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A Note on the Commutation Relations of Field Operators

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Abstract. Let $\phi(.)$ and $\psi(.)$ be two fields transforming according to finite irreducible representations of SL(2, C). Then the (anti-) commuting of two properly chosen components of $\phi(x)$ and $\psi(y)$, (x, y) varying in a domain $G \subset R^4 \times R^4$, implies the vanishing of all (anti-) commutators between any component of $\phi(x)$ and $\psi(y)$ respectively.

We consider a Wightman theory [1, 2] containing the fields $\phi(.)$ and $\psi(.)$ which are assumed to transform according to the irreducible representations $[p, q]$ and $[r, s]$ of SL(2, C) respectively. Accordingly we have

$$U(A)\,\phi(x)\,U(A)^{-1} = S_1(A^{-1})\,\phi(\Lambda(A)\,x)$$
$$U(A)\,\psi(x)\,U(A)^{-1} = S_2(A^{-1})\,\psi(\Lambda(A)\,x) \tag{1}$$

where $A \to U(A)$ is the unitary continuous representation of SL(2, C) in the Hilbert space $\mathcal{H}$ on which the fields act as operator-valued distributions. (1) and the following equations hold on a dense linear set $D \subset \mathcal{H}$ in the sense of distribution theory. The field operators as well as $U(A)$ map D into D. $A \to S_1(A) = (A^{\otimes p})_{sym} \otimes (\bar{A}^{\otimes q})_{sym}$ is the irreducible representation of SL(2, C) characterized by $[p, q]$, and similar for $[r, s]$. Finally, $A \to \Lambda(A)$ is the canonical homomorphism from SL(2 ,C) onto $L_+^\uparrow$, explicitly $\Lambda^\mu_\nu(A) = 1/2\; Tr\, \sigma_\mu\, A\, \sigma_\nu\, A^*$.

With the above-mentioned assumptions we shall prove the following

Theorem: If the (anti-) commutator

$$[\phi_{0,q}(x), \psi_{r,0}(y)]_{(\overline{+})} \equiv \phi_{0,q}(x)\,\psi_{r,0}(y) \overset{(+)}{-} \psi_{r,0}(y)\,\phi_{0,q}(x) \tag{2}$$

between the distinguished components $\phi_{0,q}(x)$ and $\psi_{r,0}(y)$ vanishes, (x, y) varying in the domain $G \subset R^4 \times R^4$, then

$$[\phi_{h,k}(x), \psi_{m,n}(y)]_{(\overline{+})} = 0,\quad (x, y) \in G \tag{3}$$

for all components $\phi_{h,k}(x)$ and $\psi_{m,n}(y)$. (Instead of the usual dotted and undotted spinor indices with values 1 or 2 we use the numbers h and k to characterize the components of $\phi(.)$, k (h) being the number of (un-)dotted indices of value 1.)

Proof: We insert the following one-parametric subgroups of SL(2, C)

$$A_1(\lambda) = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}\quad A_2(\lambda) = \begin{pmatrix} 1 & -i\lambda \\ 0 & 1 \end{pmatrix}\quad A_3(\lambda) = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}\quad A_4(\lambda) = \begin{pmatrix} 1 & 0 \\ -i\lambda & 1 \end{pmatrix} \tag{4}$$

into (1) and get, taking the derivative at $\lambda = 0$

$$\begin{aligned}
[M_1, \phi_{h,k}(x)]_- &= D_1(x)\, \phi_{h,k}(x) + h\, \phi_{h-1,k}(x) + k\, \phi_{h,k-1}(x) \\
[M_2, \phi_{h,k}(x)]_- &= D_2(x)\, \phi_{h,k}(x) + i\, h\, \phi_{h-1,k}(x) - i\, k\, \phi_{h,k-1}(x) \\
[M_3, \phi_{h,k}(x)]_- &= D_3(x)\, \phi_{h,k}(x) + (p-h)\, \phi_{h+1,k}(x) + (q-k)\, \phi_{h,k+1}(x) \\
[M_4, \phi_{h,k}(x)]_- &= D_4(x)\, \phi_{h,k}(x) + i\,(p-h)\, \phi_{h+1,k}(x) - i\,(q-k)\, \phi_{h,k+1}(x) \qquad (5)
\end{aligned}$$

and similar Equations (5') for $\psi_{m,n}$. $i\, M_k$, $k = 1, 2, 3, 4$, are the self-adjoint generators of the one-parametric unitary groups $U_k(\lambda) = U(A_k(\lambda))$; M_k maps D into D [1]. $D_k(x)$ are linear differential operators of the form $\sum_{\mu\nu} \alpha_k^{\mu\nu}\, x_\mu\, \partial_\nu$, $\alpha_k^{\mu\nu} = -\alpha_k^{\nu\mu}$.

For arbitrary operators X, Y, Z mapping D into D, the following identity holds on D:

$$[X, [Y, Z]_-]_\varrho + \varrho\, [Z, [Y, X]_-]_\varrho = [Y, [X, Z]_\varrho]_- \qquad (6)$$

where

$$[A, B]_\varrho = A\, B + \varrho\, B\, A\,, \quad \varrho = \pm\,.$$

Applying (6) to $\psi_{m,n}(y)$, M_i, $\phi_{h,k}(x)$ we get

$$[\psi_{m,n}(y), [M_i, \phi_{h,k}(x)]_-]_\varrho + \varrho\, [\phi_{h,k}(x), [M_i, \psi_{m,n}(y)]_-]_\varrho = 0 \qquad (7)$$

if

$$[\phi_{h,k}(x), \psi_{m,n}(y)]_\varrho = 0\,, \quad (x, y) \in G \qquad (8)$$

holds. Together with (8), also the equations

$$[D_i(x)\, \phi_{h,k}(x), \psi_{m,n}(y)]_\varrho = 0\,, \quad (x, y) \in G \qquad (9)$$

$$[\phi_{h,k}(x), D_i(y)\, \psi_{m,n}(y)]_\varrho = 0\,, \quad (x, y) \in G \qquad (10)$$

are valid, G being a domain.

Inserting (5), (5') into (7) and taking into account (9), (10) we are left with the equations

$$\begin{aligned}
&h[\psi_{m,n}(y), \phi_{h-1,k}(x)]_\varrho \pm k[\psi_{m,n}(y), \phi_{h,k-1}(x)]_\varrho \\
&\quad + \varrho\, m[\phi_{h,k}(x), \psi_{m-1,n}(y)]_\varrho \pm \varrho\, n[\phi_{h,k}(x), \psi_{m,n-1}(y)]_\varrho = 0 \qquad (11)\ (12)
\end{aligned}$$

$$\begin{aligned}
&(p-h)\, [\psi_{m,n}(y), \phi_{h+1,k}(x)]_\varrho \pm (q-k)\, [\psi_{m,n}(y), \phi_{h,k+1}(x)]_\varrho \\
&\quad + \varrho\, (r-m)\, [\phi_{h,k}(x), \psi_{m+1,n}(y)]_\varrho \pm \varrho\, (s-n)\, [\phi_{h,k}(x), \psi_{m,n+1}(y)]_\varrho = 0\,. \qquad (13)\ (14)
\end{aligned}$$

Adding and subtracting (11) and (12), (13) and (14), leads to

$$h[\psi_{m,n}(y), \phi_{h-1,k}(x)]_\varrho + \varrho\, m[\phi_{h,k}(x), \psi_{m-1,n}(y)]_\varrho = 0 \qquad (15)$$

$$k[\psi_{m,n}(y), \phi_{h,k-1}(x)]_\varrho + \varrho\, n[\phi_{h,k}(x), \psi_{m,n-1}(y)]_\varrho = 0 \qquad (16)$$

$$(p-k)\, [\psi_{m,n}(y), \phi_{h+1,k}(x)]_\varrho + \varrho\, (r-m)\, [\phi_{h,k}(x), \psi_{m+1,n}(y)]_\varrho = 0 \qquad (17)$$

$$(q-k)\, [\psi_{m,n}(y), \phi_{h,k+1}(x)]_\varrho + \varrho\, (s-n)\, [\phi_{h,k}(x), \psi_{m,n+1}(y)]_\varrho = 0 \qquad (18)$$

(15)–(18) are valid only if (8) holds. This is the case for $h = 0$, $k = q$ and $m = r$, $n = 0$ according to the assumption in the theorem. Therefore, making use of (15), (8) holds for $h = 0$, $k = q$ and $m = r - 1$, $n = 0$. Again using (15), (8) holds for $h = 0$, $k = q$, $m = r - 2$, $n = 0$ and so on. (8) being valid now for $h = 0$, $k = q$, $n = 0$ and all m, repeated use of (17) extends the validity of (8) to $k = q$, $n = 0$, all h, all m. (16) extends this result to $n = 0$, all h, all k, all m and finally, by (18) we end with the statement of the theorem.

Remark: Usually one considers the domain $G = \{(x, y)/(x - y)^2 < 0\}$. In this case the vanishing of either the commutator or the anticommutator between components of field operators at space-like separated points is called locality. Our theorem shows that locality need be assumed only between $\phi_{0,q}(x)$ and $\psi_{r,0}(y)$.

In the cases $\psi(.) = \phi^*(.)$ [1–3], $\psi(.) = \phi(.)$ [1, 2, 4] the choice $\varrho = (-1)^{p+q+1}$ is enforced by the positivity condition. In any other case ϱ is arbitrary, but there always exist sufficiently many symmetries with the help of which new fields can be defined such that $\varrho = \min\{(-1)^{p+q+1}, (-1)^{r+s+1}\}$ [1, 2, 5].

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