N/D Equations in the Pole Approximation [1] [2]

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Summary. One-channel N/D equations are considered. We propose a method allowing the construction of a convergent sequence of pole approximations for the left-hand cut. Then, the convergence of the corresponding sequence of approximate amplitudes is proved under specific conditions. This convergence holds especially in the physical region. Finally, the physical meaning of some abnormal cases is discussed.

I. Introduction

Since about 10 years, the partial-wave dispersion relations are revealed as a main tool of elementary particle physics. In spite of the difficulties encountered on justifying or applying them, they have provided the description of intermediate energy scattering processes with a very useful frame.

Independently to the questions linked with the dynamical interpretation of the left-hand singularities, the purely mathematical problem of deducing a partial-wave scattering amplitude from the left-hand discontinuity and the inelastic factor is not yet completely clarified. Nevertheless, different practical methods have been proposed to solve this problem. The N/D formalism, whose first version is due to CHEW and MANDELSTAM [1], has been the most used. The structure of the N/D equations has been studied carefully by FRYE and WARNOCK [2], touching in particular the question of existence and uniqueness of a solution, the CDD ambiguities and the inelasticity implications.

In most applications, the N/D equations are treated numerically starting from some approximations of the left-hand discontinuity (and inelasticity). The 'pole approximation method' consists in replacing part of the left-hand singularities by a finite number of poles. Generally, this technique is applied to the less known distant part of the cut, with residues at substituting poles as adjustable parameters [3–18]. If the left-hand singularities as a whole are simulated by poles, the N/D equations reduce to an elementary algebraic problem, the solution of which is written easily in terms of the poles parameters.

This work is mainly devoted to the study of the pole approximation in the N/D formalism. More precisely, a scheme of successive approximations is built which

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[2]) Part of a Thesis entitled 'Equations N/D et approximation par des pôles', submitted in fulfilment of the requirements for the degree of Docteur ès sciences, Université de Lausanne, 1967.

amounts to replace the left-hand discontinuity by a sum of Dirac contributions, the number of which grows indefinitely with the order of the approximation. In this manner, the spectral integral over the left-hand discontinuity is represented by a sequence of meromorphic functions generating, by pure algebraic manipulation, a corresponding sequence of approximate (though unitary!) amplitudes. These meromorphic functions are determined univocally by their identification with successive approximants (Padé approximants) of some continued fraction. Then, the convergence properties of these approximants in a cut plane allows to establish (in a sense which will be made clear) the convergence of the sequence of approximate amplitudes to the exact solution of N/D equations.

The existence of a *convergent* method of this kind seems to us interesting for three reasons:

1) It gives a reliable frame to the numerous pole approximations used up to now, and specify their validity conditions (though the only case where all the left-hand singularities are replaced by poles will be treated, the procedure clearly applies, with some minor changes, to the more frequent situation in which only a part of these singularities is subjected to the approximation).

2) The method in itself constitures an effective procedure of resolution by successive approximations, the practical convergence of which may be faster than that of the customary procedure (using Fredholm method) [19].

3) It may be easier to establish certain properties of the amplitude when the left-hand singularities are reduced to a finite number of poles [20]. The convergence theorem allows then to extend them to the general case (on condition however that the asymptotic behaviour of the amplitude is not involved). This gives some hope to understand the qualitative relations between the left-hand discontinuity (the 'forces') and the physical scattering amplitude in a clearer manner.

In Section II we formulate our basic assumptions and state the N/D algorithm in the convenient form. In Section III we present the method which allows to construct a convergent sequence of pole approximations for the left-hand cut contribution to the amplitude (the 'potential'). The convergence of the corresponding sequence of approximate amplitudes is established in Sections IV and V, with a particular attention paid to the physical region. The conditions in which this proof is valid are, roughly, those that assure as well the equivalence of the N/D equations with a Fredholm equation. Section VI is devoted to the interpretation of exceptional cases where the method apparently fails to work. Also, a necessary condition for the occurence of narrow resonances is given. Our conclusions are finally summarized in Section VII, and some technical points are referred to the Appendices.

II. N/D Formalism and Fredholm Equation

We consider the elastic scattering of two spinless equal mass particles. Let $T_l(z)$ be the set of partial wave amplitudes, expressed in terms of $z = q^2$, the squared center-of-mass momentum (the mass of the particles is set equal to 1). These amplitudes, as functions of the complex variable z, satisfy the following well known conditions (as resulting for example from the Mandelstam representation):

a) *Analyticity*

$T_l(z)$ s a meromorphic function in the z-plane with the two cuts $[-\infty, -a]$, $[0, \infty]$ $(a > 0)$. Its possible poles are real (on the physical sheet).

The physical amplitude is boundary-value of $T_l(z)$ for $z = x + i\,0$ $(x \geqslant 0)$.

b) '*Reality*'

$T_l(z^*) = T_l^*(z)$.

c) *Unitarity*

$$T_l\,(x + i\,0) = \sqrt{\frac{x+1}{x}}\;\frac{1}{2\,i}\;[\eta_l(x)\;e^{2\,i\,\delta_l(x)} - 1] \quad (x \geqslant 0)$$

where: $\delta_l(x)$ real

$\eta_l(x) = 1$ for $0 \leqslant x \leqslant z_I$ with $z_I > 0$

$0 \leqslant \eta_l(x) \leqslant 1$ for $x > z_I$

(we assume the existence of an elastic domain).

d) *Threshold behaviour*

$$T_l(z) \underset{z \to 0}{=} 0(z^l)\;.$$

In our problem, the left-hand discontinuity:

$$\phi_l(x) \equiv \frac{1}{2\,i}\;[T_l\,(x + i\,0) - T_l\,(x - i\,0)] = Im\;T_l\,(x + i\,0) \quad (x \leqslant -a)$$

is assumed to be known, as well as the inelastic factor, i.e. the ratio of the total and elastic partial wave cross sections:

$$R_l(x) \equiv \frac{\sigma_{tot,\,l}}{\sigma_{e\,l,\,l}} = 2\;\frac{1 - \eta_l\cos 2\,\delta_l}{1 - 2\,\eta_l\cos 2\,\delta_l + \eta_l^2}\;.$$

Clearly, the information about the inelasticity given by η_l is not equivalent to that given by R_l. The former possibility, systematically used by FRYE and WARNOCK [2, 21], is not suitable for our purposes and we choose the latter. There is no reason which prevents R_l to have infinite values at points x where $\sigma_{e\,l,\,l} = \sigma_{in,\,l} = 0$. Yet we exclude this eventuality and take for granted that $R_l(x)$ is bounded on $[0, \infty]$.

Now, the unitarity condition can be written:

c′) $$Im\;[1/T_l\,(x + i\,0)] = -\sqrt{\frac{x}{x+1}}\;R_l(x) \quad (x \geqslant 0)$$

where: $R_l(x) = 1$ for $0 \leqslant x \leqslant z_I$, $\quad 1 \leqslant R_l(x) \leqslant I_l$ for $x > z_I$.

We introduce the N/D representation in the form:

$$T_l(z) = z^l\;\frac{N_l(z)}{D_l(z)} \tag{II.1}$$

where $N_l(z)$ is a 'real' holomorphic function in the z-plane with the left-hand cut and $D_l(z)$ a 'real' meromorphic function in the z-plane with the right-hand cut. Conditions a), b) and d) are then automatically satisfied. In order to assure the unitarity condition c′) and that $T_l(z)$ produces the given left-hand discontinuity, one has only to impose:

$$\begin{cases} Im\, D_l\,(x + i\,0) = -\,x^l \sqrt{\dfrac{x}{x+1}}\; R_l(x)\, N_l(x) & \text{for } x \geqslant 0\,, \\ Im\, N_l\,(x + i\,0) = x^{-l}\, \phi_l(x)\; D_l(x) & \text{for } x \leqslant -a\,. \end{cases}$$

The resulting N/D equations are, when ignoring subtractions and CDD poles:

$$\begin{cases} N_l(z) = \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{-a} dx\, \dfrac{\phi_l(x)\; D_l(x)}{x^l\,(x-z)}\,, & \text{(II.2)} \\ D_l(z) = 1 - \dfrac{z}{\pi} \displaystyle\int_0^{\infty} dx\, x^{l-1} \sqrt{\dfrac{x}{x+1}}\; R_l(x)\, \dfrac{N_l(x)}{x-z}\,. & \text{(II.3)} \end{cases}$$

(for convenience, we have normalized the function $D_l(z)$ to 1 at the point $z = 0$).

If the first integral above converges, clearly $\lim_{z\to\infty} |z\, N(z)| = \text{const.} \neq 0$ or ∞ for a wide class of functions $\phi_l(x)$. This causes the second integral to diverge as soon as $l \geqslant 2$, and subtractions are then necessary. In order to avoid inessential complications due to additional parameters, we shall deal explicitly only with the P-wave ($l = 1$), free from CDD poles, refering to Appendix 1 for the treatment of partial waves $l \geqslant 2$ and $l = 0$ (as usual, the S-wave must be considered separately).

In the following, we shall have to require the equivalence of equations (II.2–3) with a Fredholm integral equation. This forbids subtractions in the P-wave N/D equations too (see Appendix I). Thus, suppressing the index $l = 1$:

$$\begin{cases} T(z) = z\, \dfrac{N(z)}{D(z)}\,, & \text{(II.4)} \\ N(z) = \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{-a} dx\, \dfrac{\phi(x)\; D(x)}{x\,(x-z)}\,, & \text{(II.5)} \\ D(z) = 1 - \dfrac{z}{\pi} \displaystyle\int_0^{\infty} dx\, \varrho(x)\, \dfrac{N(x)}{x-z}\,, & \text{(II.6)} \end{cases}$$

where:

$$\varrho(x) = \sqrt{\frac{x}{x+1}}\; R(x)$$

is a known continuous factor which, according to c′), satisfy the following conditions:

$$\begin{cases} 0 < \varrho(x) < I \quad \text{for } x > 0\,, \\ \varrho(x) \underset{x\to 0}{\simeq} \sqrt{x}\,. \end{cases} \qquad \text{(II.7)}$$

Later on, another (weak) assumption will be made about this 'kinematical' factor (see Section V).

Introducing (II.6) into (II.5), one obtains the integral equation for N:

$$N(z) = B(z) + \int_0^\infty dz'\, K(z, z')\, N(z')\,, \tag{II.8}$$

in which the inhomogeneous term:

$$B(z) = \frac{1}{\pi} \int_a^\infty dx\, \frac{\phi(-x)}{x\,(x+z)} \tag{II.9}$$

and the kernel:

$$K(z, z') = \frac{\varrho(z')}{\pi^2} \int_a^\infty dx\, \frac{\phi(-x)}{(x+z)\,(x+z')} = \frac{\varrho(z')}{\pi} \left[\frac{z\,B(z) - z'\,B(z')}{z-z'}\right] \tag{II.10}$$

are now given.

The function $B(z)$, holomorphic in the z-plane with the cut $[-\infty, -a]$, will be called the 'potential'.

The asymptotic behaviour o $\phi(-x)$ must be compatible with the convergence of the integrals contained in equations (II.9–10). But we have to impose a stronger condition: *the discontinuity* $\phi(-x)$ *must be such that equation* (II.8) *is a Fredholm equation, in the strict sense*. Only in this case our approximation scheme has been proved to be convergent. We assume the following conditions on $\phi(-x)$ (with some obvious integrability properties):

A) $\phi(-x)$ has a finite number of changes of sign on $[a, \infty]$,

B) $\phi(-x)$ is finite for all $x \in [a, \infty]$,

C) $\phi(-x) \underset{x\to\infty}{\simeq}$ const. $x^{-\alpha}(\mathrm{Log}\, x)^{-\beta}$.

Then, the necessary and sufficient condition for equation (II.8) to be a Fredholm equation is:

$$\begin{cases} \alpha > 0 \ \text{(any } \beta) \\ \text{or } \alpha = 0\,,\ \beta > 1/2 \end{cases} \tag{II.11}$$

The proof of (II.11) is almost straightforward and will be omitted. It consists in verifying the properties [22]:

i) $$K \in \mathfrak{L}^2[(0,\infty) \times (0,\infty)] : \|K\|^2 \equiv \int_0^\infty dz\, dz'\, |K(z, z')|^2 < \infty$$

(K is a kernel of Hilbert-Schmidt type),

ii) $$\int_0^\infty dz\, |K(z, z')|^2 < \infty\ \forall\, z'\,,\ \int_0^\infty dz'\, |K(z, z')|^2 < \infty\ \forall\, z\,,$$

iii) $$B \in \mathfrak{L}^2(0,\infty) : \|B\|^2 \equiv \int_0^\infty dz\, |B(z)|^2 < \infty\,.$$

As a result of the restriction (II.11), the solution(s) $N(z)$ of equation (II.8) belong(s) to the class $\mathfrak{L}^2(0, \infty)$. Moreover, it follows immediately from Schwarz's inequality that the integral (II.6) defining $D(z)$ converges for $z \notin [0, \infty]$.

III. The Potential as a Sequence of Meromorphic Functions

When the left-hand discontinuity reduces to a finite sum of Dirac contributions:

$$\phi(-x) = \sum_{n=1}^{p} \Gamma_n \, \delta\,(x - z_n) \qquad (z_n \geqslant a)\,, \tag{III.1}$$

the potential is a meromorphic function:

$$B(z) = \frac{1}{\pi} \sum_{n=1}^{p} \frac{\Gamma_n}{z_n\,(z_n + z)}\,, \tag{III.2}$$

and K a *Fredholm kernel of finite rank:*

$$K(z, z') = \frac{\varrho(z')}{\pi^2} \sum_{n=1}^{p} \frac{\Gamma_n}{(z_n + z)\,(z_n + z')}\,. \tag{III.3}$$

The integral equation (II.8) is then equivalent to an inhomogeneous system of p linear equations [22] which results as well directly from equations (II.5–6) and (III.1):

$$\left\{\begin{aligned} N(z) &= \frac{1}{\pi} \sum_{n=1}^{p} \frac{\Gamma_n\, D(-z_n)}{z_n\,(z_n + z)}\,. \\ D(z) &= 1 - \frac{z}{\pi^2} \sum_{n=1}^{p} \frac{\Gamma_n\, D(-z_n)}{z_n} \int_0^\infty dx\, \frac{\varrho(x)}{(z_n + x)\,(x - z)}\,. \end{aligned}\right. \tag{III.4}$$

Actually the functions $N(z)$ and $D(z)$ are completely determined by the p coefficients $D(-z_n)$, solution of the system:

$$\sum_{n=1}^{p} (\delta_{nm} - \Gamma_n\, f_{mn})\, D(-z_n) = 1 \qquad (m = 1, 2, \dots, p) \tag{III.5}$$

where:

$$f_{mn} = \frac{z_m}{z_n} \frac{1}{\pi^2} \int_0^\infty dx\, \frac{\varrho(x)}{(z_n + x)\,(z_m + x)}\,.$$

Our purpose is now:

a) to construct a sequence of meromorphic functions $B_p(z)$ ($p = 1, 2, \dots$) of the type (III.2) converging to the potential $B(z)$ (Section III),

b) to show that the sequence of amplitudes $T_p(z) = z\, N_p(z)/D_p(z)$ calculated from $B_p(z)$ by means of equations (III.2–5) converges to the amplitude $T(z)$ resulting from the potentiel $B(z)$ (Sections IV–V).

The way we choose the sequence of $B_p(z)$ is to consider them as the successive *approximants* of a continued fraction. Under some conditions, such approximants are known to converge in a cut plane [23], generalizing then in a sense the partial sums of Taylor series, whose convergence is effective only in a circle. This is the very generalization we are looking for, since the sequence $\{B_p(z)\}$ has to converge within a domain containing the physical region.

More precisely, let $H(z)$ be a 'real' function, analytic at the origin, given by its Taylor expansion:

$$H(z) = \sum_{n=0}^{\infty} \alpha_n z^n \qquad (\alpha_n \text{ real}) . \tag{III.6}$$

We can construct a sequence of approximants:

$$H_p(z) = \cfrac{\beta_0}{1+\cfrac{\beta_1 z}{1+ \cdots \cfrac{\beta_{2p-2} z}{1+\beta_{2p-1} z}}} \qquad (p = 1, 2, \ldots) \tag{III.7}$$

uniquely determined by requiring that the $2p$ first coefficients of the Taylor expansion of $H_p(z)$ at the origin coincide with the $2p$ first coefficients α_n of the expansion (III.6). An algorithm which allows to calculate the β_n [starting from the coefficients α_n $(n = 0, 1, \ldots, 2p-1)$] in order to carry out this identification is given in Appendix 2.

It is remarkable that the β_n are independent of p, the *order* of the approximant $H_p(z)$. So one is lead to consider the $H_p(z)$ as the successive approximants of some formal continued fraction. But if one wishes to make sure that $\{H_p(z)\}$ forms a *convergent* sequence, additional restrictions must be imposed to $H(z)$ (e.g. a sufficient condition is that $H(z)$ belongs to the Herglotz class). Due to these restrictions, the direct identification of $H(z)$ with our potentiel is not possible and a further step is needed.

Before undertaking it, we give an alternative explicit form of $H_p(z)$ in terms of the coefficients α_n:

$$\left\{\begin{aligned}
H_p(z) &= \frac{X_p(z)}{Y_p(z)}\,,\\
X_p(z) &= \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_p \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{p+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{p-1} & \alpha_p & \alpha_{p+1} & \cdots & \alpha_{2p-1} \\ 0 & \alpha_0 z^{p-1} & \sum_{i=p-2}^{p-1} \alpha_{i-p+2} z^i & \cdots & \sum_{i=0}^{p-1} \alpha_i z^i \end{vmatrix} \\
Y_p(z) &= \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_p \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{p+1} \\ \vdots & \vdots & & \vdots \\ \alpha_{p-1} & \alpha_p & \cdots & \alpha_{2p-1} \\ z^p & z^{p-1} & \cdots & 1 \end{vmatrix}
\end{aligned}\right. \tag{III.8}$$

In this form, the rational fraction $H_p(z)$ is known as the *Padé approximant* $[p, p-1](z)$ of the function $H(z)$ [23, 24]. It is easy to extract from (III.8) the $2p$ equalities between derivatives at the origin:

$$H_p^{(k)}(0) = H^{(k)}(0) \qquad (k = 0, 1, \dots, 2p-1)\,, \tag{III.9}$$

which in fact determine entirely (except for a common constant factor) the polynomials $X_p(z)$ and $Y_p(z)$. The unicity property then implies the equivalence of definitions (III.7) and (III.8).

We now write the potential as:

$$B(z) = \frac{1}{\pi z}\int_a^\infty dx\,\frac{\phi(-x)}{x} - \frac{1}{\pi z}\,F(z) \tag{III.10}$$

with:

$$F(z) = \int_a^\infty dx\,\frac{\phi(-x)}{x+z}\,.$$

Such a decomposition is possible only if $\int^\infty dx\,\phi(-x)/x$ converges.

This limits the asymptotic behaviour of $\phi(-x)$ to:

$$\begin{cases} \alpha > 0 & (\text{any } \beta) \\ \text{or} \quad \alpha = 0, & \beta > 1\,, \end{cases} \tag{III.11}$$

a condition slightly more restrictive than (II.11), which we assume to be satisfied from now on.

On the other hand, the assumption A) of Section II induces the (non unique!) decomposition of $\phi(-x)$:

$$\phi(-x) = \sum_i \varepsilon_i\,\phi_i(-x) \qquad (\text{finite sum})\,,$$

where:

$$\varepsilon_i = \pm 1\,,$$

$$\begin{cases} \phi_i(-x) \text{ is a non negative function with support } [b_i, c_i] \\ (a \leqslant b_i < c_i \leqslant \infty)\,. \end{cases} \tag{III.12}$$

Likewise:

$$\begin{cases} F(z) = \sum_i \varepsilon_i\,H_i(z)\,, \\ H_i(z) = \int_{b_i}^{c_i} dx\,\frac{\phi_i(-x)}{x+z}\,. \end{cases} \tag{III.13}$$

Our approximation scheme will be applied separately to each function $H_i(z)$. Thus, consider one of them and suppress the index i:

$$H(z) = \int_b^c dx\,\frac{\phi(-x)}{x+z} \tag{III.14}$$

is a 'real' function, holomorphic in the z-plane cut along $[-c, -b]$. Moreover, it belongs to the Herglotz class in the sense that, from (III.12), $Im\, H(z)/Im\, z < 0\ \forall\, z \notin [-c, -b]$. This property is essential for the validity of the following argument.

Let $H_p(z)$ be the approximant of order p of $H(z)$ constructed by means of equations (III.6) and (III.8).

Theorem III.1: The sequence $\{H_p(z)\}$ converges uniformly to $H(z)$ over every compact domain $\mathcal{D}$ whose distance from the cut $[-c, -b]$ is positive.

To establish this theorem, we need an important lemma. It asserts that the functions $H_p(z)$, do not only approximate $H(z)$ arbitrarily well in the neighbourhood of the origin, but also enjoy the same 'degree' of analyticity in the whole z-plane. Its somewhat lengthy proof is postponed to Appendix 3.

Lemma III.1: The approximants $H_p(z)$ have the form:

$$H_p(z) = \sum_{n=1}^{p} \frac{\Gamma_n^p}{z + z_n^p} \tag{III.15}$$

with:

$$\left.\begin{array}{l} b < z_n^p < c \\ \Gamma_n^p > 0 \end{array}\right\}\ \forall\, p = 1, 2, \ldots;\ n = 1, 2, \ldots, p\,. \tag{III.16}$$

Proof of the theorem:

From analyticity properties of $H(z)$ and Lemma III.1, the expansions:

$$H(z) = \sum_{n=0}^{\infty} \alpha_n z^n\,,$$

$$H_p(z) = \sum_{n=0}^{2p-1} \alpha_n z^n + \sum_{n=2p}^{\infty} \alpha_n^p z^n$$

converge in a (closed) circle C centered at $z = 0$ with radius $R < b$ (Fig. 1).

The function $H(z)$ is bounded over C:

$$|H(z)| \leqslant M$$

and the sequence $\{H_p(z)\}$ is uniformly bounded there:

$$|H_p(z)| \leqslant P \qquad (p = 1, 2, \ldots)\,.$$

Indeed, for $z \in C$ and using (III.15–16):

$$|H_p(z)| = \left|\sum_{n-1}^{p} \frac{\Gamma_n^p}{z + z_n^p}\right| < \operatorname*{Sup}_{\substack{z \in C \\ x \in [b, c]}} \left|\frac{1}{1 + \frac{z}{x}}\right| \sum_{n=1}^{p} \frac{\Gamma_n^p}{z_n^p} = \frac{b}{b-R}\, \alpha_0\,.$$

Thus, we can write the Cauchy inequalities:

$$|\alpha_m| \leqslant \frac{M}{R^m}\,,\quad |\alpha_m^p| \leqslant \frac{P}{R^m}\ (p = 1, 2, \ldots)$$

and deduce:

$$|H_p(z) - H(z)| < \sum_{m=0}^{\infty} \left(\frac{P}{R^{2p+m}} + \frac{M}{R^{2p+m}}\right) |z|^{2p+m} \qquad \forall z \in C .$$

Let C' be a circle centered at $z = 0$ with radius $R' < R$. We get:

$$|H_p(z) - H(z)| < (P + M) \frac{R}{R - R'} \left(\frac{R'}{R}\right)^{2p} \qquad \forall z \in C' .$$

Hence $\{H_p(z)\} \to H(z)$ uniformly over C'.

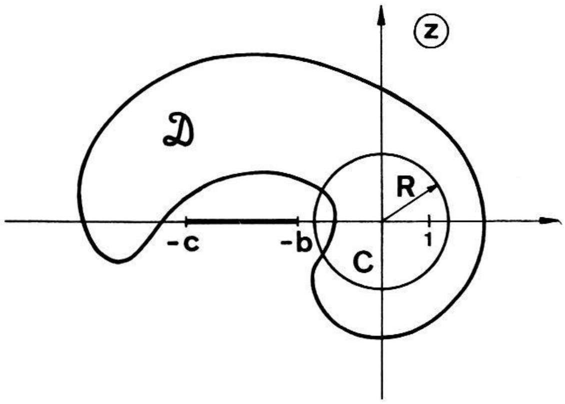

Figure 1
Proof of Theorem III.1.

The distance of $\mathcal{D}$ from $[-c, -b]$ being positive, the sequence $\{H_p(z)\}$ is uniformly bounded over $\mathcal{D}$ too:

$$|H_p(z)| \leqslant Q\, \alpha_0 \qquad \forall z \in \mathcal{D} \qquad (p = 1, 2, \ldots) ,$$

$$Q = \operatorname*{Sup}_{\substack{z \in D \\ x \in [b, c]}} \left| \frac{1}{1 + \frac{z}{x}} \right| < \infty .$$

If the domain $\mathcal{D}$ is connected and if its intersection with C' contains an infinity of points, we meet the hypothesis of Vitali's theorem [25]. It follows that $\{H_p(z)\} \to H(z)$ uniformly over $\mathcal{D}$. The extension to an arbitrary compact $\mathcal{D}$ is trivial. q.e.d.

Refering to the decomposition (III.13), equation (III.15) becomes:

$$H_{p,i}(z) = \sum_{n=1}^{p} \frac{\Gamma_n^{p,i}}{z + z_n^{p,i}} \qquad (b_i < z_h^{p,i} < c_i, \Gamma_n^{p,i} > 0) ,$$

and we can define the approximant of order p of the complete function $F(z)$ by:

$$F_p(z) = \sum_i \varepsilon_i\, H_{p,i}(z) .$$

Therefore, by changing the meaning of the index p:

$$F_p(z =) \sum_{n=1}^{p} \frac{\Gamma_n^p}{z + z_n^p} ,$$

with:

$$F_p^{(k)}(0) = F^{(k)}(0) \qquad (k = 0, 1, \dots, 2P - 1),$$
$$z_n^p > 0 \qquad \forall p, n = 1, 2, \dots, p. \tag{III.17}$$

In (III.17), P is an integer growing indefinitely with p, which depends on the decomposition (III.13) and the distribution of the poles of $F_p(z)$ between the various $H_{p,i}(z)$.

According to equation (III.10), the approximants $B_p(z)$ of the potential will be defined by:

$$B_p(z) = \frac{1}{\pi z} \int_a^\infty dx \frac{\phi(-x)}{x} - \frac{1}{\pi z} F_p(z), \tag{III.18}$$

which gives, taking (III.17) into account (for $k = 0$):

$$B_p(z) = \frac{1}{\pi} \sum_{n=1}^{p} \frac{\Gamma_n^p}{z_n^p (z_n^p + z)} \quad (z_n^p > a). \tag{III.19}$$

Definition (III.18) implies the validity of Theorem III.1 directly when the unctions $H_p(z)$ and $H(z)$ are replaced by $B_p(z)$ and $B(z)$ respectively, and the cut $[-c, -b]$ by $[-\infty, -a]$.

Thus we have achieved our first goal: the construction of a sequence of rational fractions of the type (III.2) converging to the potentiel $B(z)$ over the whole cut plane.

IV. Convergence of the Functions *N* and *D*

From the convergent sequence $\{B_p(z)\}$, two related sequences, $\{N_p(z)\}$ and $\{D_p(z)\}$, are computable by means of equations (III.4) and (III.5). The functions $N_p(z)$ are meromorphic in the z-plane with poles lying on $[-\infty, -a]$, whereas the functions $D_p(z)$ are holomorphic in the z-plane cut along $[0, \infty]$. The approximate amplitudes $T_p(z) = z\, N_p(z)/D_p(z)$ are thus meromorphic in this cut plane with poles given by the poles $-z_n^p$ of $N_p(z)$ ('force' poles) and the zeroes of $D_p(z)$. Each amplitude satisfies exactly the unitarity condition.

To go over from the convergence $\{B_p(z)\} \to B(z)$ to that of the amplitudes, some technical points have first to be investigated. Actually, this section contains essential preliminaries to the proof of Theorem V.3. First we set (for real positive z, z'):

$$\begin{cases} b(z) = \sqrt{\varrho(z)}\; B(z), \\ k(z, z') = \sqrt{\dfrac{\varrho(z)}{\varrho(z')}}\; K(z, z') = \dfrac{1}{\pi} \sqrt{\varrho(z)}\; \dfrac{z\, B(z) - z'\, B(z')}{z - z'} \sqrt{\varrho(z')} \end{cases} \tag{IV.1}$$

$$n(z) = \sqrt{\varrho(z)}\; N(z), \tag{IV.2}$$

in order to obtain from equation (II.8) an integral equation with symmetrical kernel:

$$n(z) = b(z) + \int_0^\infty dz'\, k(z, z')\, n(z'), \quad k(z, z') = k(z', z). \tag{IV.3}$$

By virtue of properties (II.7) of $\varrho(z)$, equation (IV.3) is a Fredholm equation, too ($b \in \mathfrak{L}^2$, $k \in \mathfrak{L}^2$).

Hereafter, we frequently write equations such as (II.8) or (IV.3) in operator form:

$$N = B + K\,N\,, \tag{IV.4}$$

$$n = b + k\,n\,. \tag{IV.5}$$

Notice that equations (IV.4) and (IV.5) (as well as their homogeneous version) are strictly equivalent since, as easily seen: $n \in \mathfrak{L}^2 \leftrightarrow N \in \mathfrak{L}^2$.

The kernels K and k being compact operators acting in the Hilbert space $\mathfrak{L}^2(0, \infty)$, the set of their characteristic values is at most countably infinite [22, 26]. Furthermore they are real since $K(z, z')$ is real and k hermitian. We denote by $\{\lambda_\mu;\, x_\mu(z)\}$ the characteristic system of k, i.e. the set of characteristic values (spectrum) and associated orthonormal characteristic functions, defined by:

$$\lambda_\mu(k\,x_\mu) = x_\mu\,, \qquad x_\mu \in \mathfrak{L}^2 \qquad (\mu = 1, 2, \ldots)\,,$$

$$(x_\mu, x_\sigma) \equiv \int\limits_0^\infty dz\, x_\mu(z)\, x_\sigma(z) = \delta_{\mu\sigma}$$

(a degenerate characteristic value with rank r is counted r times).

According to the Fredholm theorem [22], equations (IV.4) and (IV.5) have a unique (real) solution in $\mathfrak{L}^2$ if 1 does not belong to the spectrum of K (respectively k). The corresponding homogeneous equations have at least one non trivial solution in $\mathfrak{L}^2$ if 1 belongs to this spectrum. From the remark following equation (IV.5), an exhaustive set $\{\lambda_\mu;\, x_\mu(z)/\varrho(z)\}$ of characteristic values of K and related (non orthonormal) characteristic functions corresponds to the characteristic system $\{\lambda_\mu;\, x_\mu(z)\}$. The spectra of the kernels K and k coincide.

Now we put:

$$K_p(z, z') = \frac{\varrho(z')}{\pi}\, \frac{z\, B_p(z) - z'\, B_p(z')}{z - z'} = \frac{\varrho(z')}{\pi^2} \sum_{n=1}^{p} \frac{\Gamma_n^p}{(z + z_n^p)\,(z' + z_n^p)}\,. \tag{IV.6}$$

Let:

$$N_p(z) = \frac{1}{\pi} \sum_{n=1}^{p} \frac{\overline{\Gamma}_n^p}{z_n^p\,(z + z_n^p)} \tag{IV.7}$$

be the solution of the equation:

$$N_p = B_p + K_p\, N_p. \tag{IV.8}$$

This solution results as well from equations (III.4) and (III.5). Equation (IV.8) is then equivalent to the system for the $\overline{\Gamma}_n^p$:

$$\sum_{m=1}^{p} (\delta_{n\,m} - \Gamma_n^p\, f_{n\,m}^p)\, \overline{\Gamma}_m^p = \Gamma_n^p \qquad (n = 1, 2, \ldots, p)\,, \tag{IV.9}$$

where:

$$f_{n\,m}^p = \frac{z_n^p}{z_m^p}\, \frac{1}{\pi^2} \int\limits_0^\infty dx\, \frac{\varrho(x)}{(x + z_n^p)\,(x + z_m^p)}\,.$$

The symmetrized equation (IV.8) reads:

$$n_p = b_p + k_p\, n_p\,, \tag{IV.10}$$

with:

$$\begin{cases} b_p(z) = \sqrt{\varrho(z)}\; B_p(z)\,, \\ k_p(z, z') = \sqrt{\dfrac{\varrho(z)}{\varrho(z')}}\; K_p(z, z') = k_p(z', z)\,, \end{cases} \tag{IV.11}$$

$$n_p(z) = \sqrt{\varrho(z)}\; N_p(z)\,. \tag{IV.12}$$

The spectrum $\{\lambda_n^p\}$ $(n = 1, 2, \dots, p)$ of the kernels K_p and k_p is made up of the p characteristic values of the matrix with elements $\Gamma_n^p f_{nm}^p$ $(n, m = 1, 2, \dots, p)$. The system (IV.9) is regular when $1 \notin \{\lambda_n^p\}$.

Finally, let us introduce kernels without kinematical factors:

$$\begin{cases} \tilde{k}(z, z') = \dfrac{1}{\sqrt{\varrho(z)\,\varrho(z')}}\; k(z, z') = \dfrac{1}{\pi}\; \dfrac{z\,B(z) - z'\,B(z')}{z - z'}\,, \\ \tilde{k}_p(z, z') = \dfrac{1}{\sqrt{\varrho(z)\,\varrho(z')}}\; k_p(z, z')\,. \end{cases} \tag{IV.13}$$

These are analytic functions in the topological product of z- and z'-planes cut along $[-\infty, -a]$.

Theorem IV.1: Let $\mathcal{D}$ and $\mathcal{D}'$ be any compacts whose distances from the cuts $[-\infty, -a]$ in the z- and z'-planes are positive. Then $\{\tilde{k}_p(z, z')\} \to \tilde{k}(z, z')$ uniformly over $\mathcal{D} \times \mathcal{D}'$.

Proof:

After superposing the z- and z'-planes, consider a simple contour C surrounding $\mathcal{D}$ and $\mathcal{D}'$, whose distance from the cut $[-\infty, -a]$ is positive and whose distance from $\mathcal{D} \cup \mathcal{D}'$ is $\delta > 0$.

Since $B(z)$ and $B_p(z)$ are holomorphic within C, one gets:

$$B(z) = \frac{1}{2\,i\,\pi} \oint_C dw\, \frac{B(w)}{w - z} \qquad \forall\, z \in \mathcal{D}\,,$$

$$\tilde{k}(z, z') = \frac{1}{2\,i\,\pi^2} \oint_C dw\, \frac{w\,B(w)}{(w - z)\,(w - z')} \qquad \forall\, z \in \mathcal{D},\, z' \in \mathcal{D}'\,,$$

and similar formulas for $B_p(z)$ and $\tilde{k}_p(z, z')$.

Hence:

$$|\tilde{k}_p(z, z') - \tilde{k}(z, z')| = \frac{1}{2\,\pi^2} \left| \oint_C dw\, \frac{w\,[B_p(w) - B(w)]}{(w - z)\,(w - z')} \right|.$$

But:

$$\left.\begin{matrix} |w - z| \geqslant \delta \\ |w - z'| \geqslant \delta \end{matrix}\right\} \quad \forall\, w \in C,\, z \in \mathcal{D},\, z' \in \mathcal{D}'\,,$$

and, as a consequence of Theorem IV.1:

$$|w\,[B_p(w) - B(w)]| < \varepsilon_c(p) \qquad \left(\lim_{p \to \infty} \varepsilon_c(p) = 0\right).$$

Thus, if L_c is the length of the contour C:

$$|\tilde{k}_p(z, z') - \tilde{k}(z, z')| < \frac{L_c}{2\pi^2 \delta^2} \varepsilon_c(p) ,$$

which completes the proof. q.e.d.

We also need the *strong* convergence of vectors b_p and operators k_p in $\mathfrak{L}^2$. It is stated, with other useful properties in:

Theorem IV.2:

α) $\lim_{p \to \infty} \| b_p - b \| = 0$,

β) $\lim_{p \to \infty} \| k_p - k \| = 0$,

γ) $\int_0^\infty dz' \, |\tilde{k}_p(z, z') - \tilde{k}(z, z')|^2$ converges to 0 uniformly over $\mathfrak{D}$,

δ) $\int_0^\infty dz' \, |\tilde{k}_p(z, z')|^2$ is uniformly bounded with respect to p and $z \in \mathfrak{D}$.

As the proofs of the different parts are very similar, we sketch here only the most difficult one β).

Putting:

$$\Delta k_p(z, z') = k_p(z, z') - k(z, z') = \Delta k_p(z', z) ,$$

we have:

$$\| \Delta k_p \|^2 = 2 \int_0^L dz \int_0^z dz' \, |\Delta k_p(z, z')|^2 + 2 \int_L^\infty dz \int_0^z dz' \, |\Delta k_p(z, z')|^2 .$$

But, from Theoıem IV.1:

$$|\Delta k_p(z, z')| < \varepsilon_L(p) \quad \text{for} \quad 0 \leqslant z' \leqslant z \leqslant L ,$$

and from Minkowski's inequality:

$$\int_L^\infty dz \int_0^z dz' \, |\Delta k_p(z, z')|^2 \leqslant \left\{ \left[\int_L^\infty dz \int_0^z dz' \, |k_p(z, z')|^2 \right]^{\frac{1}{2}} + \left[\int_L^\infty dz \int_0^z dz' \, |k(z,z')|^2 \right]^{\frac{1}{2}} \right\}^2 .$$

Moreover, since $k \in \mathfrak{L}^2$:

$$\int_L^\infty dz \int_0^z dz' \, |k(z, z')|^2 \equiv \omega^2(L) \to 0 \quad \text{when} \quad L \to \infty .$$

Hence:

$$\| \Delta k_p \|^2 < L^2 \, \varepsilon_L^2(p) + 2 \left\{ \omega(L) + \left[\int_L^\infty dz \int_0^z dz' \, |k_p(z, z')|^2 \right]^{\frac{1}{2}} \right\}^2 .$$

To avoid inessential complications, we shall assume here all the Γ_n^p to be positive (i.e. $\phi(-x) \geqslant 0 \ \forall \, x \geqslant a$). Then, owing to equations (IV.6), (IV.11) and (II.7).

$$\int_L^\infty dz \int_0^\infty dz' \, |k_p(z, z')|^2 < \frac{I^2}{\pi^4} \sum_{n, m=1}^{p} \Gamma_n^p \Gamma_m^p \int_L^\infty \frac{dz}{(z + z_n^p)(z + z_m^p)} \int_0^z \frac{dz'}{(z' + z_n^p)(z' + z_m^p)} .$$

A weaker form of this inequality is obtained by using Schwarz's inequality in order to factorize successively the two last integrals with respect to the indices n and m. This gives:

$$\int_L^\infty dz \int_0^z dz' \, |k_p(z, z')|^2 < \frac{I^2}{\pi^4} \left[\sum_{n=1}^{p} \frac{\Gamma_n^p}{\sqrt{z_n^p (L + z_n^p)}} \right]^2 .$$

The proof could be achieved easily if one could secure:

$$\lim_{L\to\infty} \lim_{p\to\infty} \left[\sum_{n=1}^{p} \frac{\Gamma_n^p}{\sqrt{z_n^p (L + z_n^p)}} \right]^2 = 0 . \qquad \text{(IV.14)}$$

This property is by no means evident, for the convergence to 0 of the bracket when $L \to \infty$ is not necessarily uniform with respect to p.

To establish (IV.14), we use the set of equations for z_n^p and Γ_n^p resulting from equations (III.9), (III.14) and (III.15):

$$\sum_{n=1}^{p} \frac{\Gamma_n^p}{(z_n^p)^{k+1}} = \int_a^\infty dx \, \frac{\phi(-x)}{x^{k+1}} \qquad (k = 0, 1, \dots, 2p - 1) . \qquad \text{(IV.15)}$$

Let us introduce the variable $t = 1/x$, and set:

$$\psi(t) = \int_0^t \frac{dt'}{t'} \, \phi\left(-\frac{1}{t'}\right),$$

$$\psi_p(t) = \sum_{n=1}^{p} \frac{\Gamma_n^p}{z_n^p} \, \theta\left(t - \frac{1}{z_n^p}\right).$$

According to the assumption C) of Section II, with (III.11):

$$\psi(t) < \text{const.} \int_0^t dt' \, t'^{\alpha-1} \left(\text{Log} \frac{1}{t'}\right)^{-\beta},$$

and $\psi(t)$ is a positive non decreasing function over the *closed* interval $[0, 1/a]$. (There is no loss of generality to take $a > 1$.)

As a consequence of equations (IV.15), $\psi_p(t)$ is solution of the reduced 'Hausdorff moment problem' [27, 28]:

$$\int_0^{1/a} d\psi_p(t) \, t^k = \int_0^{1/a} d\psi(t) \, t^k \qquad (k = 0, 1, \dots, 2p - 1) .$$

We can thus compare the sum:

$$\sum_{n=1}^{p} \frac{\Gamma_n^p}{\sqrt{z_n^p (L + z_n^p)}} = \int_0^{1/a} d\psi_p(t) \sqrt{\frac{1}{1 + L\,t}}$$

to the integral:

$$\int_0^{1/a} d\psi(t) \sqrt{\frac{1}{1+L\,t}}$$

by means of the problem of 'approximate quadratures' considered by SHOHAT and TAMARKIN [27]. As easily seen, all assumptions of Theorem 4.6 (Loc. cit. p. 120) hold true, so that:

$$\left| \int_0^{1/a} d\,[\psi_p(t) - \psi(t)] \sqrt{\frac{1}{1+L\,t}} \right| < \delta_L(p) \qquad (\lim_{p\to\infty} \delta_L(p) = 0)\,,$$

or:

$$\sum_{n=1}^{p} \frac{\Gamma_n^p}{\sqrt{z_n^p\,(L+z_n^p)}} < \delta_L(p) + \int_0^{1/a} \frac{dt}{t}\,\phi\left(-\frac{1}{t}\right) \sqrt{\frac{1}{1+L\,t}} = \delta_L(p) + \omega(L) \;\; (\to \text{(IV.14)})\,.$$

Hence:

$$\|\Delta k_p\|^2 < L^2\,\varepsilon_L^2(p) + 2\left[\omega(L) + \frac{I}{\pi^2}\,\delta_L(p)\right]^2,$$

$$\lim_{p\to\infty} \|\Delta k_p\| \leqslant \sqrt{2}\,\omega(L)\,,$$

and finally, as L is arbitrary:

$$\lim_{p\to\infty} \|\Delta k_p\| = 0 \qquad \text{q.e.d.}$$

Now we derive the strong convergence of the sequence $\{n_p(z)\}$. For this, two lemmas are required which hold only if 1 does not belong to the spectrum of K. In the remainder of this section (as also in the next one), we shall assume this condition to be fulfilled; the case $1 \in \{\lambda_\mu\}$ will be discussed in Section VI.

Lemma IV.1:

$$\text{Let } \chi_p = \operatorname*{Inf}_{n=1,\ldots,p} |\lambda_n^p - 1|\,.$$

Then:

$$\liminf_{p\to\infty} \chi_p > 0 \text{ if } 1 \notin \{\lambda_\mu\}\,.$$

Lemma IV.2: Let r_p be the resolvent of the kernel k_p [22]. If $1 \notin \{\lambda_\mu\}$, there is a positive integer P such that the sequence:

$$\{\|r_p\|\} \qquad (p = P, P+1, \ldots)$$

is bounded.

The proofs are not very instructive and we omit them.

Theorem IV.3: If $1 \notin \{\lambda_\mu\}$, $\lim_{p\to\infty} \|n_p - n\| = 0$.

Proof:

According to Lemma IV.1, $1 \notin \{\lambda_n^p\}$ when $p \geqslant P$, and from the definition of the resolvent r_p, equation (IV.10) has the solution:

$$n_p = b_p + r_p\,b_p\,.$$

Likewise, an equivalent form of equation (IV.5) is:

$$n = (b + \Delta k_p\, n) + k_p\, n\,,$$

and the 'solution' reads:

$$n = (b + \Delta k_p\, n) + r_p\, (b + \Delta k_p\, n)\,.$$

Hence:

$$n_p - n = (1 + r_p)\, (b_p - b + \Delta k_p\, n)\,.$$

Then, using Minkowski's and Schwarz's inequalities:

$$\| n_p - n \| < (1 + \| r_p \|)\, (\| b_p - b \| + \| n \|\, \| \Delta k_p \|)\,.$$

At this stage, the theorem results from Theorem IV.2 α), β) and Lemma IV.2. q.e.d.

Now we are in a position to deduce the local convergence of the sequences $\{N_p(z)\}$, $\{D_p(z)\}$ ($p = P, P+1, \ldots$). Obviously the functions $D_p(z)$ are defined by:

$$D_p(z) = 1 - \frac{z}{\pi} \int_0^\infty dx\, \varrho(x)\, \frac{N_p(x)}{x - z} = 1 - \frac{z}{\pi^2} \sum_{n=1}^{p} \frac{\overline{\Gamma}_n^p}{z_n^p} \int_0^\infty dx\, \frac{\varrho(x)}{(x + z_n^p)\, (x - z)} \qquad \text{(IV.16)}$$

the system (IV.9) which provides the $\overline{\Gamma}_n^p$ being now regular for $p \geqslant P$.

Theorem IV.4:

α) $\{N_p(z)\} \to N(z)$ uniformly over any compact $\mathcal{D}$ whose distance from the cut $[-\infty, -a]$ is positive,

β) $\{D_p(z)\} \to D(z)$ uniformly over any compact $\mathcal{O}$ whose distance from the cut $[0, \infty]$ is positive.

Due to the similarity of the arguments, we give the proof of β) only. One deduces from equations (II.6) and (IV.16), when using equations (IV.2) and (IV.12):

$$D_p(z) - D(z) = -\frac{z}{\pi} \int_0^\infty dx\, \sqrt{\varrho(x)}\, \frac{n_p(x) - n(x)}{x - z}$$

so that:

$$| D_p(z) - D(z) | < \frac{|z|}{\pi} \sqrt{I}\, \| n_p - n \| \left[\int_0^\infty \frac{dx}{|x - z|^2} \right]^{1/2}.$$

But:

$$\int_0^\infty \frac{dx}{|x - z|^2} = \frac{\pi - |\phi|}{|z| \sin |\phi|} \quad \text{where } \phi = \arg z \quad (-\pi \leqslant \phi \leqslant \pi)\,,$$

and the proof is complete if we take Theorem IV.3 into account and observe that $0 < \phi_0 < |\phi| \leqslant \pi$ when $z \in \mathcal{O}$. q.e.d.

V. Convergence of Scattering Amplitudes in the Physical Region

The convergence of the approximate amplitudes $T_p(z)$ to the exact one $T(z)$ is a trivial consequence of Theorem IV.4 for regular points z not on the cuts $[-\infty, -a]$, $[0, \infty]$. We have now to show how this convergence can be extended to the physical region $[0, \infty]$.

In order to give a general form to the final result, we first produce a statement which exhibits the connection between possible dynamical poles of $T(z)$ and those of approximations $T_p(z)$. This theorem may be of some use for the locating of bound states, ghosts and even resonances (see the remark following Theorem V.3).

Theorem V.1: Let $\mathcal{O}$ be any compact whose distance from the cut $[0, \infty]$ is positive, and let $z_1, \dots, z_k$ be the zeroes of $D(z)$ within $\mathcal{O}$ (we suppose $\mathcal{O}$ to be such that its boundary does not meet any zero of $D(z)$).

Then the zeroes of $D_p(z)$ within $\mathcal{O}$ have the points $z_1, \dots, z_k$ as limitpoints, and only these, when $p \to \infty$. Moreover there exists a p_0 such that $D_p(z)$ has the same number of zeroes as $D(z)$ within $\mathcal{O}$ when $p \geqslant p_0$ (multiplicities included).

This proposition is a variant of a theorem due to Hurwitz [25] and its proof will be omitted.

General arguments suggest that the singularities of partial-wave scattering amplitudes lying on the positive real axis form a discrete set without accumulation point, namely the set of branch-points corresponding to the various production thresholds [29]. If the amplitudes actually have to be analytic almost everywhere in the physical region, this must reflect in some regularity property of the inelastic factor $\varrho(x)$. So we can expect that the sequence $\{T_p(z)\}$ will converge over $[0, \infty]$ as well.

Because of the almost phenomenological context of oui study, we do not intend to derive specific conditions about $\varrho(x)$, but shall be content with the following plausible assumption, compatible with the threshold behaviours:

The factor $\varrho(x)$ fulfills the *Lipschitz condition:*

$$|\varrho(x_1) - \varrho(x_2)| \leqslant A\, |x_1 - x_2|^\mu \quad \forall\ x_1, x_2 \in [0, \infty] \text{ with } 0 < \mu \leqslant 1\,. \tag{V.1}$$

Let us now introduce:

$$\tilde{\varrho}(z) = \frac{z}{\pi} \int_0^\infty dx\, \frac{\varrho(x)}{x\,(x-z)}\,, \tag{V.2}$$

which is a 'real' function, holomorphic in the cut z-plane, such that:

$$Im\,[\tilde{\varrho}\,(x + i\,0)] = \varrho(x) \qquad (x \geqslant 0)\,. \tag{V.3}$$

We next define:

$$G(z) = D(z) + \tilde{\varrho}(z)\, z\, N(z)\,, \qquad G_p(z) = D_p(z) + \tilde{\varrho}(z)\, z\, N_p(z)\,. \tag{V.4}$$

From equations (II.3) and (V.3), these auxiliary functions are holomorphic in the z-plane cut along $[-\infty, -a]$. Using the property:

$$|\tilde{\varrho}(z)\,|\, z < B_{\mathcal{D}} \qquad \forall\ z \in \mathcal{D} = \text{compact whose distance from } [-\infty, -a] \text{ is positive} \tag{V.5}$$

which results from (V.1), (V.2) and from a theorem due to Muskhelishvili [30], we can deduce the following lemma, the proof of which is given in Appendix 4.

Lemma V.1: $\{G_p(z)\} \to G(z)$ uniformly over any $\mathcal{D}$.

This allows to strengthen Theorem IV.4 β):

Theorem V.2: The sequence $\{D_p(z)\}$ converges uniformly to $D(z)$ over *any* compact.

Proof:

First, the theorem is true for a compact $\mathcal{D}$ whose distance from $[-\infty, -a]$ is positive, as resulting from application of Theorem IV.4 α), Lemma V.1 and bound (V.5) to the inequality:

$$|D_p(z) - D(z)| \leqslant |G_p(z) - G(z)| + |\tilde{\varrho}(z)\, z|\, |N_p(z) - N(z)|\, .$$

Next, it is also true for any compact, because of Theorem IV.4 β). q.e.d.

Now we can assert that the sequence of $T_p(z) = z\, N_p(z)/D_p(z)$ converges uniformly to $T(z) = z\, N(z)/D(z)$ over any compact $\mathcal{E}$ whose distance from $[-\infty, -a]$ and from the zeroes of $D(z)$ is positive (Fig. 2). This follows from Theorem IV.4 α), Theorem V.2 and from uniform lower boundedness of $D_p(z)$ over $\mathcal{E}$ for $p \geqslant P$ (as a consequence of Theorem V.1 and V.2).

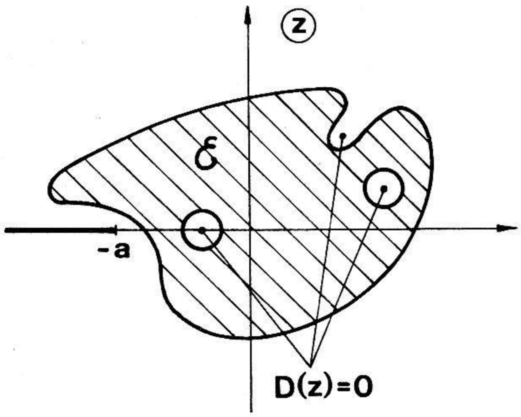

Figure 2

Example of a domain where the sequence of amplitudes $T_p(z)$ converges uniformly.

Two further comments may be useful to state our result precisely:

1) The convergence $\{T_p(z)\} \to T(z)$ extends automatically to the second sheet of the elastic right-hand cut $[0, z_I]$. This comes from the simple explicit form of the analytic continuation $T_{II}(z)$ of the amplitude onto this sheet:

$$T_{II}(z) = z\, \frac{N(z)}{D(z) + 2\, i\, z \sqrt{z/z+1}\; N(z)}\, .$$

2) Convergence difficulties could arise if $D(z)$ would vanish at some real positive point z_0, due to the unboundedness of $T_p(z)$ in the neighbourhood of z_0 (even though $T(z_0) < \infty$, as required by unitarity). In fact there is no reason to worry about such a phenomenon, for we shall see that $D(z)$ cannot vanish on $[0, \infty]$ when $1 \notin \{\lambda_\mu\}$ (Theorem VI.1).

Thus, the following statement summarizes our results:

Theorem V.3: Let $\mathcal{R}$ be the Riemann surface made up with the physical sheet and the second sheet of the elastic right-hand cut $[0, z_I]$.

When $1 \notin \{\lambda_\mu\}$, the sequence of amplitudes $\{T_p(z)\}$ converges uniformly to $T(z)$ over any compact $\mathcal{C}$ whose distance from the 'dynamical' singularities of $T(z)$ (cuts $[-\infty, -a]$, $[-\infty, -1]$ and poles) is positive.

Moreover, $T_p(z)$ and $T(z)$ have the same number of poles within $\mathcal{C}$ for p large enough, and the poles of $T_p(z)$ tend toward those of $T(z)$ when $p \to \infty$.

We can still notice that $\{T_p(z)\}$ actually converges to $T(z)$ over any domain in which $T(z)$ and $T_p(z)$ have a continuation, as a consequence of Vitali's theorem (cf. proof of Theorem III.1). This is particularly the case for the continuation of $T(z)$ and $T_p(z)$ (if allowed) onto the unphysical sheets connected with the various inelastic thresholds. Therefore, the $T_p(z)$ may be suitable approximations for the research of 'inelastic' resonances too.

VI. Poles at Subtraction Point and Extinct Poles

Now we have to deal with the situation where the spectrum $\{\lambda_\mu\}$ of K contains the value 1. We choose to go over continuously from the normal case $1 \notin \{\lambda_\mu\}$ to the pathological one $1 \in \{\lambda_\mu\}$ by multiplying the discontinuity $\phi(-x)$ with a 'coupling constant' λ and inspecting the behaviour of the solutions $N(z;\lambda)$, $D(z;\lambda)$ in the limit $\lambda \to 1$.

When substituting $\lambda\,\phi(-x)$ for $\phi(-x)$ in equation (IV.3), one gets the usual form of a Fredholm equation with parameter λ:

$$\frac{n(z;\lambda)}{\lambda} = b(z) + \lambda \int_0^\infty dz'\, k(z,z')\, \frac{n(z';\lambda)}{\lambda}\,. \qquad \text{(VI.1)}$$

The kernel $k(z,z')$ being continuous for $z \geqslant 0$, $z' \geqslant 0$, its resolvent is given by the relatively uniformly absolutely convergent series [22]:

$$r(z,z';\lambda) = k(z,z') + \lambda \sum_\mu \frac{x_\mu(z)\, x_\mu(z')}{\lambda_\mu\,(\lambda_\mu - 1)} \qquad (\lambda \notin \{\lambda_\mu\})\,, \qquad \text{(VI.2)}$$

and the (unique) solution of equation (VI.1) by:

$$\frac{n(z;\lambda)}{\lambda} = b(z) + \lambda \int_0^\infty dz'\, r(z,z';\lambda)\, b(z')\,. \qquad \text{(VI.3)}$$

Let r be the rank of the characteristic value 1. We order the λ_μ so that:

$$\lambda_1 = \lambda_2 = \ldots = \lambda_r = 1 \qquad (\lambda_\mu \neq 1 \text{ if } \mu > r)\,,$$

and denote by $x_1(z), x_2(z), \ldots, x_r(z)$ r orthonormal characteristic functions belonging to the value 1.

We deduce from equations (VI.2), (VI.3) and (II.6):

$$N(z;\lambda) = \lambda\, B(z) + \lambda^2 \int_0^\infty dz'\, K(z,z')\, B(z')$$
$$+ \frac{\lambda^3}{\sqrt{\varrho(z)}} \left[\frac{1}{1-\lambda} \sum_{m=1}^{r} b_m\, x_m(z) + \sum_{\mu>r} \frac{b_\mu\, x_\mu(z)}{\lambda_\mu\,(\lambda_\mu - \lambda)}\right], \qquad \text{(VI.4)}$$

$$D(z;\lambda) = D_2(z;\lambda) - \frac{\lambda^3 z}{\pi} \left[\frac{1}{1-\lambda} \sum_{m=1}^{r} b_m \int_0^\infty dz'\, \frac{\sqrt{\varrho(z')}\, x_m(z')}{z'-z}\right.$$
$$\left. + \sum_{\mu>r} \frac{b_\mu}{\lambda_\mu\,(\lambda_\mu - \lambda)} \int_0^\infty dz'\, \frac{\sqrt{\varrho(z')}\, x_\mu(z')}{z'-z}\right], \qquad \text{(VI.5)}$$

where.

$$b_\mu = \int_0^\infty dx\, x_\mu(z)\, b(z) \equiv (x_\mu, b)\,, \qquad b_m = \ldots\,,$$

and where the function:

$$D_2(z;\lambda) = 1 - \frac{\lambda\, z}{\pi} \int_0^\infty dz'\, \frac{\varrho(z')\, B(z')}{z'-z} - \frac{\lambda^2\, z}{\pi} \int_0^\infty dz'\, \frac{\varrho(z')}{z'-z} \int_0^\infty dz''\, K(z', z'')\, B(z'')$$

is the second order perturbative solution of the integral equation for $D(z;\lambda)$.

Let us first assume that not all b_m vanish. Then, according to equations (VI.4) and (VI.5), $N(z;\lambda) \to \infty$ and $D(z;\lambda) \to \infty$ for $\lambda \to 1$, while the amplitude remains finite:

$$T(z;1) = -\frac{\pi}{\sqrt{\varrho(z)}} \frac{\sum_{m=1}^{r} b_m\, x_m(z)}{\sum_{m=1}^{r} b_m \int_0^\infty dz' \sqrt{\varrho(z')}\, x_m(z')/(z'-z)}\,.$$

Besides, equation (VI.5) shows that a zero $z_0(\lambda)$ of $D(z;\lambda)$ reaches the origin when $\lambda \to 1$:

$$z_0(\lambda) \underset{\lambda\to 1}{=} \frac{\pi\,(1-\lambda)}{\sum_{m=1}^{r} b_m \int_0^\infty dz' \sqrt{\varrho(z')}\, x_m(z')/z'} + 0\,[(1-\lambda)^2]\,.$$

This holds when the origin is chosen as a subtraction point for $D(z;\lambda)$ (as done for convenience from the very beginning). More generally, for an arbitrary subtraction point $\bar{z}$, it is easily seen that $D(z;\lambda)$ produces a zero $z_0(\lambda)$ such that:

$$z_0(\lambda) \underset{\lambda\to 1}{=} \bar{z} + 0\,(1-\lambda)\,.$$

Thus, it appears generally that *a pole of the amplitude (a bound state) goes through the subtraction point of the D-function when λ takes the value* $1 \in \{\lambda_\mu\}$. This allows to reduce the research of a possible bound state with energy $2\sqrt{\bar{z}+1}$ to an eigenvalue problem for the operator k.

Concerning our approximation scheme, the difficulties of convergence which could arise from the presence of a bound state at $z = 0$ (cf. Theorem VI.2) will be removed by a shift of the subtraction point eliminating the characteristic value 1.

A more interesting situation occurs when $b_1 = b_2 = \ldots = b_r = 0$.

The functions:

$$N(z) = \lim_{\lambda\to 1} N(z;\lambda)\,, \qquad D(z) = \lim_{\lambda\to 1} D(z;\lambda)$$

ore then well defined and determine a particular solution of the N/D equations. Other solutions can be obtained by adding to $N(z)$ the general solution of the homogeneous version of equation (II.8). It results in a multiplicity of N- and D-solutions which does not disappear with a shift of the subtraction point.

Nevertheless, it can be shown explicitely that such an ambiguity for $N(z)$ and $D(z)$ does not affect their quotient: *the amplitude $T(z)$ is always determined uniquely.*

The reason for this is that the arbitrary parts of N and D can be factorized and cancel out. The appearance of *coincident arbitrary zeroes* in N- and D-functions is a possible mechanism for such a delusive undeterminacy [20], as shown by the following proposition.

Theorem VI.1: If the functions $N(z)$ and $D(z)$ both vanish at a point $z = z_0$, then $1 \in \{\lambda_\mu\}$.

Proof:

Assuming for definiteness that z_0 is real, we can introduce a family of new N- and D-functions:

$$N_\alpha(z) = \frac{\alpha\, z - z_0}{z - z_0}\, N(z)\ , \qquad D_\alpha(z) = \frac{\alpha\, z - z_0}{z - z_0}\, D(z)\ , \tag{VI.6}$$

where α is a real parameter.

The new functions (VI.6) are 'real', regular at $z = z_0$, and have the same analyticity properties as the old ones (in particular $D_\alpha(z)$ is free from CDD poles). Moreover:

$$\begin{cases} \dfrac{N_\alpha(z)}{D_\alpha(z)} = \dfrac{T(z)}{z}\ , \\ N_\alpha(z) \underset{|z| \to \infty}{\simeq} \alpha\, N(z)\ , \quad D_\alpha(z) \underset{|z| \to \infty}{\simeq} \alpha\, D(z) \quad (\alpha \neq 0)\ , \\ D_\alpha(0) = 1\ . \end{cases}$$

Accordingly, $N_\alpha(z)$ and $D_\alpha(z)$ are solutions of the same N/D equations as $N(z)$ and $D(z)$ for every non vanishing value of α.

Furthermore:

$$N(z) \in \mathfrak{L}^2 \Rightarrow N_\alpha(z) \in \mathfrak{L}^2\ ,$$

so that the solution of equation (II.8) cannot be unique in this case, which implies $1 \in \{\lambda_\mu\}$. q.e.d.

Notice that the occurence of coincident zeroes in N- and D-functions (i.e. 'exctinct poles' for $T(z)$) has been claimed as a possible way to resolve the difficulties encounrered if the Pomeranchuk trajectory crosses the zero angular momentum line at a negative value of center-of-mass energy [31].

Going back to our approximation scheme, we find it fails to work any more. This appears clearly from some specific examples [32], and also from the theorem quoted below, which is a counterpart of Lemma IV.1.

Theorem VI.2: If $1 \in \{\lambda_\mu\}$, it is possible to extract from the sequence of spectra $\{\lambda_n^p\}$ a sequence of characteristic value $\lambda_{n_p}^p$. $(p = 1, 2, \ldots)$ such that $\lim_{p \to \infty} \lambda_{n_p}^p = 1$.

We omit the proof.

Then the canonical representation of the resolvent kernel r_p [22]:

$$r_p(z, z') = k_p(z, z') + \int_0^\infty dz''\, k_p(z, z'')\, k_p(z'', z') + \sum_{n=1}^{p} \frac{x_n^p(z)\, x_n^p(z')}{(\lambda_n^p)^2\, (\lambda_n^p - 1)}$$

shows that $\lim_{p \to \infty} r_p(z, z') = \infty$. This produces the vanishing of the determinant of system (IV.9) in the limit $p \to \infty$, and causes our approximation scheme to be practically useless in that case, even if the sequence:

$$n_p(z) = b_p(z) + \sum_{n=1}^{p} \frac{(x_n^p, b_p)}{\lambda_n^p - 1} x_n(z)$$

converges, because the solutions of a linear system with nearly vanishing determinant are very instable with respect to variations of the coefficients.

As a matter of fact, the presence of a characteristic value equal or near to 1 in the spectrum of K will be revealed, above some order p, by the appearance of characteristic values $\lambda_{n_p}^p \cong 1$, resulting in a nearly vanishing determinant:

$$\text{Det}\,(\delta_{nm} - \Gamma_n^p f_{mn}^p) \cong 0\,. \tag{VI.7}$$

In so far as its 'accuracy' can be estimated, the approximate relation (VI.7) may be useful, as being a *necessary condition for the occurence of a narrow resonance*. This must be so because a slight modification of the left-hand discontinuity $\phi(-x)$ suffices to shift the pair of conjugate zeroes of $D(z)$ exactly onto the positive real axis, in such a way that the spectrum of the perturbed kernel contains the value 1.

Observe however that the relations (VI.7) are only necessary conditions: They can result from resonances, but also from antiresonances, extinct ghosts or extinct bound states.

VII. Summary

Recently, the Padé approximants technique has become of frequent use in several physical problems, particularly as a tool supplying the failing perturbation method with a consistent extension [33–36].

In this work, we showed in the frame of N/D equations how this technique applies to the construction of convergent sequences of approximate (but unitary) partial-wave amplitudes. These amplitudes involved Padé approximants referring to the 'energy' variable. The starting point consisted in replacing the left-hand singularities by a growing number of poles, a procedure used many times within purely physical contexts, i.e. without attending to its mathematical aspect.

Sufficient conditions for the convergence of the method have been given. These are slightly stronger than those insuring the equivalence of N/D equations with a Fredholm equation (compare the required asymptotic behaviours of the left-hand discontinuity, respectively equations (III.11) and (II.11)). For the intermediate case ($\alpha = 0$, $1/2 < \beta \leqslant 1$ if $l \geqslant 1$), we did not succeed to prove the convergence, nor its failure.

Even under the general conditions (II.11), we have displayed special cases where the convergence is bad. We have found at the same time that the oscillatory behaviour of the solutions may be an indication for a narrow resonance.

Turning now to the so-called 'marginal case' ($\alpha = \beta = 0$), it can be shown that our pole approximation method does not converge any more [32]. This comes from the very different nature of the mathematical problem: the fact that the N/D equations do not lead to a Fredholm equation results in a multiplicity of acceptable solutions [37–39]. We shall deal with this somewhat pathological case in a subsequent article.

We do not discuss here the various extensions of our method to more complicated realistic descriptions (different masses, inclusion of spin, many-channel formalism). These generalizations are mostly straightforward.

At the time this work was completed, two papers appeared, due to COMMON and SWEIG, dealing essentially with the same subject [40, 41]. Although the proofs given by these authors follow different lines, their pratical scheme of approximation coincide with ours, as well as their main conclusions.

It is a great pleasure to thank Professor G. WANDERS for his advice and encouragement during the course of this work.

Appendices

1. *The Partial Waves with $l \neq 1$*

When introducing an arbitrary number of subtractions, the equations (II.2–3) and (II.8–10) take the form:

$$\begin{cases} N_l(z) = P_{N-1}(z) + \dfrac{z^N}{\pi} \displaystyle\int_{-\infty}^{-a} dx \, \dfrac{\phi_l(x) \, D_l(x)}{x^{l+N} \, (x - z)} \, , \\ D_l(z) = 1 + z \, Q_{M-1}(z) - \dfrac{z^{M+1}}{\pi} \displaystyle\int_0^{\infty} dx \, \varrho_l(x) \, x^{l-M-1} \, \dfrac{N_l(x)}{x+z} \, , \end{cases} \tag{A.1}$$

$$N_l(z) = B_l(z) + \int_0^{\infty} dz' \, K_l(z, z') \, N_l(z') \, , \tag{A.2}$$

$$B_l(z) = P_{N-1}(z) + (-1)^{l+N+1} \frac{z^N}{\pi} \int_a^{\infty} dx \, \frac{\phi_l(-x)}{x^{l+N} \, (x+z)} \, [1 - x \, Q_{M-1}(-x)] \, , \tag{A.3}$$

$$K_l(z, z') = (-1)^{l+N+M+1} \, \varrho_l(z') \, z'^{\,l-M-1} \, \frac{z^N}{\pi^2} \int_a^{\infty} dx \, x^{M-N-l+1} \, \frac{\phi_l(-x)}{(x+z) \, (x+z')} \, , \tag{A.4}$$

where $P_{N-1}(z)$ and $Q_{M-1}(z)$ are arbitrary 'real' polynomials of degree $N - 1$ and $M - 1$ respectively. In equations (A.1), the possible *CDD* poles have been 'absorbed' by subtractions [42]. It follows from (A.3) that $B_l(z)/z^{N-1}$ is generally not bounded when $z \to \infty$ (whatever the asymptotic behaviour of $\phi_l(-x)$ may be). Thus the condition $B_l \in \mathfrak{L}^2$ requires $N = 0$ and $P_{N-1}(z) = 0$. Likewise $K_l \in \mathfrak{L}^2$ implies $M \geqslant l - 1$.

From now on, we have to distinguish the *S*-wave from the higher waves.

a) $l \geqslant 1$

We put $M = l - 1$ in order to secure at best the convergence of the integrals contained in (A.3) and (A.4). Then:

$$\begin{cases} B_l(z) = \dfrac{(-1)^{l+1}}{\pi} \displaystyle\int_a^{\infty} dx \, \dfrac{\phi_l(-x)}{x^l \, (x+z)} \, [1 - x \, Q_{l-2}(-x)] \, , \\ K_l(z, z') = \dfrac{\varrho_l(z')}{\pi^2} \displaystyle\int_a^{\infty} dx \, \dfrac{\phi_l(-x)}{(x+z) \, (x+z')} \, . \end{cases}$$

From these expressions, we can infer:

i) for all $l \geqslant 1$, equation (II.11) applied to $\phi_l(-x)$ gives the necessary and sufficient conditions for equation (A.2) to be a Fredholm equation,

ii) if these conditions do not hold, the convergence of (A.3) and (A.4) can still be maintained by an increase of $N-M$, i.e. by introducing '*CDD* subtractions' [42] or by choosing $M < l - 1$. But in that case, equation (A.2) (even symmetrized) is *not* a Fredholm equation,

iii) when $\phi_l(-x)$ fulfills conditions (III.11), the contents of Sections III–VII remain plainly valid, provided that one applies the pole approximation method of Section III to the function:

$$F_l(z) = \int_a^\infty dx \, \frac{\phi_l(-x)}{x+z} ,$$

from which the potential can be reconstructed by using:

$$B_l(z) = \frac{1}{\pi z^l} \left\{ \sum_{n=0}^{l-1} \left[\frac{d^n}{dz^n} \left((1 + z\, Q_{l-2})\, F_l \right) \right]_{z=0} \frac{z^n}{n!} - [1 + z\, Q_{l-2}(z)]\, F_l(z) \right\}$$

and by keeping the arbitrary coefficients of $Q_{l-2}(z)$ constant during the whole process.

b) $l = 0$

The optimal value of M is now $M = l = 0$, giving:

$$B_0(z) = -\frac{1}{\pi} \int_a^\infty dx \, \frac{\phi_0(-x)}{x+z} , \qquad K_0(z, z') = -\frac{\varrho_0(z')}{z'} \, \frac{1}{\pi^2} \int_a^\infty dx \, \frac{x\, \phi_0(-x)}{(x+z)\,(x+z')} .$$

The conditions on $\phi_0(-x)$ insuring $K_0(z, z')$ to be a Hilbert-Schmidt kernel are, instead of (II.11):

$$\alpha > 0 \ (\text{any } \beta) \quad \text{or} \quad \alpha = 0,\ \beta > 2 .$$

The pole approximation method will apply to the function:

$$F_0(z) = \pi\, z\, B_0(z) + \int_a^\infty dx\, \phi_0(-x) = \int_a^\infty dx \, \frac{x\, \phi_0(-x)}{x+z} ,$$

provided that $\alpha > 1$ or $\alpha = 1$, $\beta > 1$.

The conclusions of Sections III–VII are then unchanged.

2. *Recurrent Construction of the Continued Fraction*

Starting from the expansion (III.6), one can construct step by step the representation (III.7) by means of the following rules [23, 43].

Let $\{V_n(u)\}$ be a sequence of polynomials defined by the recurrence:

$$\begin{cases} V_0(u) = 1 , \quad V_1(u) = u - \dfrac{\alpha_1}{\alpha_0} , \\ V_{n+1}(u) = (u + \beta_{2n} + \beta_{2n+1})\, V_n(u) - \beta_{2n-1}\, \beta_{2n}\, V_{n-1}(u) \quad (n = 1, 2, \ldots) . \end{cases} \tag{A.5}$$

Introduce a linear form $\mathfrak{F}$ on the polynomials, defined by its value on the monomials:

$$\mathfrak{F}\{u_n\} = \alpha_n \qquad (n = 0, 1, \ldots) . \tag{A.6}$$

The coefficients β_n are then given by:

$$\begin{cases} \beta_0 = \alpha_0 \,,\ \beta_1 = -\dfrac{\alpha_1}{\alpha_0} \,, \\ \beta_{2n} = \dfrac{\mathfrak{F}\{u^n\, V_n(u)\}}{\mathfrak{F}\{u^{n-1}\, V_{n-1}(u)\}}\, \dfrac{1}{\beta_{2n-1}} \,, & (n = 1, 2, \ldots) \\ \beta_{2n+1} = \dfrac{\mathfrak{F}\{u^n V_{n-1}(u)\}}{\mathfrak{F}\{u^{n-1}\, V_{n-1}(u)\}} - \dfrac{\mathfrak{F}\{u^{n+1}\, V_n(u)\}}{\mathfrak{F}\{u^n\, V_n(u)\}} - \beta_{2n} \,. \end{cases} \tag{A.7}$$

Once β_m $(m = 0, 1, \ldots, 2n - 1)$ and $V_m(u)$ $(m = 0, 1, \ldots, n)$ are known, formulas (A.5–7) clearly allow to compute β_{2n}, β_{2n+1} and $V_{n+1}(u)$, as long as $\mathfrak{F}\{u^p\, V_p(u)\} \neq 0$ $(p = n - 1, n)$. These last conditions happen to be satisfied when:

$$\begin{vmatrix} \alpha_0 & \alpha_1 & \ldots \alpha_n \\ \alpha_1 & \alpha_2 & \ldots \alpha_{n+1} \\ \vdots & \vdots & \vdots \\ \alpha_n & \alpha_{n+1} & \ldots \alpha_{2n} \end{vmatrix} \neq 0 \qquad (n = 0, 1, \ldots) .$$

3. *Proof of Lemma III.1*

We have to show first that all the poles of $H_p(z)$ are simple, or that $Y_p(z)$ has p distrincts zeroes $- z_n^p$. Then obviously $H_p(z)$ is reducible to the form (III.15).

Let us proceed 'ab absurdo' by assuming that $Y_p(z)$ has a zero $- z_0$ of order $\geqslant 2$. If z_0 is real, this means:

$$Y_p(z) = (z + z_0)^2\, Z_{p-2}(z) ,$$

where $Z_{p-2}(z)$ is a polynomial of degree $p - 2$.

Then, from (III.8):

$$\begin{vmatrix} \alpha_0 & \alpha_1 \ldots \alpha_p \\ \alpha_1 & \alpha_2 \ldots \alpha_{p+1} \\ \vdots & \vdots \quad \vdots \\ \alpha_{p-1} & \alpha_p \ldots \alpha_{2p-1} \\ 1 & u \ldots u^p \end{vmatrix} = (u + u_0)^2\, Q_{p-2}(u) \qquad \left(u_0 = \frac{1}{z_0}\right) , \tag{A.8}$$

with

$$Q_{p-2}(u) = \sum_{n=0}^{p-2} q_n\, u^n \qquad (q_n \text{ real}) .$$

By using (A.8) and the linear form defined in (A.6), one gets:

$$\mathfrak{F}\{u^k\, (u + u_0)^2\, Q_{p-2}(u)\} = 0 \qquad (k = 0, 1, \ldots, p - 1) ,$$

$$\mathfrak{F}\{(u + u_0)^2\, Q_{p-2}(u)\} = \sum_{k=0}^{p-2} q_k\, \mathfrak{F}\{u^k\, (u + u_0)^2\, Q_{p-2}(u)\} = 0 .$$

Now, according to equations (III.6) and (III.14):

$$\alpha_k = (-1)^k \int_b^c dx \, \frac{\phi(-x)}{x^{k+1}} \qquad (k = 0, 1, \ldots) .$$

Hence:

$$\mathfrak{F}\{(u+u_0)^2 \, Q_{p-2}(u)^2\} = \int_b^c dx \, \frac{\phi(-x)}{x} \left[\left(u_0 - \frac{1}{x}\right) Q_{p-2}\left(-\frac{1}{x}\right)\right]^2 = 0 .$$

But the last equality is incompatible with the property (III.12). The eventuality of a complex zero is ruled out in a analogous way.

We next prove the inequalities (III.16) for $c = \infty$.

Take equation (IV.15):

$$\sum_{n=1}^{p} \frac{\Gamma_n^p}{(z_n^p)^{k+1}} = \int_b^\infty dx \, \frac{\phi(-x)}{x^{k+1}} \qquad (k = 0, 1, \ldots, 2p-1) , \tag{A.9}$$

and put:

$$\varrho_n^p = \frac{b}{z_n^p} , \; y = \frac{x}{b} , \; \nu = 2p - 1 - k , \; \Phi_p(y) = b \, \frac{\phi(-b_y)}{y^{2p}} .$$

For a fixed p, we have to show that the relations:

$$\sum_{n=1}^{p} \Gamma_n^p (\varrho_n^p)^{2p-\nu} = \int_1^\infty dy \, \Phi_p(y) \, y^\nu \qquad (\nu = 0, 1, \ldots, 2p-1) \tag{A.10}$$

imply:

$$0 < \varrho_n^p < 1 \qquad \Gamma_n^p > 0 \qquad (n = 1, 2, \ldots, p) \tag{A.11}$$

if $\Phi_p(y)$ is a non negative function with support $[1, \infty]$.

Let us put further:

$$t = y - 1 , \quad \sigma_p(t) = \Phi_p \, (t+1) ,$$

and introduce the moments:

$$\mu_\nu = \int_0^\infty dt \, \sigma_p(t) \, t^\nu \qquad (\nu = 0, 1, \ldots, 2p-1) .$$

We see that $\sigma_p(t)$ is solution of a reduced 'Stieltjes moment problem' [27, 28]. Both quadratic forms:

$$\sum_{i,j=0}^{\nu} \mu_{i+j} \, \xi_i \, \xi_j , \quad \sum_{i,j=0}^{\nu} \mu_{i+j+1} \, \xi_i \, \xi_j \qquad (\nu = 0, 1, \ldots, p-1)$$

must then be positive definite. On the other hand, by using (A.10):

$$\mu_\nu = \int_0^\infty dy \, \Phi_p(y) \, (y-1)^\nu = \sum_{n=1}^{p} \Gamma_n^p (\varrho_n^p)^{2p-\nu} \, (1 - \varrho_n^p)^\nu .$$

Hence, for any vector ξ_i:

$$\begin{cases} \sum_{n=1}^{p} \Gamma_n^p (\varrho_n^p)^{2p} \sum_{i,j=0}^{\nu} \left(\frac{1}{\varrho_n^p} - 1\right)^{i+j} \xi_i \xi_j > 0 , \\ \sum_{n=1}^{1} \Gamma_n^p (\varrho_n^p)^{2p} \sum_{i,j=0}^{\nu} \left(\frac{1}{\varrho_n^p} - 1\right)^{i+j+1} \xi_i \xi_j > 0 . \end{cases} \qquad (\nu = 0, 1, \dots , p-1)$$

Choosing $\nu = p - 1$, one obtains:

$$\sum_{n=1}^{p} \Gamma_n^p (\varrho_n^p)^{2p} (a_n^p)^2 > 0 , \qquad \sum_{n=1}^{p} \Gamma_n^p (\varrho_n^p)^{2p} \left(\frac{1}{\varrho_n^p} - 1\right) (a_n^p)^2 > 0 , \tag{A.12}$$

where:

$$a_n^p = \sum_{i=0}^{p-1} \left(\frac{1}{\varrho_n^p} - 1\right)^i \xi_i \qquad (n = 1, 2, \dots , p) . \tag{A.13}$$

Let Δ be the $p \times p$ matrix with elements $\Delta_{ni} = [(1/\varrho_n^p) - 1]^i$.

As $z_n^p \neq z_m^p$ for $n \neq m$:

$$\operatorname{Det} \Delta = \prod_{\substack{n=2,\dots,p \\ m<n}} \left(\frac{1}{\varrho_n^p} - \frac{1}{\varrho_m^p}\right) \neq 0 .$$

Thus for each $m = 1, 2, \dots , p$, a vector ξ_i can be found (by solving the system (A.13)) such that $a_n^p = \delta_{nm}$. For this particular choice of ξ_i, (A.12) gives:

$$\Gamma_m^p (\varrho_m^p)^{2p} > 0 , \qquad \Gamma_m^p (\varrho_m^p)^{2p} \left(\frac{1}{\varrho_m^p} - 1\right) > 0 ,$$

from which inequalities (A.11) follow.

When $c < \infty$, the argument can be reduced to the preceding one by a simple change of variables and functions. q.e.d.

Notice that the (non linear) system (A.9), which has the $2p$ quantities z_n^p, Γ_n^p as (unique) solution, is equivalent to the equation:

$$Y_p(z) = 0$$

giving the z_n^p, and to the linear system for the Γ_n^p:

$$\sum_{n=1}^{p} \frac{\Gamma_n^p}{(-z_n^p)^{k+1}} = -\alpha_k \qquad (k = 0, 1, \dots , p-1)$$

with non vanishing determinant.

As a consequence of the Lemma, $Y_p(0) \neq 0$. This is precisely the condition we required at the end of Appendix 2. In fact:

$$\begin{vmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_n \\ \alpha_1 & \alpha_2 & \dots & \alpha_{n+1} \\ \vdots & \vdots & & \vdots \\ \alpha_n & \alpha_{n+1} & \dots & \alpha_{2n} \end{vmatrix} > 0 \qquad (n = 0, 1, \dots) ,$$

since the quadratic form:

$$\sum_{i,j=0}^{n} \alpha_{i+j}\, \xi_i\, \xi_j = \int_b^c dx\, \frac{\phi(-x)}{x} \left[\sum_{i=0}^{n} \frac{\xi_i}{(-x)^i} \right]^2$$

is positive definite.

4. *Proof of Lemma V.1*

Referring to the proof of Theorem IV.4, we can write for $z \in \mathcal{D}$, $y = Im\, z \neq 0$:

$$|N_p(z) - N(z)| < \varepsilon_{\mathcal{D}}(p)\,, \qquad \left(\lim_{p\to\infty} \varepsilon_{\mathcal{D}}(p) = \lim_{p\to\infty} \eta_{\mathcal{D}}(p) = 0\right)$$

$$|D_p(z) - D(z)| < \frac{\eta_{\mathcal{D}}(p)}{\sqrt{|y|}}\,.$$

Thus, according to (V.4) and (V.5):

$$|G_p(z) - G(z)| \leqslant |D_p(z) - D(z)| + |\tilde{\varrho}(z)\, z|\, |N_p(z) - N(z)| < \frac{\nu_{\mathcal{D}}(p)}{\sqrt{|y|}}\,. \tag{A.14}$$

Next, a contour C enclosing $\mathcal{D}$ can be drawn such that:

i) it contains the segment '$2\,d$' (Fig. 3),

ii) its distance from the cut $[-\infty, -a]$ is positive,

iii) its distance δ from $\mathcal{D}$ is positive.

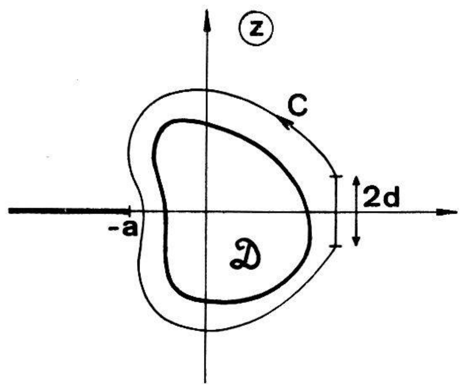

Figure 3

Proof of Lemma V.1.

Then:

$$G_p(z) - G(z) = \frac{1}{2\,i\,\pi} \left[\int_{C-2d} dz'\, \frac{G_p(z') - G(z')}{z' - z} + \int_{-d}^{d} dy\, \frac{G_p\,(x+i\,y) - G\,(x+i\,y)}{x+i\,y-z} \right] \quad (z \in \mathcal{D})\,.$$

But:

$$|z' - z| \geqslant \delta \ \text{ for } z \in \mathcal{D},\ z' \in C\,,$$

$$|G_p(z') - G(z')| < \begin{cases} \dfrac{\nu_C(p)}{\sqrt{d}} \ \text{ for } z \in C - 2\,d\,, \\[2ex] \dfrac{\nu_C(p)}{\sqrt{|y|}} \ \text{ for } z \in 2\,d\,. \end{cases} \qquad \text{(from (A.14))}$$

As a result:

$$| G_p(z) - G(z) | < \frac{\nu_C(p)}{2\pi\delta} \left(\frac{L_C}{\sqrt{d}} + 4\sqrt{d} \right) \qquad \forall\ z \in \mathcal{D},$$

and $\lim\limits_{p\to\infty} |G_p(z) - G(z)| = 0$ uniformly over $\mathcal{D}$. q.e.d.

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