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A Note on the Commutation Relations of Field Operators

by **Walter Schneider**

Department of Mathematics, Imperial College, London

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Abstract. Let $\phi(\cdot)$ and $\psi(\cdot)$ be two fields transforming according to finite irreducible representations of $SL(2, \mathbb{C})$. Then the (anti-) commuting of two properly chosen components of $\phi(x)$ and $\psi(y)$, (x, y) varying in a domain $G \subset R^4 \times R^4$, implies the vanishing of all (anti-) commutators between any component of $\phi(x)$ and $\psi(y)$ respectively.

We consider a Wightman theory [1, 2] containing the fields $\phi(\cdot)$ and $\psi(\cdot)$ which are assumed to transform according to the irreducible representations $[\rho, q]$ and $[r, s]$ of $SL(2, \mathbb{C})$ respectively. Accordingly we have

$$\begin{aligned} U(A) \phi(x) U(A)^{-1} &= S_1(A^{-1}) \phi(\Lambda(A) x) \\ U(A) \psi(x) U(A)^{-1} &= S_2(A^{-1}) \psi(\Lambda(A) x) \end{aligned} \quad (1)$$

where $A \rightarrow U(A)$ is the unitary continuous representation of $SL(2, \mathbb{C})$ in the Hilbert space \mathcal{H} on which the fields act as operator-valued distributions. (1) and the following equations hold on a dense linear set $D \subset \mathcal{H}$ in the sense of distribution theory. The field operators as well as $U(A)$ map D into D . $A \rightarrow S_1(A) = (A^{\otimes \rho})_{sym} \otimes (\bar{A}^{\otimes q})_{sym}$ is the irreducible representation of $SL(2, \mathbb{C})$ characterized by $[\rho, q]$, and similar for $[r, s]$. Finally, $A \rightarrow \Lambda(A)$ is the canonical homomorphism from $SL(2, \mathbb{C})$ onto L_+^\uparrow , explicitly $\Lambda_\nu^\mu(A) = 1/2 \text{Tr} \sigma_\mu A \sigma_\nu A^*$.

With the above-mentioned assumptions we shall prove the following

Theorem: If the (anti-) commutator

$$[\phi_{0,q}(x), \psi_{r,0}(y)]_{(\pm)} \equiv \phi_{0,q}(x) \psi_{r,0}(y)^{(\pm)} \psi_{r,0}(y) \phi_{0,q}(x) \quad (2)$$

between the distinguished components $\phi_{0,q}(x)$ and $\psi_{r,0}(y)$ vanishes, (x, y) varying in the domain $G \subset R^4 \times R^4$, then

$$[\phi_{h,k}(x), \psi_{m,n}(y)]_{(\pm)} = 0, \quad (x, y) \in G \quad (3)$$

for all components $\phi_{h,k}(x)$ and $\psi_{m,n}(y)$. (Instead of the usual dotted and undotted spinor indices with values 1 or 2 we use the numbers h and k to characterize the components of $\phi(\cdot)$, k (h) being the number of (un-)dotted indices of value 1.)

Proof: We insert the following one-parametric subgroups of $SL(2, \mathbb{C})$

$$A_1(\lambda) = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \quad A_2(\lambda) = \begin{pmatrix} 1 & -i\lambda \\ 0 & 1 \end{pmatrix} \quad A_3(\lambda) = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \quad A_4(\lambda) = \begin{pmatrix} 1 & 0 \\ -i\lambda & 1 \end{pmatrix} \quad (4)$$

into (1) and get, taking the derivative at $\lambda = 0$

$$\begin{aligned} [M_1, \phi_{h,k}(x)]_- &= D_1(x) \phi_{h,k}(x) + h \phi_{h-1,k}(x) + k \phi_{h,k-1}(x) \\ [M_2, \phi_{h,k}(x)]_- &= D_2(x) \phi_{h,k}(x) + i h \phi_{h-1,k}(x) - i k \phi_{h,k-1}(x) \\ [M_3, \phi_{h,k}(x)]_- &= D_3(x) \phi_{h,k}(x) + (p - h) \phi_{h+1,k}(x) + (q - k) \phi_{h,k+1}(x) \\ [M_4, \phi_{h,k}(x)]_- &= D_4(x) \phi_{h,k}(x) + i (p - h) \phi_{h+1,k}(x) - i (q - k) \phi_{h,k+1}(x) \end{aligned} \quad (5)$$

and similar Equations (5') for $\psi_{m,n}$. $i M_k$, $k = 1, 2, 3, 4$, are the self-adjoint generators of the one-parametric unitary groups $U_k(\lambda) = U(A_k(\lambda))$; M_k maps D into D [1]. $D_k(x)$ are linear differential operators of the form $\sum_{\mu\nu} \alpha_k^{\mu\nu} x_\mu \partial_\nu$, $\alpha_k^{\mu\nu} = -\alpha_k^{\nu\mu}$.

For arbitrary operators X, Y, Z mapping D into D , the following identity holds on D :

$$[X, [Y, Z]_-]_\varrho + \varrho [Z, [Y, X]_-]_\varrho = [Y, [X, Z]_\varrho]_- \quad (6)$$

where

$$[A, B]_\varrho = A B + \varrho B A, \quad \varrho = \pm .$$

Applying (6) to $\psi_{m,n}(y)$, M_i , $\phi_{h,k}(x)$ we get

$$[\psi_{m,n}(y), [M_i, \phi_{h,k}(x)]_-]_\varrho + \varrho [\phi_{h,k}(x), [M_i, \psi_{m,n}(y)]_-]_\varrho = 0 \quad (7)$$

if

$$[\phi_{h,k}(x), \psi_{m,n}(y)]_\varrho = 0, \quad (x, y) \in G \quad (8)$$

holds. Together with (8), also the equations

$$[D_i(x) \phi_{h,k}(x), \psi_{m,n}(y)]_\varrho = 0, \quad (x, y) \in G \quad (9)$$

$$[\phi_{h,k}(x), D_i(y) \psi_{m,n}(y)]_\varrho = 0, \quad (x, y) \in G \quad (10)$$

are valid, G being a domain.

Inserting (5), (5') into (7) and taking into account (9), (10) we are left with the equations

$$\begin{aligned} h[\psi_{m,n}(y), \phi_{h-1,k}(x)]_\varrho \pm k[\psi_{m,n}(y), \phi_{h,k-1}(x)]_\varrho \\ + \varrho m[\phi_{h,k}(x), \psi_{m-1,n}(y)]_\varrho \pm \varrho n[\phi_{h,k}(x), \psi_{m,n-1}(y)]_\varrho = 0 \end{aligned} \quad (11) \quad (12)$$

$$\begin{aligned} (p - h) [\psi_{m,n}(y), \phi_{h+1,k}(x)]_\varrho \pm (q - k) [\psi_{m,n}(y), \phi_{h,k+1}(x)]_\varrho \\ + \varrho (r - m) [\phi_{h,k}(x), \psi_{m+1,n}(y)]_\varrho \pm \varrho (s - n) [\phi_{h,k}(x), \psi_{m,n+1}(y)]_\varrho = 0. \end{aligned} \quad (13) \quad (14)$$

Adding and subtracting (11) and (12), (13) and (14), leads to

$$h[\psi_{m,n}(y), \phi_{h-1,k}(x)]_\varrho + \varrho m[\phi_{h,k}(x), \psi_{m-1,n}(y)]_\varrho = 0 \quad (15)$$

$$k[\psi_{m,n}(y), \phi_{h,k-1}(x)]_\varrho + \varrho n[\phi_{h,k}(x), \psi_{m,n-1}(y)]_\varrho = 0 \quad (16)$$

$$(p - k) [\psi_{m,n}(y), \phi_{h+1,k}(x)]_\varrho + \varrho (r - m) [\phi_{h,k}(x), \psi_{m+1,n}(y)]_\varrho = 0 \quad (17)$$

$$(q - k) [\psi_{m,n}(y), \phi_{h,k+1}(x)]_\varrho + \varrho (s - n) [\phi_{h,k}(x), \psi_{m,n+1}(y)]_\varrho = 0 \quad (18)$$

(15)–(18) are valid only if (8) holds. This is the case for $h = 0$, $k = q$ and $m = r$, $n = 0$ according to the assumption in the theorem. Therefore, making use of (15), (8) holds for $h = 0$, $k = q$ and $m = r - 1$, $n = 0$. Again using (15), (8) holds for $h = 0$, $k = q$, $m = r - 2$, $n = 0$ and so on. (8) being valid now for $h = 0$, $k = q$, $n = 0$ and all m , repeated use of (17) extends the validity of (8) to $k = q$, $n = 0$, all h , all m . (16) extends this result to $n = 0$, all h , all k , all m and finally, by (18) we end with the statement of the theorem.

Remark: Usually one considers the domain $G = \{(x, y)/(x - y)^2 < 0\}$. In this case the vanishing of either the commutator or the anticommutator between components of field operators at space-like separated points is called locality. Our theorem shows that locality need be assumed only between $\phi_{0,q}(x)$ and $\psi_{r,0}(y)$.

In the cases $\psi(\cdot) = \phi^*(\cdot)$ [1–3], $\psi(\cdot) = \phi(\cdot)$ [1, 2, 4] the choice $\varrho = (-1)^{p+q+1}$ is enforced by the positivity condition. In any other case ϱ is arbitrary, but there always exist sufficiently many symmetries with the help of which new fields can be defined such that $\varrho = \min \{(-1)^{p+q+1}, (-1)^{r+s+1}\}$ [1, 2, 5].

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