under the full group SL(2, C). Restricting ourselves to the case $m \geqslant 0$ we obtain

$$F'_{\dot\alpha_1\ldots\dot\alpha_m}(\zeta;\boldsymbol{M}) = F_{\dot\beta_1\ldots\dot\beta_m}(\zeta;\boldsymbol{g}\,\boldsymbol{M}\,\boldsymbol{g}^+)\,g^{*\dot\beta_1}{}_{\dot\alpha_1}\ldots g^{*\dot\beta_m}{}_{\dot\alpha_m}\,. \tag{4.4}$$

Infinitesimal transformations therefore generate linear combinations of functions of the type

$$F_{\alpha_1 \ldots \alpha_j \dot\beta_1 \ldots \dot\beta_{m+j}}(\zeta; \boldsymbol{M}) = \frac{\partial}{\partial M^{\alpha_1 \dot\beta_{m+1}}} \cdots \frac{\partial}{\partial M^{\alpha_j \dot\beta_{m+j}}} F_{\dot\beta_1 \ldots \dot\beta_m}(\zeta; \boldsymbol{M}) . \quad (4.5)$$

The traceless parts of these quantities transform according to $D(m/2 + j)$ under $SU(2)_M$ and we shall refer to these traceless parts as elements of the canonical basis. We have just verified that the elements of the canonical basis can be obtained from the generating functions by means of differentiation with respect to the metric $\boldsymbol{M}$. The action of SL(2, C) thus generates the following ladder of representations of SU(2): $D(m/2)$, $D(m/2 + 1)$, $D(m/2 + 2)$, ... The proof that this ladder actually exhausts the representation space can be found in Ref. [13].

Scalar Product of an Arbitrary Element with an Element of the Canonical Basis

We now turn to the problem of computing the norm of an element of the canonical basis. As a first step we investigate the more general problem of the inner product of an arbitrary element with an element of the canonical basis. To simplify the argument let us restrict ourselves to the particular representations with $m = 0$ and consider the generating function

$$F(\zeta; \boldsymbol{M}) = (\zeta\, \boldsymbol{M}\, \zeta^+)^{(i\varrho/2)-1} \quad (4.6)$$

which in this case is associated with the trivial representation $D(0)$ of $SU(2)_M$. Let $G(\zeta)$ be an arbitrary element of the representation space and consider the scalar product

$$(F, G) = \int d\mu(\zeta)\, (\zeta\, \boldsymbol{M}\, \zeta^+)^{-(i\varrho/2)-1}\, G(\zeta) = \int dz (\zeta_0\, \boldsymbol{M}\, \zeta_0^+)^{-(i\varrho/2)-1}\, G(\zeta_0) \quad (4.7)$$

where ζ_0 stands for the spinor

$$\zeta_0 = (z, 1) . \quad (4.8)$$

We now transform this integral by inserting the Gaussian representation

$$(\zeta_0\, \boldsymbol{M}\, \zeta_0^+)^{-(i\varrho/2)-1} = \frac{2}{\Gamma(1 + i\,\varrho/2)} \int_0^\infty d\lambda\, \lambda^{1+i\varrho}\, e^{-\lambda^2 (\zeta_0 \boldsymbol{M} \zeta_0^+)} . \quad (4.9)$$

The form of the exponent invites us to introduce instead of ζ_0 the spinor

$$\xi = \lambda\, \zeta_0; \quad \xi_1 = \lambda\, z; \quad \xi_2 = \lambda .$$

The integration over λ and z is equivalent to an integration over all complex values of ξ_1 and an integration over the real variable ξ_2 from 0 to infinity. In order to treat ξ_1 and ξ_2 on completely equal footing we consider instead of ξ the spinor ζ defined by

$$\zeta = \lambda\, e^{i\phi}\, \zeta_0; \quad \zeta_1 = \lambda\, e^{i\phi}\, z; \quad \zeta_2 = \lambda\, e^{i\phi} \quad (4.10)$$

with an arbitrary phase ϕ. Making use of the homogeneity of $G(\zeta)$ we have

$$(F, G) = \frac{2}{\Gamma(1 + i\,\varrho/2)} \int d\lambda\, \lambda^3\, dz\, e^{-\zeta \boldsymbol{M} \zeta^+}\, G(\zeta) . \quad (4.11)$$

The integrand is independent of ϕ. We are therefore allowed to replace the integral by its average over ϕ

$$(F, G) = \frac{1}{\pi \Gamma(1+i\varrho/2)} \int d\lambda\, \lambda^3\, d\phi\, dz\, e^{-\zeta \boldsymbol{M} \zeta^+}\, G(\zeta)\,. \tag{4.12}$$

The integration over λ, ϕ and z is equivalent to an integration over the 2 complex variables ζ_1 and ζ_2. It is straightforward to compute the Jacobian of this transformation with the result

$$d\zeta = d\lambda\, \lambda^3\, d\phi\, dz \tag{4.13}$$

where $d\zeta$ denotes the product of the differentials of real and imaginary parts of ζ_1 and ζ_2. Therefore

$$(F, G) = \frac{1}{\pi \Gamma(1+i\varrho/2)} \int d\zeta\, e^{-\zeta \boldsymbol{M} \zeta^+}\, G(\zeta) \tag{4.14}$$

and we have managed to express the scalar product in a manifestly covariant form.

The same method applies in the case of a representation $U_{m,\varrho}$ with arbitrary m. The scalar product of the generating function $F_{\dot\alpha_1 \ldots \dot\alpha_m}(\zeta; \boldsymbol{M})$ of the representation $U_{m,\varrho}$ with an arbitrary element $G(\zeta)$ is given by (we restrict ourselves to $m \geqslant 0$)

$$(F_{\dot\alpha_1 \ldots \dot\alpha_m}, G) = \frac{1}{\pi \Gamma[1/2\,(m+i\varrho+2)]} \int d\zeta\, e^{-\zeta \boldsymbol{M} \zeta^+}\, \zeta_{\alpha_1} \ldots \zeta_{\alpha_m}\, G(\zeta)\,. \tag{4.15}$$

The analogous scalar products involving elements of higher representations $D\,(m/2 + j)$ are obtained through differentiation with respect to the metric.

An Example

To evaluate the integral for a particular choice of the function $G(\zeta)$ let us again consider the representations with $m = 0$ and let us compute the expectation value of $U(\boldsymbol{g})$ in the state $F(\zeta; \boldsymbol{M})$ defined in (4.6). This expectation value is closely related to the so-called elementary spherical functions of SL(2, C), which can be found in Ref. [13].

$$G(\zeta) = U(\boldsymbol{g})\, F = F(\zeta; \boldsymbol{N})$$

$$\boldsymbol{N} = \boldsymbol{g}\, \boldsymbol{M}\, \boldsymbol{g}^+\,. \tag{4.16}$$

The matrix $\boldsymbol{N}$ is again positive definite, hermitean and unimodular. In the particular case $\boldsymbol{g} \in \mathrm{SU}(2)_M$ this expectation value represents the norm of F.

We make use of an exponential representation for the function $G(\zeta)$ to rewrite the scalar product as

$$(F, U(\boldsymbol{g})\, F) = \frac{1}{\pi} \left| \Gamma\left(1 + \frac{i\varrho}{2}\right) \right|^{-2} \int_0^\infty d\mu\, \mu^{-i\varrho/2} \int d\zeta\, e^{-\zeta \boldsymbol{M} \zeta^+ - \mu \zeta \boldsymbol{N} \zeta^+}. \tag{4.17}$$

The integration over the homogeneous variable ζ can be carried out by means of the well-known formula

$$\int d\zeta\, e^{-\zeta \boldsymbol{A} \zeta^+} = \pi^2 (\det \boldsymbol{A})^{-1} \tag{4.18}$$

with the result

$$(F, U(g), F) = \pi \left| \Gamma\left(1 + \frac{i\varrho}{2}\right) \right|^{-2} \int_0^\infty d\mu \, \mu^{-i\varrho/2} D^{-1}; \quad D = \det(\boldsymbol{M} + \mu \boldsymbol{N}) \,. \tag{4.19}$$

This integral converges for all positive definite hermitean matrices $\boldsymbol{M}$ and $\boldsymbol{N}$. Let us represent these matrices by[3])

$$\boldsymbol{M} = m^\mu \tilde{\boldsymbol{\sigma}}_\mu \,, \quad \boldsymbol{N} = n^\mu \tilde{\boldsymbol{\sigma}}_\mu \,; \quad m^\mu m_\mu = n^\mu n_\mu = 1 \,. \tag{4.20}$$

Furthermore denote the hyperbolic angle between the vectors m^μ and n^μ by χ

$$m^\mu n_\mu = \cosh\chi \,. \tag{4.21}$$

In terms of this angle we have

$$D = 1 + 2\mu \cosh\chi + \mu^2 \,. \tag{4.22}$$

The remaining definite integral over μ has the value[4])

$$(F, U(g)\, F) = N^2 \, \frac{2 \sin \varrho/2\, \chi}{\varrho \sinh \chi} \,; \; N^2 = \pi^2 \left| \Gamma\left(1 + \frac{i\varrho}{2}\right) \right|^2 \frac{\varrho}{2 \sinh \varrho\pi/2} \tag{4.23}$$

where N denotes the norm of F. If the element $\boldsymbol{g} \in \mathrm{SL}(2, \mathrm{C})$ represents a Lorentz transformation, then the angle χ introduced above is related to the relative velocity of the two reference frames connected by this transformation by

$$\mathrm{tgh}\,\chi = \frac{v}{c} \,. \tag{4.24}$$

The final expression (4.23) can readily be generalized to arbitrary elements of the canonical basis of an arbitrary representation $U_{m,\varrho}$. The corresponding integrals are of the general form[5])

$$\int d\zeta\, e^{-\zeta \boldsymbol{M} \zeta^+} \zeta_{\alpha_1} \cdots \zeta_{\alpha_n} \zeta^*_{\dot\beta_1} \cdots \zeta^*_{\dot\beta_n} (\zeta \boldsymbol{N} \zeta^+)^p$$
$$= (-1)^n \frac{\partial}{\partial M^{\alpha_1 \dot\beta_1}} \cdots \frac{\partial}{\partial M^{\alpha_n \dot\beta_n}} \int d\zeta\, e^{-\zeta \boldsymbol{M} \zeta^+} (\zeta \boldsymbol{N} \zeta^+)^p$$

and can therefore be evaluated by means of the same technique.

Finite Dimensional Representations

The methods used in this section can be applied to finite dimensional representations as well. We first demonstrate that an expression analogous to (4.15) can be given for the scalar product in the case of finite dimensional representations. In fact, consider the representation $D(j_1, j_2)$ with e.g. $j_1 \geqslant j_2$, which is defined on tensors of the type $A^{\alpha_1 \ldots \alpha_{2j_1} \dot\beta_1 \ldots \dot\beta_{2j_2}}$. The invariant scalar product of two tensors of this type is given by

$$\langle A, B \rangle = A_{\alpha_1 \ldots \alpha_{2j_1} \dot\beta_1 \ldots \dot\beta_{2j_2}} B^{\alpha_1 \ldots \alpha_{2j_1} \dot\beta_1 \ldots \dot\beta_{2j_2}} \tag{4.25}$$

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4) The expression for the elementary spherical function given in Ref. [13] contains a misprint.

5) In expressions involving differentiations with respect to the metric M the constraint $\det M = 1$ is to be imposed after the differentiations have been carried out.

where in the first factor we have lowered the indices by means of the invariant symbol[3]) $\varepsilon_{\alpha\beta}$. Now let B be an arbitrary element of the representation space and let $A_{\dot{\gamma}_1 \ldots \dot{\gamma}_\Delta}$, $(\Delta = 2j_1 - 2j_2)$, denote the elements of the smallest subspace in the reduction SL(2, C) $\supset$ SU(2)$_M$. This subspace transforms according to $D(\Delta/2)$ under SU(2)$_M$ and we have explicitly

$$A_{\dot{\gamma}_1 \ldots \dot{\gamma}_\Delta ; \alpha_1 \ldots \alpha_{2j_1} \dot{\beta}_1 \ldots \dot{\beta}_{2j_2}} = \underset{(\alpha)}{\mathsf{S}} M^{-1}_{\dot{\gamma}_1 \alpha_1} \ldots M^{-1}_{\dot{\gamma}_\Delta \alpha_\Delta} M^{-1}_{\dot{\beta}_1 \alpha_{\Delta+1}} \ldots M^{-1}_{\dot{\beta}_{2j_2} \alpha_{2j_1}} \tag{4.26}$$

where the operator $\underset{(\alpha)}{\mathsf{S}}$ projects out that part of the right hand side which is totally symmetric with respect to permutations of $\alpha_1 \ldots \alpha_{2j_1}$. (Note that we have assumed $j_1 \geqslant j_2$). We are interested in the scalar product $\langle A_{\dot{\gamma}_1 \ldots \dot{\gamma}_\Delta}, B \rangle$ and we want to show that this quantity can be expressed in terms of the homogeneous function $G(\zeta)$ associated with the element B

$$G(\zeta) = B^{\alpha_1 \ldots \alpha_{2j_1} \dot{\beta}_1 \ldots \dot{\beta}_{2j_2}} \zeta_{\alpha_1} \ldots \zeta_{\alpha_{2j_1}} \zeta^*_{\dot{\beta}_1} \ldots \zeta^*_{\dot{\beta}_{2j_2}} \tag{4.27}$$

in a manner completely analogous to (4.15) as follows

$$\langle A_{\dot{\gamma}_1 \ldots \dot{\gamma}_\Delta}, B \rangle = \gamma \int d\zeta \, e^{-\zeta \boldsymbol{M} \zeta^+} \zeta^*_{\dot{\gamma}_1} \ldots \zeta^*_{\dot{\gamma}_\Delta} G(\zeta) \,. \tag{4.28}$$

The proof is very simple. We merely have to evaluate the integral over ζ which is of the form

$$\int d\zeta \, e^{-\zeta \boldsymbol{M} \zeta^+} \zeta_{\alpha_1} \ldots \zeta_{\alpha_{2j_1}} \zeta^*_{\dot{\beta}_1} \ldots \zeta^*_{\dot{\beta}_{2j_1}} .$$

This evaluation is straightforward and the result reads

$$\pi^2 (2j_1)! \underset{(\alpha)}{\mathsf{S}} M^{-1}_{\dot{\beta}_1 \alpha_1} \ldots M^{-1}_{\dot{\beta}_{2j_1} \alpha_{2j_1}} .$$

We have thus verified (4.28) with

$$\gamma = \frac{1}{\pi^2 (2j_1)!} . \tag{4.29}$$

It is instructive to investigate the finite dimensional analog of the expectation value $(F, U(\boldsymbol{g}) F)$ considered above for the particular case of the unitary representations with $m = 0$. The finite dimensional analogs of these representations are the representations with $j_1 = j_2 = j$, since only these contain the trivial representation $D(0)$ in the reduction SL(2, C) $\supset$ SU(2). Let A denote the tensor belonging to the representation $D(0)$ contained in $D(j_1, j_2)$

$$A^{\alpha_1 \ldots \alpha_{2j} \dot{\beta}_1 \ldots \dot{\beta}_{2j}} = \underset{(\alpha)}{\mathsf{S}} M^{\alpha_1 \dot{\beta}_1} \ldots M^{\alpha_{2j} \dot{\beta}_{2j}} . \tag{4.30}$$

The expectation value of the operator $D(\boldsymbol{g})$ is then given by

$$\begin{aligned} \langle A, D(\boldsymbol{g}) A \rangle &= \gamma \int d\zeta \, e^{-\zeta \boldsymbol{M} \zeta^+} G(\zeta) \\ G(\zeta) = (\zeta \boldsymbol{N} \zeta^+)^{2j} &= (-1)^{2j} (2j)! \frac{1}{2\pi i} \oint \frac{dz}{z^{2j+1}} e^{-z(\zeta \boldsymbol{N} \zeta^+)} . \end{aligned} \tag{4.31}$$

Here N is the matrix defined above and the integral extends over a closed path in the complex z-plane around the origin. The integral over ζ can be evaluated with the result

$$\langle A, D(g)\, A\rangle = (-1)^{2j} \frac{1}{2\pi i} \oint \frac{dz}{z^{2j+1}}\, D^{-1};\; D = 1 + 2\, z \cosh\chi + z^2\,. \qquad (4.32)$$

Apart from the singularity at $z = 0$ the integrand possesses two simple poles at

$$z = -\,e^{\pm\chi}\,. \qquad (4.33)$$

The path of integration can therefore be deformed into two circles around these two poles and a circle at infinity which does not contribute. Evaluating the residues of the two poles we obtain

$$\langle A, D(g)\, A\rangle = \frac{\sinh(2\,j+1)\,\chi}{\sinh\chi}\,. \qquad (4.34)$$

This result closely resembles the corresponding expression given above for infinite dimensional unitary representations. In fact the two results are connected through analytic continuation [6]) in the group invariant j from positive half-integer j to

$$2\,j + 1 = \frac{i\,\varrho}{2}\,. \qquad (4.35)$$

This is a special case of a more general relation between matrix elements of finite dimensional representations and those of unitary representations, a relation extensively used by FRONSDAL [10]: The matrix elements of the unitary representation $U_{m,\varrho}$ of the principal series in the canonical basis are obtained from those of the finite dimensional representation $D(j_1, j_2)$ by continuing analytically in j_1 and j_2 to the points

$$2\,j_1 + 1 = \frac{1}{2}\,(-\,m + i\,\varrho) \qquad 2\,j_2 + 1 = \frac{1}{2}\,(+\,m + i\,\varrho)\,. \qquad (4.36)$$

This connection is made plausible by a comparison of the homogeneity conditions (3.15) for finite dimensional representations and (3.3) for unitary representations.

5. Clebsch-Gordan-Kernels

Before we proceed to an analysis of the more involved groups SL(n, C) let us briefly consider the reduction of direct product representations in the framework of SL(2, C), which again takes a particularly transparent form in terms of homogeneous variables.

The reduction of direct products of unitary representations of SL(2, C) has been solved by NAIMARK [16]. As far as the principal series is concerned, NAIMARK's main result is the statement that the direct product of two representations characterized by (m_1, ϱ_1) and by (m_2, ϱ_2) respectively contains the representation (m_3, ϱ_3) if and only if $m_1 + m_2 + m_3$ is even in which case the representation (m_3, ϱ_3) occurs exactly once [7]). The generalization of the well-known Clebsch-Gordan-coefficients associated with the reduction of representations of the rotation group is a Clebsch-Gordan-kernel with the following property. Let $f_1(z_1)$ and $f_2(z_2)$ be two elements of the representation

[6]) This statement concerns the normalized expectation values.

[7]) For a more precise statement of this theorem see Ref. [16].

spaces of the representations U_{m_1, ϱ_1} and U_{m_2, ϱ_2} respectively. The Clebsch-Gordan-kernel $k(z_3 \mid z_1, z_2)$ accomplishes the coupling of f_1 and f_2 to a function f which transforms according to $U_{m_3, \varrho}$

$$f(z_3) = \int dz_1\, dz_2\, k(z_3 \mid z_1, z_2)\, f_1(z_1)\, f_2(z_2)\,. \tag{5.1}$$

An explicit expression for the kernel k can be found in Ref. [16]. This explicit expression can easily be obtained making use of homogeneous variables by means of the following heuristic procedure.

Let us associate with $k(z_3 \mid z_1, z_2)$ the homogeneous kernel $K(\zeta_3 \mid \zeta_2, \zeta_1)$

$$K(\zeta_3 \mid \zeta_2, \zeta_1) = \alpha_1^*(\lambda_1)\, \alpha_2^*(\lambda_2)\, \alpha_3(\lambda_3)\, k(z_3 \mid z_1, z_2)$$
$$\zeta_i = \lambda_i(z_i, 1)\,. \tag{5.2}$$

By definition the degree of homogeneity of K is

$$K(\mu_3\, \zeta_3 \mid \mu_1\, \zeta_1, \mu_2\, \zeta_2) = \alpha_1^*(\mu_1)\, \alpha_2^*(\mu_2)\, \alpha_3(\mu_3)\, K(\zeta_3 \mid \zeta_1, \zeta_2)\,. \tag{5.3}$$

The transformation properties of k imply in the usual way the transformation properties of K. In fact, K must simply be an invariant function

$$K(\zeta_3\, g \mid \zeta_1\, g, \zeta_2\, g) = K(\zeta_3 \mid \zeta_1, \zeta_2)\,. \tag{5.4}$$

It is a simple matter to solve this invariance condition. There are only 6 algebraically independent invariants that can be formed out of 3 spinors: $r_1, r_2, r_3, r_1^*, r_2^*, r_3^*$

$$r_1 = \zeta_2\, \varepsilon\, \zeta_3 = \zeta_{2\alpha}\, \varepsilon^{\alpha\beta}\, \zeta_{3\beta} \text{ (cycl.)} \tag{5.5}$$

where $\varepsilon^{\alpha\beta}$ is the antisymmetric tensor. The homogeneity condition can be satisfied with the ansatz

$$K(\zeta_3 \mid \zeta_1, \zeta_2) = r_1^{S_1^+}\, r_2^{S_2^+}\, r_3^{S_3^+}\, r_1^{*S_1^-}\, r_2^{*S_2^-}\, r_3^{*S_3^-}$$
$$S_1^\pm = \pm \frac{1}{4}(-m_1 + m_2 - m_3) + \frac{i}{4}(\varrho_1 - \varrho_2 + \varrho_3) - \frac{1}{2}$$
$$S_2^\pm = \pm \frac{1}{4}(m_1 - m_2 - m_3) + \frac{i}{4}(-\varrho_1 + \varrho_2 + \varrho_3) - \frac{1}{2}$$
$$S_3^\pm = \pm \frac{1}{4}(m_1 + m_2 + m_3) + \frac{i}{4}(-\varrho_1 - \varrho_2 - \varrho_3) - \frac{1}{2} \tag{5.6}$$

If this kernel is reexpressed in terms of the variables z_1, z_2, z_3 one does indeed recover the expression given by NAIMARK. Note, however, that the kernel $K(\zeta_3 \mid \zeta_1, \zeta_2)$ may be used directly to carry out the reduction of direct products of elements of the canonical basis. Let us restrict ourselves to the case $m_1 = m_2 = m_3 = 0$ and let $F_1(\zeta_1; \boldsymbol{M}_1)$ and $F_2(\zeta_2; \boldsymbol{M}_2)$ denote the generating functions of U_{m_1, ϱ_1} and U_{m_2, ϱ_2}. The homogeneous function $F(\zeta_3)$ belonging to the representation U_{m_3, ϱ_3} in the direct product is then given by

$$F(\zeta_3) = N \int d\zeta_1\, d\zeta_2\, e^{-\zeta_1 \boldsymbol{M} \zeta_1^+ - \zeta_2 \boldsymbol{M} \zeta_2^+}\, K(\zeta_3 \mid \zeta_1, \zeta_2)\,. \tag{5.7}$$

This procedure can easily be extended to representations of SL(n, C). For applications of this method the reader is referred to Refs. [6, 17].

6. Principal Series of Unitary Representations of SL(n, C)

We now take up the generalization of the method explained in the previous sections to the more general and more involved case of the group SL(n, C). Let us briefly review the GELFAND-NAIMARK theory [7] of the principal series of unitary representations of SL(n, C). The group SL(n, C) admits of several pairs of subgroups of the type Z and K considered in the case of SL(2, C). In fact let $(n_1, n_2, \ldots, n_r)$ be some partition of the number n such that

$$n_1 + n_2 + \cdots + n_r = n; \; 1 \leqslant r \leqslant n; \; n_p \text{ positive integers.}$$

Given this partition, divide each $n \times n$ matrix $\boldsymbol{s}$ into r^2 blocks

$$\boldsymbol{s} = \{s^p_q\}; \; p, q = 1, \ldots, r$$

where the block $\boldsymbol{s}^p_q$ is a rectangular matrix with n_p rows and n_q columns. The subgroup Z belonging to the given partition then consists of elements of the form

$$\boldsymbol{z} = \{\boldsymbol{z}^p_q\}; \; \boldsymbol{z}^p_q = 0 \text{ if } p < q; \; \boldsymbol{z}^p_p = \text{unit matrix}$$

and the elements of the complementary group K are characterized by

$$\boldsymbol{k} = \{\boldsymbol{k}^p_q\}; \; \boldsymbol{k}^p_q = 0 \text{ if } p > q .$$

Almost all elements of the group SL(n, C) can again be decomposed uniquely[8]) as a product of elements of K and Z

$$\boldsymbol{s} = \boldsymbol{k}\,\boldsymbol{z} .$$

As an illustration consider the partition $(n-1, 1)$, i.e. $r = 2$, $n_1 = n - 1$, $n_2 = 1$. In this case the elements of the subgroups Z and K are of the form

$$\boldsymbol{z} = \left(\begin{array}{c|c} \mathbf{1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline z_1 \ldots z_{n-1} & 1 \end{array}\right) \qquad \boldsymbol{k} = \left(\begin{array}{c|c} \tilde{\boldsymbol{k}} & \begin{matrix} k^1_n \\ \vdots \end{matrix} \\ \hline 0 \ldots 0 & \boldsymbol{k}^n_n \end{array}\right). \tag{6.1}$$

To each of the possible partitions of n there belongs a class of unitary representations. These representations are defined on functions $f(\boldsymbol{z})$ on the subgroup Z as in the case of SL(2, C) and the definition of the representation $U(\boldsymbol{g})$ reads

$$f \to f' = U(\boldsymbol{g})\, f \qquad f'(\boldsymbol{z}) = \alpha(\boldsymbol{k}')\, f(\boldsymbol{z}') \tag{6.2}$$

where the elements $\boldsymbol{k}'$ and $\boldsymbol{z}'$ are again the unique resolution of the matrix $\boldsymbol{z}\,\boldsymbol{g}$

$$\boldsymbol{k}'\,\boldsymbol{z}' = \boldsymbol{z}\,\boldsymbol{g} . \tag{6.3}$$

The above definition of $U(\boldsymbol{g})$ furnishes a representation of SL(n, C) provided $\alpha(\boldsymbol{k})$ is of the form

$$\alpha(\boldsymbol{k}) = \varkappa_1^{a_1^+} \ldots \varkappa_r^{a_r^+} \; {\varkappa_1^*}^{a_1^-} \ldots {\varkappa_r^*}^{a_r^-} \tag{6.4}$$

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[8]) For a characterization of the set of measure zero of matrices s for which this decomposition is not possible see Ref. [7].

where the quantities $\varkappa_p$ are the determinants of the blocks of the matrix $\boldsymbol{k}$ situated along the diagonal

$$\varkappa_p = \det \boldsymbol{k}_p^p \tag{6.5}$$

and satisfy

$$\varkappa_1 \varkappa_2 \ldots \varkappa_r = 1 \,. \tag{6.6}$$

In the example considered above

$$\varkappa_1 = \det \tilde{\boldsymbol{k}};\ \varkappa_2 = k_n^n \,. \tag{6.7}$$

The representation is required to be unitary in the scalar product

$$(f, g) = \int dz\, f^*(\boldsymbol{z})\, g(\boldsymbol{z}) \,. \tag{6.8}$$

Here dz stands for the product of the differentials of real and imaginary parts of all nontrivial elements of $\boldsymbol{z}$.

This requirement fixes the absolute value of $\alpha(\boldsymbol{k})$ whereas the phase is restricted only by the requirement that the function $\alpha(\boldsymbol{k})$ be one-valued on K. This leads to the expression

$$a_p^{\pm} = \mp \frac{1}{2}\, m_p + \frac{i}{2}\, \varrho_p + \frac{1}{2}\,(n_p + n_1) - \sum_{q=1}^{p} n_q \tag{6.9}$$

where $(m_1, m_2, \ldots, m_r)$ are integers and $(\varrho_1, \varrho_2, \ldots, \varrho_r)$ are arbitrary real parameters. Clearly these parameters are redundant since the transformation

$$m_p \to m_p + m_0;\ \varrho_p \to \varrho_p + \varrho_0 \tag{6.10}$$

leaves the function $\alpha(\boldsymbol{k})$ and hence the representation $U(\boldsymbol{g})$ unchanged. This ambiguity in the choice of the variables m_p and ϱ_p can be removed by the normalization

$$m_1 = \varrho_1 = 0 \,. \tag{6.11}$$

With this normalization the exponents $a_1^{\pm}$ vanish such that $\alpha(\boldsymbol{k})$ does not involve $\varkappa_1$, or $\varkappa_1^*$. We shall in the following adhere to this normalization. The representations of the principal series associated with the partition $(n_1, \ldots, n_r)$ – referred to as the degeneracy type of the representations belonging to this class – are therefore labelled by $r - 1$ integers, $(m_2, \ldots, m_r)$ and $r - 1$ real parameters $(\varrho_2, \ldots, \varrho_r)$. In the particular case of the partition $(n - 1, 1)$ considered above the representations are labelled by one integer m and one real parameter ϱ and the multiplier $\alpha(\boldsymbol{k})$ takes the form

$$\alpha(\boldsymbol{k}) = (k_n^n)^{(1/2)(-m+i\varrho-n)}\, (k_n^{*\,n})^{(1/2)(+m+i\varrho-n)} \,. \tag{6.12}$$

To conclude this short review we mention the following equivalence theorem [7]:

Two representations characterized by the degeneracy types $(n_1, \ldots, n_r)$ and $(\tilde{n}_1, \ldots, \tilde{n}_{\tilde{r}})$ and the parameters $(m_2, \ldots, m_r)$, $(\varrho_2, \ldots, \varrho_r)$ and $(\tilde{m}_2, \ldots, \tilde{m}_{\tilde{r}})$, $(\tilde{\varrho}_2, \ldots, \tilde{\varrho}_{\tilde{r}})$ are equivalent if and only if

(1) $r = \tilde{r}$

(2) there exists a permutation $1, \ldots, r \to i_1, \ldots, i_r$ such that $\tilde{n}_p = n_{i_p}$;

$$\tilde{m}_p = m_{i_p} + m_0;\ \tilde{\varrho}_p = \varrho_{i_p} + \varrho_0 \,. \tag{6.13}$$

In the case of SL(2, C) this theorem reduces to the statement mentioned in Section 2.

7. Representations of SL(n, C) in Terms of Homogeneous Functions

As in the case of SL(2, C) the representations of the groups SL(n, C) can alternatively be described in terms of homogeneous functions which we introduce as follows. Let $\boldsymbol{s}$ be a unimodular $n \times n$ matrix, i.e. $\boldsymbol{s} \in \mathrm{SL}(n, C)$, and let s^A_B denote its matrix elements ($A, B = 1, \ldots, n$). We define the homogeneous function $F(\boldsymbol{s})$ associated with the element $f(\boldsymbol{z})$ of the Gelfand-Naimark representation by

$$F(\boldsymbol{s}) = \alpha(\boldsymbol{k})\, f(\boldsymbol{z}); \quad \boldsymbol{s} = \boldsymbol{k}\, \boldsymbol{z}\,. \tag{7.1}$$

The transformation law for the element $f(\boldsymbol{z})$ implies a simple transformation law for the homogeneous function

$$F \to F' = U(\boldsymbol{g})\, F \qquad F'(\boldsymbol{s}) = F(\boldsymbol{s}\, \boldsymbol{g})\,. \tag{7.2}$$

Furthermore from the fact that K is a subgroup we conclude that

$$F(\boldsymbol{k}_0\, \boldsymbol{s}) = \alpha(\boldsymbol{k}_0)\, F(\boldsymbol{s}); \quad \boldsymbol{k}_0 \in K\,. \tag{7.3}$$

This is the generalized homogeneity condition. The explicit solution of this condition will be given below. To get some feeling for the meaning of this constraint we consider the two following special cases. First, suppose that $\boldsymbol{k}_0$ vanishes below the diagonal and that its diagonal elements are unity, whereas the elements above the diagonal are arbitrary. In this case $\alpha(\boldsymbol{k}_0) = 1$ and the constraint implies that the function $F(\boldsymbol{s})$ is unchanged if to the A^{th} row vector of $\boldsymbol{s}$ an arbitrary linear combination of the row vectors below it are added. The function $F(\boldsymbol{s})$ therefore depends on $\boldsymbol{s}$ only through those combinations which are independent under such a transformation, i.e. through the determinants $\Delta(j)$

$$\Delta(j)_{A_1 \ldots A_j} = \det \begin{vmatrix} s^{n-j+1}_{A_1} & \ldots & s^{n-j+1}_{A_j} \\ s^n_{A_1} & \ldots & s^n_{A_j} \end{vmatrix} \quad j = 1, \ldots, n-1 \tag{7.4}$$

which are antisymmetric with respect to permutations of $A_1, \ldots, A_j$. In particular we note that $F(\boldsymbol{s})$ is independent of the first row of $\boldsymbol{s}$. Clearly under the transformations (7.2) these determinants transform according to the finite dimensional antisymmetric tensor representation of rank j.

Next consider a diagonal matrix $\boldsymbol{k}_0$. In this case the row vectors of the matrix $\boldsymbol{k}_0\, \boldsymbol{s}$ are multiples of the corresponding row vectors of $\boldsymbol{s}$ and therefore the generalized homogeneity condition (7.3) implies that $F(\boldsymbol{s})$ is a homogeneous function of these vectors with a degree of homogeneity given by the multiplier $\alpha(\boldsymbol{k}_0)$. – These two special cases do not exhaust the full subgroup K except in the case $r = n$, $n_1 = n_2 = \ldots n_r = 1$, the so-called non-degenerate representations. For the so-called degenerate representations, for which at least one of the elements n_p of the partition is different from 1, the homogeneity condition is stronger. The homogeneous functions are then allowed to depend only on those particular determinants whose rank corresponds in a specific way to the partition of the representation. In this general case the solution of the homogeneity condition reads explicitly

$$\begin{aligned} &F(\boldsymbol{s}) = F[\Delta(l_1), \ldots, \Delta(l_{r-1})] \\ &F[\lambda_1 \Delta(l_1), \ldots, \lambda_{r-1} \Delta(l_{r-1})] \\ &\quad = \lambda_1^{\alpha_1^+} \ldots \lambda_{r-1}^{\alpha_{r-1}^+} \lambda_1^{*\alpha_1^-} \ldots \lambda_{r-1}^{*\alpha_{r-1}^-} F[\Delta(l_1), \ldots, \Delta(l_{r-1})] \end{aligned} \tag{7.5}$$

where we have used the notation

$$l_p = n_r + n_{r-1} + \cdots + n_{r-p+1}; \; p = 1, \ldots, r \tag{7.6}$$

for the rank of the tensors appearing in F. The degrees of homogeneity are given by

$$\alpha_p^{\pm} = \mp \frac{1}{2} (m_{r-p+1} - m_{r-p}) + \frac{i}{2} (\varrho_{r-p+1} - \varrho_{r-p}) - \frac{1}{2} (n_{r-p+1} + n_{r-p}) \,. \tag{7.7}$$

To summarize this discussion we note the following features of the homogeneous functions

1) The independent variables of the homogeneous functions are the row vectors of the matrix $\boldsymbol{s}$:

$$F = F(s^2, s^3, \ldots, s^n); \; s^A = (s_1^A, \ldots, s_n^A) \,.$$

2) Under the action of SL(n, C) these row vectors are transformed according to

$$s^A \rightarrow s^A \, \boldsymbol{g}; \; F(s^2, \ldots, s^n) \rightarrow F(s^2 \, \boldsymbol{g}, \ldots, s^n \, \boldsymbol{g}) \,. \tag{7.8}$$

This transformation law is independent of the particular representation under consideration.

3) The homogeneous functions depend on the row vectors of $\boldsymbol{s}$ only through the particular combinations occuring in the antisymmetric tensors $\Delta(j)$ defined in (7.4). Furthermore, if the representation is degenerate, only those tensors occur whose rank is related to the partition of the representation by (7.6).

4) The function $F(\boldsymbol{s})$ is a homogeneous function of the tensors $\Delta(j)$ and, therefore a homogeneous function of the row vectors $s^2, \ldots, s^n$ with a degree of homogeneity specified by the parameters $(m_2, \ldots, m_r)$ and $(\varrho_2, \ldots, \varrho_r)$ characterizing the representation.

5) Finally we note that the relation between the homogeneous function $F(\boldsymbol{s})$ and the corresponding element in the Gelfand-Naimark representation is specified by (7.1). This relation is based on the resolution of the implicit definition of the variables $\boldsymbol{k}$ and $\boldsymbol{z}$ in terms of $\boldsymbol{s}$

$$\boldsymbol{s} = \boldsymbol{k} \, \boldsymbol{z} \,. \tag{7.9}$$

As we know that $F(\boldsymbol{s})$ depends on the matrix $\boldsymbol{s}$ only through the tensors $\Delta(j)$ it must be possible to express both the matrix $\boldsymbol{z}$ as well as the quantities $\varkappa_1, \ldots, \varkappa_r$ appearing in the multiplier $\alpha(\boldsymbol{k})$ through these tensors. Such an expression can indeed be found in the book by Gelfand and Naimark [7]. It reads as follows. Let

$$j_p = n_1 + \cdots + n_{p-1} \,. \tag{7.10}$$

Then the nontrivial elements of the matrix $\boldsymbol{z}$ are given by

$$z^{j_p + a}_{\;b} = (-1)^{a+1} \frac{\Delta \, (n - j_p)_{b, j_p+1, j_p+2, \ldots, n}}{\Delta \, (n - j_p)_{j_p+1, j_p+2, \ldots n}} \tag{7.11}$$

where in the numerator the index $j_p + a$ is omitted. For fixed p the indices a and b cover the range

$$1 \leqslant a \leqslant n_p; \; 1 \leqslant b \leqslant j_p + n_p$$

and p runs from 2 to r.

On the other hand the determinants $\varkappa_1, \ldots, \varkappa_r$ are given by

$$\varkappa_p = \frac{\Delta\,(n-j_p)_{j_p+1,\ldots,n}}{\Delta\,(n-j_{p+1})_{j_{p+1}+1,\ldots,n}}\,; \quad p = 1, \ldots, r-1\,; \quad \varkappa_r = \Delta(1)_n\,. \tag{7.12}$$

Examples

As a first example consider again the representations belonging to the partition $(n-1, 1)$. The form of the matrices $\boldsymbol{z}$ and $\boldsymbol{k}$ has been given in (6.1). According to (7.5) the corresponding homogeneous variables reduce to one single n-component vector

$$\Delta(1)_A = s^n_A;\ F = F[\Delta(1)] \tag{7.13}$$

and we note that the explicit expressions (7.11) and (7.12) reduce to

$$z^n_A = \frac{\Delta(1)_A}{\Delta(1)_n} = \frac{s^n_A}{s^n_n}\,;\ \varkappa_2 = \Delta(1)_n = s^n_n\,. \tag{7.14}$$

As another illustration consider the partition $(1, n-2, 1)$. In this case there are two sets of homogeneous variables

$$\Delta(1)_A = s^n_A$$

$$\Delta\,(n-1)_{A_1\ldots A_{n-1}} = \det \begin{vmatrix} s^2_{A_1} & \ldots & s^2_{A_{n-1}} \\ \vdots & & \vdots \\ s^n_{A_1} & \ldots & s^n_{A_{n-1}} \end{vmatrix}\,. \tag{7.15}$$

It is convenient to use instead of the antisymmetric symbol $\Delta\,(n-1)_{A_1\ldots A_{n-1}}$ the dual quantity

$$\Delta\,(n-1)^A = \frac{1}{(n-1)\,!}\,\varepsilon^{AA_1\ldots A_{n-1}}\,\Delta\,(n-1)_{A_1\ldots A_{n-1}} = \varepsilon^{AA_1\ldots A_{n-1}}\,s^2_{A_1}\ldots s^n_{A_{n-1}}\,. \tag{7.16}$$

We again state the explicit expressions for the GELFAND-NAIMARK variables

$$z^n_A = \frac{\Delta(1)_A}{\Delta(1)_n}\,;\ z^B_1 = -\,\frac{\Delta\,(n-1)^B}{\Delta\,(n-1)^1}\,. \tag{7.17}$$

Note that the tensors $\Delta(1)$ and $\Delta(n-1)$ satisfy the identity

$$\Delta\,(n-1)^A\,\Delta(1)_A = 0\,. \tag{7.18}$$

This reflects the fact that the tensors $\Delta(j)$ are in general redundant. They are not suitable as independent variables, since their definition (7.4) implies a number of constraints of the type (7.18). Of course these constraints are identically satisfied if one considers not the tensors $\Delta(j)$ but the row vectors of the matrix $\boldsymbol{s}$ as independent variables and this is the reason why we shall use the elements of the matrix $\boldsymbol{s}$ rather than the tensors $\Delta(j)$ as variables of integration in the following sections.

Finite Dimensional Representations

It is again instructive to investigate the role of the homogeneous functions for finite dimensional representations. The general irreducible finite dimensional representation of SL(n, C) can be characterized by a Young tableau [18]. In order to simplify the notation let us restrict ourselves to the simplest group which possesses nontrivial Young tableaus, SL(3, C). The irreducible finite dimensional representations of SL(3, C) are associated with tensors of the type

$$T^{A_1 \dots A_k B_1 C_1 \dots B_l C_l \dot{D}_1 \dots \dot{D}_m \dot{E}_1 \dot{F}_1 \dots \dot{E}_n \dot{F}_n} . \tag{7.19}$$

This tensor is antisymmetric in each of the pairs $B_i\, C_i$ and $\dot{E}_k\, \dot{F}_k$ whereas it is symmetric with respect to permutations of the indices $A_1, \dots, A_k$, or $\dot{D}_1, \dots, \dot{D}_m$ and with respect to permutations of the pairs $B_1\, C_1, \dots, B_l\, C_l$ or $\dot{E}_1\, \dot{F}_1, \dots, \dot{E}_n\, \dot{F}_n$. Furthermore it satisfies the mixed symmetry condition

$$T^{A_1 \dots B_1 C_1 \dots} + T^{B_1 \dots C_1 A_1 \dots} + T^{C_1 \dots A_1 B_1 \dots} = 0 . \tag{7.20}$$

and a similar relation for the dotted indices.

These symmetry conditions are automatically taken into account by associating with this tensor a homogeneous function $F[\Delta(1), \Delta(2)]$ as follows

$$F[\Delta(1), \Delta(2)] = T^{A_1 \dots B_1 C_1 \dots \dot{D}_1 \dots \dot{E}_1 \dot{F}_1 \dots} \Delta(1)_{A_1} \dots \Delta(2)_{B_1 C_1} \dots \Delta^*(1)_{\dot{D}_1} \dots \Delta^*(2)_{\dot{E}_1 \dot{F}_1} \cdots \tag{7.21}$$

The polynomial in the homogeneous variables automatically projects out the symmetric parts and furthermore takes care of the mixed symmetry conditions due to the fact that $\Delta(1)$ and $\Delta(2)$ are not independent but satisfy

$$\Delta(1)_A\, \Delta(2)_{BC} + \Delta(1)_B\, \Delta(2)_{CA} + \Delta(1)_C\, \Delta(2)_{AB} = 0 . \tag{7.22}$$

The degree of homogeneity of $F[\Delta(1), \Delta(2)]$ again specifies the representation completely. For the particular finite dimensional representations for which the tensor T happens to contain no antisymmetric pairs, the associated homogeneous function is independent of $\Delta(2)$. This corresponds to representations associated with the partition (2,1). Alternatively if single indices are absent, F is independent of $\Delta(1)$ and the representation corresponds to the partition (1,2).

We note that in the case of SL(2, C) the only allowed nontrivial partition is (1, 1) and the homogeneous functions are always functions of only one spinor variable ζ. This corresponds of course to the fact that the tensor representations of SL(2, C) do not contain antisymmetric pairs, simply because an object $T^{\alpha\beta}$ antisymmetric in α and β is proportional to $\varepsilon^{\alpha\beta}$ and hence is an invariant with respect to SL(2, C). The same reason excludes triples of antisymmetric pairs in the case of SL(3, C).

The generalization of these homogeneous functions to the finite dimensional representations of SL(n, C) is obvious; we shall omit it to avoid crowding of indices. Instead we now turn to the generalization of the covariant technique to the reduction of representations of SL(n, C).

8. The Reduction SL(n, C) $\supset$ SU(n)

In this section we want to show that the covariant technique used in the case of SL(2, C) for the reduction with respect to SU(2) can be generalized to SL(n, C). Before we deal with the general case let us illustrate the method by means of the simplest example, the degeneracy type $(n-1, 1)$.

Representations of Degeneracy Type $(n-1, 1)$

This case presents the closest analogy with SL(2, C). We have seen in Section 7 that this degeneracy type corresponds to homogeneous functions $F[\Delta(1)]$ of a single n-vector $\Delta(1)_A$ and we have noted in Section 6 that the representations associated with this degeneracy type can be labelled by one integer m and one real parameter ϱ. To simplify the writing we use the symbol ζ instead of $\Delta(1)$

$$\zeta_A = \Delta(1)_A = s_A^n .$$

Furthermore we denote the metric characterizing the subgroup $\mathrm{SU}(n)_M$ by the $n \times n$ positive definite, hermitean, unimodular matrix M^{AB} such that

$$\boldsymbol{g}\, \boldsymbol{M}\, \boldsymbol{g}^+ = \boldsymbol{M}; \quad \boldsymbol{g} \in \mathrm{SU}(n)_M .$$

Using the same arguments as in the case of SL(2, C) we conclude that the generating function, i.e. the element associated with the smallest representation of $\mathrm{SU}(n)_M$ is of the form (we assume $m \geqslant 0$)

$$F_{\dot{A}_1 \dots \dot{A}_m}(\zeta; \boldsymbol{M}) = \zeta^*_{\dot{A}_1} \dots \zeta^*_{\dot{A}_m} (\zeta\, \boldsymbol{M}\, \zeta^+)^{1/2(-m+i\varrho-n)} . \tag{8.1}$$

The scalar product of this element of the canonical basis with an arbitrary element of the representation space can be treated in complete analogy with the corresponding procedure in the case of SL(2, C) and one arrives at a result analogous to (4.15)

$$(F_{\dot{A}_1 \dots \dot{A}_m}, G) = \frac{1}{\pi\, \Gamma[1/2\,(m+i\varrho+n)]} \int d\zeta\, \zeta_{\dot{A}_1} \dots \zeta_{\dot{A}_m}\, e^{-\zeta \boldsymbol{M} \zeta^+}\, G(\zeta) . \tag{8.2}$$

Here the integration extends over real and imaginary parts of all n components of ζ.

The General Case

We now consider an arbitrary partition. The homogeneous functions associated with the given representation are of the type

$$F(\boldsymbol{s}) = F[\Delta(l_1), \dots, \Delta(l_{r-1})] \tag{8.3}$$

and the degree of homogeneity of F is given by (7.5). In particular the generating function of the canonical basis of the representation is of the form

$$F(\boldsymbol{s}; \boldsymbol{M}) = P(\boldsymbol{s})\, D_1^{\alpha_1} \dots D_{r-1}^{\alpha_{r-1}} . \tag{8.4}$$

Here we have suppressed the tensor indices of F which are carried on the right hand side by the polynomial P. This polynomial is the generalization of the monomial $\zeta^*_{\dot{A}_1} \dots \zeta^*_{\dot{A}_m}$ appearing in the special case of the partition $(n-1, 1)$. The general expression reads

$$P(\boldsymbol{s}) = P_1[\Delta(l_1)] \dots P_{r-1}[\Delta(l_{r-1})]$$

$$P_p[\Delta(l_p)] = \begin{cases} \Delta^*(l_p)_{\dot{A}^1_1 \dots \dot{A}^1_{l_p}} \dots \Delta^*(l_p)_{\dot{A}^{s_p}_1 \dots \dot{A}^{s_p}_{l_p}}, & s_p \geqslant 0 \\ \Delta(l_p)_{A^1_1 \dots A^1_{l_p}} \dots \Delta(l_p)_{A^{|s_p|}_1 \dots A^{|s_p|}_{l_p}}, & s_p < 0 . \end{cases} \tag{8.5}$$

The integer s_p generalizes the parameter m occuring in Equation (8.1); the value of s_p can be read off from (7.5)

$$s_p = m_{r-p+1} - m_{r-p} . \tag{8.6}$$

The 'unitarizing factors' D_p in the above definition of the generating function generalize the quantity $(\zeta\, \boldsymbol{M}\, \zeta^+)$ which appears in the generating function (8.1) associated with the partition $(n-1, 1)$:

$$D_p = \frac{1}{l_p!} \Delta(l_p)_{A_1 \ldots A_{l_p}} M^{A_1 \dot{B}_1} \ldots M^{A_{l_p} \dot{B}_{l_p}} \Delta^*(l_p)_{\dot{B}_1 \ldots \dot{B}_{l_p}} . \tag{8.7}$$

The exponents α_p can again be read off from (7.5):

$$\alpha_p = \begin{cases} \alpha_p^+ & \text{if } s_p \geqslant 0 \\ \alpha_p^- & \text{if } s_p \leqslant 0 \end{cases} . \tag{8.8}$$

The generalization of our method to arbitrary representations is contained in the statement that the scalar product of the generating function $F(\boldsymbol{s}, \boldsymbol{M})$ with an arbitrary element G of the representation space is given by the covariant formula

$$(F, G) = \Gamma \int \tilde{d}s\, e^{-\tilde{Tr}(\boldsymbol{s} \boldsymbol{M} \boldsymbol{s}^+)}\, P^*(\boldsymbol{s})\, G(\boldsymbol{s}) . \tag{8.9}$$

Here $G(\boldsymbol{s})$ is the homogeneous function representing the element G. The integration extends over real and imaginary parts of the elements of the matrix $\boldsymbol{s}$ except those belonging to the top row

$$\tilde{d}s = \prod_{A=2}^{n} \prod_{B=1}^{n} d(Re\, s_B^A)\, d(Im\, s_B^A) . \tag{8.10}$$

Note that since the integration does not include the first row of $\boldsymbol{s}$ the restriction $\det \boldsymbol{s} = 1$ is immaterial. Analogously the symbol $\tilde{Tr}$ indicates

$$\tilde{Tr}(\boldsymbol{s}\, \boldsymbol{M}\, \boldsymbol{s}^+) = \sum_{A=2}^{n} (s^A\, \boldsymbol{M}\, s^{A^+}) \tag{8.11}$$

where the quantity s^A stands for the row vector

$$s^A = (s_1^A, \ldots, s_n^A) . \tag{8.12}$$

A proof of the above general formula together with an explicit expression for the constant Γ is given in the appendix.

Clearly, in the case of SL(2, C) the formula reduces to the result given in Section 4. For the degenerate representations of SL(n, C) belonging to the partition $(n-1, 1)$ we recover (8.2), since in this case both G and P depend only on the bottom row of $\boldsymbol{s}$; the integrations over the $n-2$ other rows can be carried out in a trivial fashion.

9. Summary

We have shown that the problems associated with the reduction of the representations of SL(n, C) with respect to its maximal compact subgroup can be treated in terms of homogeneous variables. In particular we have expressed the scalar product of an arbitrary element of the representation with the generating function of the canonical basis (cf. 8.9). This expression explicitly displays the transformation properties under the group SL(n, C). The heuristic value of this reformulation of the theory of GELFAND and NAIMARK lies in the fact that the computations involved can be carried out explicitly if the representations are highly degenerate. The method is particularly useful in the framwork of the theory of infinite multiplets as proposed by BUDINI and FRONSDAL [1], on the one hand because the method automatically produces covariant results, on the other hand, because the representations involved in the theory of BUDINI and FRONSDAL are indeed highly degenerate. In this context rather elaborate applications of the method can be found in Ref. [6].

Appendix

To prove the formula (8.9) we decompose the variables of integration, $\boldsymbol{s}$, in the canonical way

$$\boldsymbol{s} = \boldsymbol{k}\, \boldsymbol{z}$$

and demonstrate that the integration over the variables $\boldsymbol{k}$ can be carried out explicitly. Let

$$\tilde{ds} = \gamma\, \tilde{dk}\, dz \tag{A.1}$$

where dz denotes the product of the differentials of real and imaginary parts of the nontrivial elements of the matrix $\boldsymbol{z}$, whereas $\tilde{dk}$ stands for the analogous product involving the variables $\boldsymbol{k}$, excluding the elements of the first row of $\boldsymbol{k}$. The Jacobian γ may be computed in a straightforward fashion with the result

$$\gamma = \gamma(\boldsymbol{k}) = \prod_{p=2}^{r} |\varkappa_p|^{2 j_p} \tag{A.2}$$

where $\varkappa_p$ denotes the determinants (6.5) associated with the matrix $\boldsymbol{k}$, and the exponents j_p are defined in (7.10). We now carry out the transformation of variables $\boldsymbol{s} \to \boldsymbol{k}, \boldsymbol{z}$ in the integrand of (8.9), recalling the connection between the homogeneous functions and the GELFAND-NAIMARK representation

$$G(\boldsymbol{s}) = \alpha(\boldsymbol{k})\, G(\boldsymbol{z}) \qquad P(\boldsymbol{s}) = \beta(\boldsymbol{k})\, P(\boldsymbol{z})\,. \tag{A.3}$$

The degree of homogeneity $\beta(\boldsymbol{k})$ of the polynomial $P(\boldsymbol{s})$ can be read off from (8.5). The result of the above substitution therefore reads

$$\begin{aligned} (F, G) &= \int dz\, I(\boldsymbol{z}\, \boldsymbol{M}\, \boldsymbol{z}^+)\, P^*(\boldsymbol{z})\, G(\boldsymbol{z}) \\ I(\boldsymbol{A}) &= \Gamma \int \tilde{dk}\, \delta(\boldsymbol{k})\, e^{-\tilde{Tr}(\boldsymbol{k} \boldsymbol{A} \boldsymbol{k}^+)}\,. \end{aligned} \tag{A.4}$$

The function $\delta(\boldsymbol{k})$ is given by

$$\begin{aligned} \delta(\boldsymbol{k}) &= \alpha(\boldsymbol{k})\, \beta^*(\boldsymbol{k})\, \gamma(\boldsymbol{k}) = \prod_{q=2}^{r} |\varkappa_p|^{2\delta_p} \\ \delta_p &= \frac{1}{2} \sum_{q=2}^{p} |m_q - m_{q-1}| + \frac{i}{2}\, \varrho_p + \frac{1}{2}\, (n_1 - n_p)\,. \end{aligned} \tag{A.5}$$

We are left with the task of evaluating the integral $I(\boldsymbol{A})$ over the subgroup K. Above all we are interested in the dependence of $I(\boldsymbol{A})$ on the matrix $\boldsymbol{A}$. Information on the function $I(\boldsymbol{A})$ can be obtained without evaluating the integral explicitly. In fact, consider the transformation $\boldsymbol{k} \to \boldsymbol{k}\,\boldsymbol{k}_0$ with $\boldsymbol{k}_0$ some fixed element of the subgroup K. The Jacobian of this transformation is given by

$$\tilde{d}(k\,k_0) = \varepsilon(\boldsymbol{k}_0)\,\tilde{d}k$$

$$\varepsilon(\boldsymbol{k}) = \prod_{p=2}^{r} |\varkappa_p\,\varkappa_{p+1} \ldots \varkappa_r|^{2n_p} . \tag{A.6}$$

The integral therefore possesses the invariance property

$$I(\boldsymbol{A}) = \delta(\boldsymbol{k}_0)\,\varepsilon(\boldsymbol{k}_0)\,I(\boldsymbol{k}_0\,\boldsymbol{A}\,\boldsymbol{k}_0) . \tag{A.7}$$

This property determines the function $I(\boldsymbol{A})$ up to a constant, because every hermitean positive definite and unimodular matrix A can be represented in the form

$$\boldsymbol{A} = \boldsymbol{k}_1\,\boldsymbol{k}_1^+ \qquad \boldsymbol{k}_1 \in K .$$

It therefore suffices to write down one particular solution of this relation. Such a solution is given by

$$I(\boldsymbol{s}\,\boldsymbol{M}\,\boldsymbol{s}^+) = D_1^{\alpha_1} \ldots D_{r-1}^{\alpha_{r-1}} . \tag{A.8}$$

In this expression we have for convenience represented the matrix $\boldsymbol{A}$ in the form $\boldsymbol{s}\,\boldsymbol{M}\,\boldsymbol{s}^+$. The functions $D_p(\boldsymbol{s};\,\boldsymbol{M})$ and the exponents α_p are defined in (8.7) and (8.8). In (A.8) we have chosen the normalization $I(\mathbf{1}) = 1$. This normalization fixes the constant Γ. We have thus verified (8.9).

Finally we evaluate the constant Γ, which is given by the integral

$$\Gamma^{-1} = \int \tilde{d}k\,\delta(\boldsymbol{k})\,e^{-\tilde{T}r\,\boldsymbol{k}\boldsymbol{k}^+} . \tag{A.9}$$

The function $\delta(\boldsymbol{k})$ involves only the determinants $\varkappa_p$ ($p = 2, \ldots, r$). The integration over the elements of the matrix $\boldsymbol{k}$ that do not belong to the blocks k_p^p ($p = 2, \ldots, r$) can therefore be carried out in a trivial fashion. There are ω of these elements

$$\omega = \frac{1}{2}\left(n^2 - \sum_{p=2}^{r} n_p^2\right) - n + n_1 \tag{A.10}$$

each giving rise to a factor π. The remaining integral then factorizes as follows

$$\Gamma^{-1} = \pi^{\omega} \prod_{p=2}^{r} J(n_p, \delta_p) . \tag{A.11}$$

Here, the symbol $J(n, \delta)$ stands for

$$J(n, \delta) = \int dg\,|\det \boldsymbol{g}|^{2\delta}\,e^{-Tr\,\boldsymbol{g}\boldsymbol{g}^+} . \tag{A.12}$$

In this expression the integration extends over all complex $n \times n$ matrices $\boldsymbol{g}$. For the special case $n = 1$ this integral is essentially the Γ-function

$$J(1, \delta) = \pi \, \Gamma(\delta + 1) .$$

In fact, even for $n > 1$, $J(n, \delta)$ has properties very similar to those of the Γ-function and we shall exploit these to determine the integral. First of all we note that $J(n, \delta)$ is analytic in $Re\, \delta > -1$. Furthermore, it follows from Schwartz' inequality that $J(n, \delta)$ is logarithmically convex for real δ

$$J(n, \delta_1)\, J(n, \delta_2) \geqslant J\left(n, \frac{1}{2}(\delta_1 + \delta_2)\right). \tag{A.13}$$

Finally we note the recurrence relation

$$J(n, \delta + 1) = (\delta + n)\,(\delta + n - 1) \ldots (\delta + 1)\, J(n, \delta) . \tag{A.14}$$

This relation can be obtained by considering instead of $J(n, \delta)$ the analogous integral with Tr $\boldsymbol{g}\, \boldsymbol{g}^+$ replaced by Tr $\boldsymbol{g}\, \boldsymbol{N}\, \boldsymbol{g}^+$; differenting the new integral n times with respect to $\boldsymbol{N}$ one arrives at (A.14).

We now recall a well-known theorem by H. Bohr [19]. This theorem states that up to a constant the Γ-function is the only logarithmically convex solution of the recurrence relation (A.14) for $n = 1$. Bohr's proof can be generalized without any essential modifications to the general case $n \geqslant 1$ with the result that up to a factor there is only one logarithmically convex solution of (A.14). The dependence of $J(n, \delta)$ on δ is therefore given by

$$J(n, \delta) = \Gamma(\delta + n)\, \Gamma(\delta + n - 1) \ldots \Gamma(\delta + 1)\, k_n . \tag{A.15}$$

The constant k_n can be evaluated by choosing $\delta = 0$. The integral $J(n, 0)$ has the value π^{n^2}. The final result therefore reads

$$\Gamma^{-1} = \pi^{(1/2)\, n(n-2) + (1/2)\, n_1^2} \prod_{p=2}^{r} \pi^{(1/2)\, n_p^2}\, \frac{\Gamma(\delta_p + n_p) \ldots \Gamma(\delta_p + 1)}{\Gamma(n_p) \ldots \Gamma(1)} . \tag{A.16}$$

One verifies that this expression indeed reproduces the normalization constant given for the partition $(n - 1, 1)$ in (8.2).

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