On the structure of quantal proposition systems

On the Structure of Quantal Proposition Systems [1])

by **J. M. Jauch** and **C. Piron**
Institut de Physique Théorique University of Geneva, Switzerland

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Abstract. It is shown that the axiom of atomicity and the covering law can be justified on the basis of a new and more satisfactory notion of state and the existence of ideal measurements of the first kind. These two axioms are thereby given a satisfactory justification in terms of empirical facts known about micro-systems. Furthermore the new notion of state introduced here does not involve any probability statements and there is therefore no difficulty attributing it to individual systems, which was not possible with the notion heretoforth used in quantum mechanics.

1. Introduction

One of the central problems in the foundation of quantum mechanics concerns the question to what extent the theory, as we know it today, is determined by the empirical facts that we observe in microsystems. Such a question does not have a precise answer, since it is clear that empirical facts alone do not determine a theory. Indeed the theory can only be constructed from the raw material of the facts by a process of induction which proceeds from a finite number of observations to an axiomatically formulated mathematical structure supplemented by the rules of interpretation. The best that one can hope to do then is to rule out certain of these structures on the basis of empirical evidence. One can never really *verify* a theory, one can only *falsify* it.

The axiomatic construction of the theory has the great advantage in that the theoretical structure and its rules of interpretation are introduced explicitly and the empirical foundation of the theory is thereby much easier to identify. If the theory is essentially determined by the axioms and if such a theory is falsified by a test of one of its consequences then (provided the mathematical deductions are free from error) one or several of the axioms must be modified.

Recent attempts [1] to reconstruct conventional quantum mechanics by such an axiomatic approach have shown, that quantum mechanics in Hilbert space can only be deduced if, in addition to empirically well supported axioms, certain additional axioms are introduced, which heretofore have not had a good support with empirical facts. We mean the two (related) axioms of *atomicity* and the *covering law*.

By a more careful analysis of the concept of the *state* of a physical system it has been possible to improve on this aspect and to give a better justification of these two

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[1]) This is a revised version of a widely distributed preprint by C. Piron, *Sur l'interprétation des treillis orthocomplémentés.*

axioms. At the same time it has become possible to clarify the notion of *state* and that of a *physical property*. The latter notion is closely related to that of *element of physical reality* introduced by EINSTEIN, ROSEN and PODOLSKY in the discussion of their paradox which bears their names [2].

It is significant that these three authors came to the conclusion that the notion of state as used in quantum mechanics cannot meaningfully be attributed to an individual system and that it is a statistical concept, applicable only to a suitably chosen assembly of systems. This criticism is justified for the usual definition of state in terms of a state vector (or Schrödinger function) in Hilbert space. However we shall show in this paper that a modified definition of state can be meaningfully applied to an individual system which represents all the properties (or elements of reality) provided the propositions of that system are an atomic lattice.

It is perhaps interesting to point out that this new notion of state, although fully quantum mechanical in its connotations, resembles the classical notion of that concept. In both areas of physics, classical and quantal, the state can only be determined by a statistical procedure, as in all physical measurements. Nevertheless in both areas it is possible and useful to define a notion of state which would correspond to an idealized set of measurements of infinite precision.

This possibility counters effectively some of the criticisms which have been formulated by several physicists and philosophers in various ways concerning the conventional notion of state and its implication for that of 'physical reality'.

Equally important for the reconstruction of conventional quantum mechanics is the axiom which we have called the *covering law*. It is shown that this axiom is closely related to the possibility of an *ideal measurement*, where 'ideal' will have to be properly defined. There is no doubt that such measurements are possible in many situations, or more precisely, that such measurements can be often simulated by actual (and therefore not ideal) measurements to any desired degree of accuracy. The covering law, which formerly had to be postulated ad hoc, obtains thereby a high degree of plausibility.

2. Yes-No Experiments

The properties of a physical system are determined by measurements. A certain class of measurements play a particularly important rôle in the establishment of the physical properties of a system. It is the experiment with only two possible results which may be denoted by 1 or 0 (yes or no). We denote such experiments by Greek letters α, β, γ, ... and shall refer to them as yes-no experiments.

If α is a yes-no experiment then there exists another one, denoted by α^ν, obtained from α by inverting the results yes and no. Thus if the result of α is 'yes' that of α^ν is 'no' and vice versa. It is clear that α^ν can be measured with the same physical equipment as that used for the measurement of α and that $(\alpha^\nu)^\nu = \alpha$.

If α_i ($i \in I$, same index set) is any family of yes-no experiments then one can define another such experiment, denoted by $\Pi\, \alpha_i$ by the following procedure: One chooses at random one of the α_i ($i \in I$) and measures it. The result is the value of $\Pi\, \alpha_i$. It follows that

$$(\Pi\, \alpha_i)^\nu = \Pi\, \alpha_i^\nu\ .$$

There exists a trivial and an absurd yes-no experiment denoted by I and ϕ, respectively. The first consists of the 'experiment' which verifies that the system exists and the second is $\phi = I^{\nu}$.

3. Properties of a System

We shall say that the yes-no experiment α is 'true' if a measurement of α will give the result yes with certainty. For the time being we are not concerned with the question how we can produce systems for which a given yes-no experiment is known to be 'true' nor how we obtain this knowledge.

It is an empirical fact that certain pairs α, β of yes-no experiments have the property

$$\alpha \text{ true} \Rightarrow \beta \text{ true}.$$

If this is the case, we write $\alpha < \beta$. This relation is a partial preorder relation, that is it satisfies the properties

$$(1) \quad \alpha < \alpha\,, \qquad (2) \quad a < \beta \text{ and } \beta < \gamma \Rightarrow \alpha < \gamma\,.$$

If two yes-no experiments α_1 and α_2 satisfy the relation $\alpha_1 < \alpha_2$ and $\alpha_2 < \alpha_1$ we shall call them *equivalent* and we denote it by $\alpha_1 \sim \alpha_2$. This relation is an equivalence relation, that is, it satisfies

$$(1) \quad \alpha \sim \alpha\,, \qquad (2) \quad \alpha \sim \beta \Rightarrow \beta \sim \alpha\,, \qquad (3) \quad \alpha \sim \beta \text{ and } \beta \sim \gamma \Rightarrow \alpha \sim \gamma\,.$$

Let α be any yes-no experiment. We denote by $a \equiv \{\alpha\}$ the class of all such experiments which are equivalent to α and we call it a *proposition*. Thus

$$a = \{\alpha_i \mid \alpha_i \sim \alpha\} = \{\alpha\}\,.$$

If α is true, then every $\alpha_i \sim \alpha$ is true too. Hence we see the proposition a is true if and only if any (and therefore all) of the $\alpha \in a$ are true. If the proposition a is true we shall call it a *property* of the system. We write $a \subset b$ if $\alpha \in a$, $\beta \in b$ and $\alpha < \beta$.

If α_i is a family of yes-no experiments all of which are true then $\Pi\,\alpha_i$ is true too. We denote by $\bigcap_i a_i$ the equivalence class $\{\Pi\,\alpha_i\}$ which contains the yes-no experiment $\Pi\,\alpha_i$. It follows from the definition that $\bigcap_i a_i$ depends only on the equivalence classes $\{\alpha_i\}$ and not on the representatives of these classes. Hence the notation is justified. Thus if a_i are properties of a system then $\cap\, a_i$ is a property too.

If $b \subset \cap\, a_i$ then it follows from the definition that this is equivalent to $b \subset a_i$:

$$b \subset \cap\, a_i \Leftrightarrow b \subset a_i \quad \forall\, i \in I\,.$$

Thus $\cap\, a_i$ is the *greatest lower bound* of the propositions a_i.

Similarly we can define the *least upper bound* by setting $\cup\, a_i \equiv \bigcap_{a_i \subset x} x$ and verify that it satisfies

$$\cup\, a_i \subset b \Leftrightarrow a_i \subset b \quad \forall\, i \in I\,.$$

If L denotes the set of all propositions we have $\phi = \bigcap_{x \in L} x$ and $I = \bigcup_{x \in L} x$.

We have thus proved the

Theorem: The set of all propositions is a complete lattice.

4. The Complements

Two propositions a and b are said to be *complements* of one another if they satisfy

$$a \cap b = \phi \text{ and } a \cup b = I .$$

For a lattice which satisfies the distributive laws

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) ,$$
$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c) .$$

the complement, if it exists, is unique.

The lattices which are encountered in quantal systems do not satisfy the distributive law and there exist usually many complements. Among these different complements we can still distinguish one, called the *compatible complement*, by the following:

Definition: The complement b is said to be a *compatible complement* of a if there exists a yes-no experiment $\alpha \in a$ such that $\alpha^\nu \in b$. We denote a compatible complement by a'.

All the known physical systems have the property that every proposition has a compatible complement. We therefore formulate the

Axiom C: For every proposition $a \in L$ there exists at least one compatible complement a'.

The lattices which satisfy the axiom C are still too general for quantal systems. The essential physically motivated axiom [1] which limits this generality is

Axiom P: If $a \subset b$ then the sublattice generated by (a, b, a', b') is Boolean.

It follows from this axiom that $a \subset b \Leftrightarrow a' \subset b'$ so that the mapping $a \mapsto a'$ is an orthocomplementation. Furthermore the lattice L is weakly modular, that is we have

$$a \subset b \Rightarrow a \cup (a' \cap b) = b .$$

It is now possible to introduce the fundamental notion of *compatibility* by the following

Definition: Two propositions $a, b \in L$ are said to be *compatible* ($a \leftrightarrow b$) if the sublattice generated by (a, b, a', b') is Boolean.

In classical systems any pair of propositions is compatible. The greater richness of quantal systems appears through the presence of proposition pairs which are not compatible.

5. The States

The classical notion of state is so familiar that it has influenced much of our thinking about quantal systems. A classical system is described by a number of real variables which define the *phase-space* of the system and a state is determined by a *point* in this space.

The propositions of a classical system can be identified with the subsets of the phase space with inclusion as the ordering relation. For any given state (identified with a point P in phase space) there exists then a class of propositions which are true in the sense defined before. They are in fact all subsets which contain the point P.

For quantal systems the phase space does not exist, but the property of states expressed in the last paragraph still persists and can be used as the defining property of states.

Guided by this analogy we are led to the

Definition: A state of a system is the set S of all true propositions of the system:

$$S = \{x \mid x \in L \text{ , } x \text{ true}\} .$$

The following remarks should clarify the meaning of this definition.

The definition is meant to imply that the state is a property of an individual system and not of a statistical ensemble of such systems. This was not possible in previous definitions of the state which involved probabilities (or probability amplitudes). Indeed, a probability is meaningful only with reference to a statistical ensemble. The definition we have given above refers only to true propositions, that is to what we have called properties of the system, and there is no objection in attributing these properties to an individual system.

We shall in fact assume that every individual system, be it an isolated system or a member of a statistical ensemble, is in a definite state as defined above.

It is important to distinguish the state of a system from the amount of information available about the system. This distinction is already important for classical systems and its appears again here for quantal systems. We attribute to every system a state in the sense defined above quite independently whether this state has been measured. We may think of the state as containing the maximal amount of information that is possible concerning an individual system. Thus we shall postulate that two states S_1 and S_2 cannot be subsets of one another.

The states defined here correspond to the so-called 'pure' states of quantum mechanics. In the view that we adopt here every individual system is in a pure state. Mixtures are only properties of statistical ensembles.

The following properties are elementary consequences of the definitions given earlier.

(1) If $x \in S$ and $x \subset y$ then $y \in S$.

(2) If $x, y \in S$ then $x \cap y \in S$.

(2′) If $x_i \in S$ $(i \in I)$ then $\bigcap_{i \in I} x_i \in S$.

(3) $\phi \notin S$, $I \in S$ for every state S .

(4) For any $x \in L$, $x \neq \phi$ there exists at least one state S such that $x \in S$.

The meaning of the least property is that a proposition x is different from ϕ if there exists at least one procedure which gives to the system the property x.

From the above it follows that for every state S, $e \equiv \bigcap_{x \in S} x$ is also contained in S and that it is an atom. Indeed if $y \subset e$ and $y \neq \phi$ then there exists a state S_0 such that $y \in S_0$. It follows then that $S \subset S_0$ so that S could not be a state. This contradicts the hypothesis. We have thus proved the

Theorem: For every state S, $e = \bigcap_{x \in S} x$ is an atom and $e \in S$.

From this we obtain the

Corollary: Every $a \neq \phi$ contains at least one atom e. In order to verify this it suffices to consider a state S such that $a \in S$. The proposition $e = \bigcap_{x \in S} x$ is then an atom and $e \subset a$.

A lattice with this property is said to be atomic. Thus we have motivated the

Axiom A_1: The lattice of propositions is atomic.

The preceding considerations show that every state may be represented by an atom e. The set of all the atoms is identical with the set of all the states. The state S associated with the atom e is the set

$$S = \{x \mid e \subset x\}.$$

In the analogy to the classical systems and the phase space, the atoms of L may be considered as the 'phase space' of the quantal system.

It is seen that the analogy to the phase space suggested here brings this new definition of states of quantal system much closer to the classical notion of states. In fact one of the essentially non-classical aspects of the states of quantal systems appears now if we consider the evolution of states in time. Classically the evolution of states is given by a transformation of phase space which maps every point of that space into another one.

This type of evolution may also occur in quantum mechanics and it is that evolution which is described in the Hilbert space formalism by a Schrödinger equation. We shall call it Schrödinger-type evolution of states. In the lattice-theoretic formulation a Schrödinger-type evolution is generated by a continuous automorphism of the lattice.

However in quantum mechanics one encounters other types of evolutions which play an equally important rôle. They are in fact at the root of most of the paradoxes in quantum mechanics. A state of a quantal system may also evolve according to a stochastic process. As we know from the examples studied in connection with the measuring process this always occurs if such a system is part of another quantal system with which it interacts. The unavoidable occurrence of probabilities in quantum mechanics is entirely due to this stochastic evolution of systems in interaction.

6. Ideal Measurements

A measurement a is said to be *ideal* if every true proposition compatible with a is also true after the measurement.

A measurement of a is called of the *first kind* if the answer yes implies a true immediately after the measurement.

We shall suppose that for every proposition a there shall exist ideal measurements of the first kind.

Consider now a system in the state S defined by the atom e and let a be any proposition. We consider an ideal measurement of the first kind of a and ask what the state is going to be after such a measurement.

We consider the proposition $y = e \cup a'$. Since $e \subset y$ we have $y \in S$. Furthermore $y \leftrightarrow a$. Since $y \leftrightarrow a'$ (Axiom P) it follows $y \leftrightarrow a$ by the definition of compatibility.

The set S_a of propositions which are true immediately after such ideal measurement, in the case of answer yes, is therefore the set that is implied both by $y = e \cup a'$ and by a. Thus we are led to the conclusion that

$$S_a = \{x \mid (e \cup a') \cap a \subset x\} .$$

Now two possibilities may a priori arise. The first one is $(e \cup a') \cap a = \phi$. But this contradicts our hypotheses because this implies $e \subset a'$, which means that the proposition a' is true and then that the answer yes, as result of the measurement a, is impossible. The second possibility is $(e \cup a') \cap a \neq \phi$. The state after the measurement of a is then this set S_a which is maximal if and only if

$$e_a = (e \cup a') \cap a$$

is an atom. Thus we are led to the following conclusion:

For every $a \in L$ and every atom $e \in L$ the proposition $(e \cup a') \cap a$ is either ϕ or an atom.

It is now easy to show that this result is equivalent with the covering law. Consider an element $b \in L$ and an atom $e \in L$. Let x be such that

$$b \subset x \subset e \cup b .$$

It follows from this that

$$\phi \subset x \cap b' \subset (e \cup b) \cap b' .$$

Since x is compatible with b, as well as b', $\phi = x \cap b'$ implies $x \subset b$ thus $x = b$. Hence if $x \neq b$ then $x \cap b' \neq \phi$. Since $(e \cup b) \cap b'$ is an atom we have then $x \cap b' = (e \cup b) \cap b'$ from which follows $e \cup b = ((e \cup b) \cap b') \cup b = (x \cap b') \cup b = x$.

Thus we have established

Axiom A_2 (covering law):

For every proposition $b \in L$ and any atom $e \in L$, $b \subset x \subset e \cup b$ implies either $x = b$ or $x = e \cup b$.

References

[1] J. M. Jauch, *Foundations of Quantum Mechanics* (Addison-Wesley 1968), chap. 5, where further references are listed.

[2] A. Einstein, P. Podolksky and N. Rosen, Phys. Rev. *47*, 777 (1935).