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# Models of Local Current Algebra and Symmetry Breaking

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*Abstract.* It is shown that a known model of local current algebra in the infinite momentum limit can be generalized to a model which includes symmetry breaking solutions. The model can further be extended to allow transitions between states of different total isospin value. The methods of generalization apply to higher algebras as well as to the $SU_2$-model treated here.

## 1. Introduction

The development of current algebra has stressed the importance of one particle solutions of current algebras at infinite momentum. Such solutions have been given by Leutwyler [1] and independently by Gell-Mann, Horn and Weyers [2] and by Kleinert [3]. Their models seem however not to satisfy in two aspects: first they fulfil the one particle saturation assumption in a very formal manner, their space of state vectors containing – in addition to a discrete set of physical one particle states – a continuum of states with spacelike momentum, and second they are submitted to stringend simplifications, exhibiting e.g. strong symmetry, which means that they fail to realize the original idea of current algebra i.e. 'group structure without symmetry'. The first point has been cleared by Leutwyler [4]. He has proved that the occurence of ghost-states is by no means fatal to the solutions: in the sense of an asymptotic expansion in a mass splitting parameter, the ghosts may be ignored. Concerning the second point, we are now going to show, that some of the restrictions imposed to the models are not necessary.

In section 2 we give a brief survey of Leutwyler's model, in section 3 we construct a symmetry breaking solution and in section 4 we discuss a model which allows the particles to make transitions between states of a different isospin value.

## 2. The Hydrogen-model

It has recently been shown [5] that Leutwyler's hydrogen model is equivalent to a model based on an infinite component field equation. We will now shortly recall some properties of the model in the field formulation. A detailed description of the field model is given in Ref. [5].

The Lagrangian of the model is

$$L(x) = \frac{1}{2}\, \partial^\nu \psi^+(x)\, \partial_\nu \psi(x) - i\, \mu\, \psi^+(x)\, \Gamma^\nu\, \partial_\nu \psi(x) + M_0^2\, \psi^+(x)\, \psi(x) \, . \tag{2.1}$$

$\psi(x)$ is a field with infinitely many components $\psi^n(x)$. The four hermitian matrices $\Gamma_n^{\nu m}$ act on the index $n$ of that field.

This Lagrangian gives rise to a field equation

$$\{\Box + 2\, i\, \mu\, \Gamma^\nu\, \partial_\nu + M_0^2\}\, \psi(x) = 0 \tag{2.2}$$

and to real conserved current:

$$j^\nu(x) = \psi^+(x)\, \{i\, \overleftrightarrow{\partial^\nu} - 2\, \mu\, \Gamma^\nu\}\, \psi(x)\, . \tag{2.3}$$

The $\psi$-space is determined – by requiring proper transformation properties for $\psi$, $j^\nu$ and $\Gamma^\nu$ under Lorentztransformations – to be one of the two Majorana representations of SL(2, C). We will only consider the half spin representation, where the $\psi$-space consists of an infinite spin-ladder

$$\psi = \psi_{j,m}\, ,\ j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \quad m = -j, \dots, +j\, . \tag{2.4}$$

A convenient set of solutions of (2.2) is given by plane waves

$$\psi_{j,m}(x) = e^{-i p x}\, \Phi(p, j, m) \tag{2.5}$$

with

$$\Phi(p, j, m) = U(\Lambda)\, \Phi(p_s, j, m)\, , \quad \Lambda\, p_s = p\, , \quad p_s = (M, 0, 0, 0)\, ,$$

where $U(\Lambda)$ stands for the Majorana representation of the homogeneous Lorentz-group. The expressions (2.5) are found to solve the differential equation (2.2) if the states $\Phi(p_s, j, m)$ are the states of the canonical basis [5, 6, 7],

$$\Phi(p_s, j, m) = |j, m)\, . \tag{2.6}$$

The physically acceptable solutions belong to the mass spectrum

$$M = \mu \left(j + \frac{1}{2}\right) + \left\{\mu^2 \left(j + \frac{1}{2}\right)^2 + M_0^2\right\}^{1/2} . \tag{2.7}$$

Yet, these solutions alone are not complete. They form a complete set together with ghost states belonging to negative $p^2$.

This model is formally generalized by impressing to the fields an internal degree of freedom (isospin), assuming however that one is always dealing with one set of $(2\, T + 1)$ identical fields (with fixed $T$), labeled by an isospin index $T_3$. The current operator (2.3) then takes the form

$$j_k^\nu(x) = \psi^+(x)\, T_k\, \{i\, \overleftrightarrow{\partial^\nu} - 2\, \mu\, \Gamma^\nu\}\, \psi(x) \tag{2.8}$$

where $T_k$ $(k = 1, 2, 3)$ are isospinmatrices acting on the index $T_3$ of the fields. If the $\psi$ are now formally quantized according to

$$[\psi_{n,T_3}(x), \Pi_{n',T_3'}(y)]_{(x_0 = y_0)} = [\psi^+_{n,T_3}(x), \Pi^+_{n',T_3'}(y)]_{(x_0 = y_0)} = i\, \delta^3\, (\boldsymbol{x} - \boldsymbol{y})\, \delta_{T_3 T_3'}\, \delta_{n\, n'} \tag{2.9}$$

with

$$\Pi_{T_3}(x) = \psi^+_{T_3}(x)\, \{\overleftarrow{\partial_0} - i\, \mu\, \Gamma_0\} \tag{2.10}$$

the currents $j_k^0(x)$ of (2.8) satisfy an isovector current algebra

$$[j_i^0(x), j_k^0(y)]_{(x_0=y_0)} = i\, \varepsilon_{ikl}\, j_l^0(x)\, \delta^3\,(\boldsymbol{x}-\boldsymbol{y})\ . \tag{2.11}$$

One-particle matrix elements of these currents are calculated classically, using the plane wave solutions (2.5), (2.6), (2.7). They are shown to saturate the algebra (2.11) in the infinite momentum limit in the above mentioned sense of an asymptotic expansion in $\mu$.

## 3. Symmetry Breaking

We will now generalize the model, as to allow symmetry breaking. For the sake of simplicity we still consider an $SU_2$-algebra of the type (2.11). Our method would however also apply to an $SU_3$-algebra or, in a model containing axial currents, even to an $SU_3 \times SU_3$-algebra [2, 8].

*a) Definition of the Model*

We will still deal with particles of just one fixed isospin value $T$. Yet, abandoning $\boldsymbol{T}$-conservation – only $T_3$ will further be conserved – we will avoid mass degeneracy with respect to $T_3$. We will keep to the simplest case, treating a system with $T = 1/2$. Its lowest spin values ($j = 1/2$) can be imagined to represent a $P$- and an $N$-state. In analogy to (2.1) we write the Lagrangian as

$$\begin{aligned} L(x) &= L_P(x) + L_N(x) \\ &= \frac{1}{2}\, \partial^\nu\, \psi_P^+(x)\, \partial_\nu\, \psi_P(x) - i\, \mu_P\, \psi_P^+(x)\, \Gamma^\nu\, \partial_\nu\, \psi_P(x) + M_{0P}^2\, \psi_P^+(x)\, \psi_P(x) \\ &+ \frac{1}{2}\, \partial^\nu\, \psi_N^+(x)\, \partial_\nu\, \psi_N(x) - i\, \mu_N\, \psi_N^+(x)\, \Gamma^\nu\, \partial_\nu\, \psi_N(x) + M_{0N}^2\, \psi_N^+(x)\, \psi_N(x)\ . \end{aligned} \tag{3.1}$$

Corresponding to this Lagrangian we have now two field equations of the type (2.2), one for $\psi_P$ and one for $\psi_N$, $P$ and $N$ standing for $T_3 = \pm 1/2$. The Lagrangian and the field equations can be written in a simple way with the help of two diagonal $2 \times 2$ matrices

$$\hat{\mu} = \begin{pmatrix} \mu_P & 0 \\ 0 & \mu_N \end{pmatrix} \text{ and } \hat{M}_0^2 = \begin{pmatrix} M_{0P}^2 & 0 \\ 0 & M_{0N}^2 \end{pmatrix}$$

and with a spinor field $\psi(x) = \begin{pmatrix} \psi_P(x) \\ \psi_N(x) \end{pmatrix}$. They then read

$$L(x) = \frac{1}{2}\, \partial^\nu\, \psi^+(x)\, \mathbf{1}\, \partial_\nu\, \psi(x) - i\, \psi^+(x)\, \hat{\mu}\, \Gamma^\nu\, \partial_\nu\, \psi(x) + \psi^+(x)\, \hat{M}_0^2\, \psi(x) \tag{3.2}$$

and

$$\{\Box\, \mathbf{1} + 2\, i\, \Gamma^\nu\, \partial_\nu\, \hat{\mu} + \hat{M}_0^2\}\, \psi(x) = 0\ . \tag{3.3}$$

If we choose $\mu_P \neq \mu_N$ and $M_{0P}^2 \neq M_{0N}^2$ we are now really handling two different fields $\psi_{T_3}$. One can again look for a solution of the field equations in terms of plane waves:

$$\psi_{T_3}(x; p, j, m) = e^{-ipx}\, \Phi_{T_3}(p, j, m) \tag{3.4}$$

and one finds that the two components $\psi_{T_3}$ belong to different mass spectra:

$$M_P = \mu_P \left(j + \frac{1}{2}\right) + \left\{\mu_P^2 \left(j + \frac{1}{2}\right)^2 + M_{0P}^2\right\}^{1/2}$$

$$M_N = \mu_N \left(j + \frac{1}{2}\right) + \left\{\mu_N^2 \left(j + \frac{1}{2}\right)^2 + M_{0N}^2\right\}^{1/2} . \quad (3.5)$$

*b) Currents and Current Algebra*

The model admits of a conserved isospin current

$$j_3^\nu(x) = \psi^+(x) \left(\frac{1}{2} [T_3, i \overleftrightarrow{\partial}^\nu \mathbf{1} - 2 \hat{\mu} \Gamma^\nu]_+\right) \psi(x) . \quad (3.6)$$

We assign to it two more components, putting

$$j_k^\nu(x) = \psi^+(x) \left(\frac{1}{2} [T_k, i \overleftrightarrow{\partial}^\nu \mathbf{1} - 2 \hat{\mu} \Gamma^\nu]_+\right) \psi(x) . \quad (3.7)$$

If we now carry on a formal quantization of the fields, according to (2.9) – (2.10) being spezified as $\Pi_{T_3}(x) = \psi_{T_3'}^+(x) \{\overleftarrow{\partial}_0 \delta_{T_3' T_3} - i \hat{\mu}_{T_3' T_3} \Gamma_0\}$ – we can easily verify that the currents $j_k^0(x)$ satisfy the current algebra (2.11). We emphasize that only the third component of the currents (3.7) is conserved and classically attached to the model. The other two components have been defined in such a way that the currents $j_k^0$ ($k = 1, 2, 3$) form a current algebra and that all the currents of (3.7) turn to the conserved currents (2.8) in the symmetry limit, i.e. for $\mu_P = \mu_N$ and $M_{0P}^2 = M_{0N}^2$.

One should now examine, if the one particle matrix elements of these currents saturate the current algebra at infinite momentum. The situation is that of Leutwyler's model: there is occurence of ghost states, there are transitions between physical states and ghost states, but for small $j$ and for small $\mu_{T_3}$ the one particle model is a good approximation of current algebra at infinite momentum. This justifies to interpret the currents of the model as the physical currents of weak and electromagnetic interaction. The matrix elements of these currents may thus be of some interest.

*c) Matrix Elements and Form Factors*

We can compute matrix elements, e.g. between a $P$- and an $N$-state and compare them with the expression

$$\langle P(p') \,|\, j_k^\nu(0) \,|N(p)\rangle = (2\pi)^{-3} \left(\frac{M_P M_N}{\omega_P \omega_N}\right)^{1/2} T_{kN}^P$$
$$\times \bar{u}_P(p') \{F_1^{PN}(q^2) \gamma^\nu + F_2^{PN}(q^2) \sigma^{\nu\mu} q_\mu + F_3^{PN}(q^2) q^\nu\} u_N(p) \quad (3.8)$$
$$(q = p - p')$$

in order to determine the form factors $F_i^{PN}$.

The one-particle matrix elements are again calculated without the formalism of second quantization, using plane waves. With (3.7), (3.4), (2.5) and (2.6) we get

$$\langle p', j', m', T_3' |\, j_k^\nu(x) \,| p, j, m, T_3\rangle = T_{kT_3}^{T_3'} N^{j' T_3'} N^{j T_3} e^{i(p'-p)x}$$
$$\times (j', m' \,|\, U^+ (L(p')) \{p^\nu + p^{\nu'} - (\mu_{T_3'} + \mu_{T_3}) \Gamma^\nu\} U(L(p)) \,|\, j, m) . \quad (3.9)$$

$N^{jT_3}$ and $N^{j'T'_3}$ are normalization constants. If we normalize the state vectors in such a way that the inner product is given by

$$\langle p', j', m', T'_3 | p, j, m, T_3 \rangle = \delta^3 (\mathbf{p} - \mathbf{p}') \, \delta_{jj'} \, \delta_{mm'} \, \delta_{T_3 T'_3} ,$$

we find for these constants the following expressions:

$$N^{jT_3} = (2\pi)^{-3/2} (2 p_0)^{-1/2} M_{T_3}^{1/2} \left\{ \mu_{T_3}^2 \left( j + \frac{1}{2} \right)^2 + M_{0T_3}^2 \right\}^{-1/4}$$

(cf. Ref. [15], equation (11.8)).

The evaluation of the SL(2, C)-matrix elements

$$(j', m' \,|\, U^+(L(p')) \{ p^\nu + p^{\nu'} - (\mu_{T'_3} + \mu_{T_3}) \Gamma^\nu \} U(L(p)) \,|\, j, m)$$

can be performed using methods given by Leutwyler and Gorgé [9]. Since the form-factors $F_i^{PN}$ depend only on the momentum transfer $q^2 = (p - p')^2$, the coordinate system can conveniently be chosen, to give the operators $U^+(L(p'))$, $U(L(p))$ a simple form.

We just quote the results one gets by comparing (3.8) with (3.9):

$$F_1^{PN}(q^2) = \frac{(4 M_N M_P)^{3/2}}{(M_N + M_P)^3} \left( 1 - \frac{q^2}{(M_N + M_P)^2} \right)^{-3/2} [(M_N - \mu_N)(M_P - \mu_P)]^{-1/2}$$
$$\times \left\{ \frac{M_N + M_P}{2} + \frac{\mu_N + \mu_P}{2} - \frac{3}{4} (\mu_N + \mu_P) \left( 1 - \frac{q^2}{(M_N + M_P)^2} \right)^{-1} \right\}$$

$$F_2^{PN}(q^2) = - \frac{(4 M_N M_P)^{3/2}}{(M_N + M_P)^4} \left( 1 - \frac{q^2}{(M_N + M_P)^2} \right)^{-3/2} [(M_N - \mu_N)(M_P - \mu_P)]^{-1/2}$$
$$\times \left\{ \frac{M_N + M_P}{2} - \frac{3}{4} (\mu_N + \mu_P) \left( 1 - \frac{q^2}{(M_N + M_P)^2} \right)^{-1} \right\}$$

$$F_3^{PN}(q^2) = \frac{(4 M_N M_P)^{3/2}}{(M_N + M_P)^5} (M_N - M_P) \left( 1 - \frac{q^2}{(M_N + M_P)^2} \right)^{-5/2}$$
$$\times [(M_N - \mu_N)(M_P - \mu_P)]^{-1/2} \frac{3}{4} (\mu_N + \mu_P) .$$

For the symmetry case ($M_N = M_P$, $\mu_N = \mu_P$), $F_1^{PN}$ and $F_2^{PN}$ coincide with the expressions given earlier [4], while $F_3^{PN}$ vanishes. $F_1^{PN}$ and $F_2^{PN}$ give a qualitative description of the experimental situation. It is agreeable that the mass difference enters linearly into the expression for $F_3^{PN}$. The contribution of $F_3^{PN}$-terms to scattering is however too small compared to electric $F_1^{PN}$- and magnetic $F_2^{PN}$-contributions to allow experimental statements on the shape of $F_3^{PN}$.

## 4. A Model without Isospin Factorization

Up to here our models have only allowed transitions between one-particle states of the same isospin value $T$. The isospin dependence of the matrix elements could therefore simply be factored out:

$$\langle p', N', T'_3 | j_k | p, N, T_3 \rangle = T_{kT_3}^{T'_3} \langle p', N' | j | p, N \rangle$$

(cf. (3.9)).

However, we know that processes like the electroproduction of $N^*$ $(P + e \to N^{3/2, 3/2+} + e)$ exist. And they are easily explained: the electromagnetic current is supposed to be the sum of an isoscalar and the third component of an isovector, admitting thus both $\Delta T = 0$- and $|\Delta T| = 1$-transitions. If the isovector currents of our current algebra model claim to be realistic, they must have non vanishing $|\Delta T| = 1$-matrix elements. To introduce them, e.g. for the special case $T = 1/2 \leftrightarrow T = 3/2$, we start with a field $\psi_\alpha$ of six isovector components. They may be labelled according to the scheme

| $\alpha$ | $T$ | $T_3$ |
|---|---|---|
| 1 | 1/2 | 1/2 |
| 2 | 1/2 | −1/2 |
| 3 | 3/2 | 3/2 |
| 4 | 3/2 | 1/2 |
| 5 | 3/2 | −1/2 |
| 6 | 3/2 | −3/2 |

(4.1)

Each isospin component $\psi_\alpha$ is itself an infinite-dimensional Majorana field. This means that our model is characterized by a Lagrangian of the type (3.2) and by field equations of the type (3.3). $\hat{\mu}$ and $\hat{M}_0^2$ now being $6 \times 6$-matrices. The three isospin currents of the model read

$$j_k^\nu(x) = \psi^+(x) \left( \frac{1}{2} [T_k, i \overleftrightarrow{\partial}^\nu \mathbf{1} - 2 \hat{\mu} \Gamma^\nu]_+ \right) \psi(x) \tag{4.2}$$

where $T_k$ are $6 \times 6$-matrices obeying

$$[T_i, T_k] = i \varepsilon_{ikl} T_l .$$

(4.1) immediately gives the form of $T_3$:

$$T_3 = \frac{1}{2} \begin{pmatrix} 1 & & & & & \\ & -1 & & & 0 & \\ & & 3 & & & \\ & & & 1 & & \\ & 0 & & & -1 & \\ & & & & & -3 \end{pmatrix} . \tag{4.3}$$

Only the third isospin component of the current (4.2) is generally conserved. We know that our model can formally be quantized and that it satisfies the current algebra (2.11).

We will now add to our currents (4.2) a term which allows $|\Delta T| = 1$-transitions. This term must be such that the current algebra is still valid and that the conserved currents are still conserved. The last point is automatically granted with an ansatz

$$J_k^\nu = j_k^\nu + k_k^\nu , \qquad k_i^\nu(x) = \partial_\mu \{\psi^+(x)\, S_i^{\nu\mu}\, \psi(x)\} , \tag{4.4}$$

$S_i^{\nu\mu}$ $(i = 1, 2, 3)$ acting in isospin space as well as in SL(2, C) space. According to the skew-symmetric character of $S_i^{\nu\mu}$, $k_i^\nu$ is conserved:

$$\partial_\nu k_i^\nu = 0 .$$

Evaluating commutators one then finds that the current $J_k^\nu$ of (4.4) satisfies a current algebra

$$[J_i^0(\boldsymbol{x}, t), J_k^0(\boldsymbol{y}, t)] = i\, \varepsilon_{ikl}\, J_l^0(\boldsymbol{x}, t)\, \delta^3\, (\boldsymbol{x} - \boldsymbol{y})$$

if the tensors $S_{\underset{i}{\sim}}^{\nu\mu}$ obey

$$[T_i, S_{\underset{k}{\sim}}^{0l}] = i\, \varepsilon_{ikn}\, S_{\underset{n}{\sim}}^{0l}\,. \tag{4.5}$$

$S_{\underset{i}{\sim}}^{\mu\nu}$ must thus behave under rotations in isospace like a vector. Such quantities can be constructed. They are Clebsch-Gordan coefficients.

To look for an explicit form of $S_{\underset{i}{\sim}}^{\nu\mu}$ one may write it as

$$S_{\underset{i}{\sim}}^{\mu\nu} = \begin{pmatrix} 0 & t_i \\ t_i^+ & 0 \end{pmatrix} M^{\mu\nu} \tag{4.6}$$

the first matrix on the right hand side acting in isospace, the second in SL(2, C)-space. A possible choice for $M^{\mu\nu}$ are the generators of the Lorentzgroup in the Majorana representation. $t_i$ can be determined in such a way that they satisfy (4.5). The current (4.4) then allows transitions between $T = 1/2$ and $T = 3/2$ states!

For the special case of the electroproduction of $N^*$ the current of interest is $J_3^\nu$. Equation (4.5) then simply reads

$$[T_3, S_{\underset{3}{\sim}}^{0l}] = 0\,.$$

Comparing this with (4.3) one immediately realizes that the isospin part of $S_{\underset{3}{\sim}}^{\mu\nu}$ in (4.6) has non vanishing (1,4)-, (4,1)-, (2,5)- and (5,2)-elements which perform the desired $T = 3/2 \leftrightarrow T = 1/2$-transitions.

We still want to notice that the $|\Delta T| = 1$-transitions described here occur also in the case that the model exhibits $\boldsymbol{T}$-conservation.

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