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Space-time Symmetry of Transverse Electromagnetic Plane Waves

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Synopsis. A way of determining the relativistic symmetry group of an electromagnetic field tensor that admits a Fourier expansion is summarized in a set of rules where the concepts of spectrum and of spectral group are introduced.

This approach is applied to the case of linearly, circularly and elliptically polarized transverse electromagnetic (TEM) waves. The group of the symmetry translations (called primitive translations), the point group and a set of associated non-primitive translations are explicitly given in each of the above three cases. From these groups one easily derives the symmetry group in space and time of a TEM wave, which is a non-symmorphic subgroup of the Poincaré group, i.e. a non-split extension of the group of primitive translations by the point group. The limit of infinite wave length is discussed and the results are shown to be consistent with previous ones relative to the symmetry of uniform electromagnetic fields.

1. Introduction

In the frame of a general programme in which physical phenomena are considered from the point of view of their symmetry in space and time, it is evident that electromagnetism is worthy of special attention. It is not at all surprising that electromagnetic fields in empty space have relativistic symmetries, as has already been shown in the case of uniform fields [1].

This paper represents the next step: the investigation of the symmetry group of a transverse electromagnetic plane wave (TEM). The relativistic symmetry group of a uniform field is the semi-direct product of the group T of all space-time translations by the point group of the field (the homogeneous symmetry group), the latter being considered as subgroup of the group of automorphisms of the abelian group T. This is no longer so for a TEM wave, whose symmetry group is not symmorphic.

Clearly, only a proper subgroup of T leaves the field invariant. The elements of this subgroup are called primitive translations, a terminology derived from that of crystallographic space groups. Note however that there are continuous primitive translations also.

Other translations than the primitive ones occur in the symmetry group G, but only together with non-trivial homogeneous transformations (here Lorentz transformations) and are called, therefore, non-primitive translations. The non-symmorphic character of G appears in the fact that these non-primitive translations cannot be transformed away (as a whole) by any change in the coordinate system [2]. This means that the symmetry group of a TEM wave is not simply the semi-direct product of a group of space-time translations by a group of Lorentz transformations.

2. Symmetry Conditions and General Definitions

In the Minkowski space, an orthonormal basis e_α ($\alpha = 0, 1, 2, 3$), is chosen, with metric tensor $g_{\alpha\beta}$, where $-g_{00} = g_{11} = g_{22} = g_{33} = 1$ and $g_{\alpha\beta} = 0$ for $\alpha \neq \beta$.

The Poincaré group IO(3,1) is the semi-direct product of the group T of all translations in space and time and the Lorentz group O(3,1) (considered as subgroup of the group of automorphisms of the abelian group T). The elements of IO(3,1) can thus be written as (t, L) where the translational part t is an element of T, the homogeneous part L is an element of O(3,1) and the multiplication law is given by:

$$(t_2, L_2)\,(t_1, L_1) = (t_2 + L_2\, t_1, L_2\, L_1)\,. \tag{2.1}$$

Under the action of an element $g = (t, L)$ of IO(3,1) an electromagnetic field tensor $F^{\alpha\beta}(x)$ transforms into another $\bar{F}^{\alpha\beta}(x)$ according to [3]:

$$\bar{F}^{\alpha\beta}(x) = (g_{op}\, F^{\alpha\beta})\,(x) = g[F^{\alpha\beta}(g^{-1}\, x)] \tag{2.2}$$

where, as already discussed in a previous paper [1]:

$$g[F^{\alpha\beta}(x)] = \dot{L}^{\alpha}_{\mu}\, L^{\beta}_{\nu}\, F^{\mu\nu}(x) \tag{2.3}$$

with:

$$\dot{L}^{\alpha}_{\mu} = \begin{cases} L^{\alpha}_{\mu} \text{ if } L^{0}_{0} > 0\,, \\ -L^{\alpha}_{\mu} \text{ if } L^{0}_{0} < 0\,, \end{cases} \tag{2.4}$$

and

$$g[x] = (t, L)\, x = L\, x + t\,. \tag{2.5}$$

The condition for g to be a symmetry, i.e. to leave the field invariant, is naturally: $\bar{F}^{\alpha\beta}(x) = F^{\alpha\beta}(x)$ and can be expressed by the relation:

$$F^{\mu\nu}\,(L\, x + t) = \dot{L}^{\mu}_{\alpha}\, L^{\nu}_{\beta}\, F^{\alpha\beta}(x)\,. \tag{2.6}$$

The relativistic symmetry group G of the electromagnetic field $F^{\alpha\beta}(x)$ is the largest subgroup of IO(3,1) which, according to (2.6), leaves the field tensor invariant.

The set of all homogeneous parts of the elements of G is a subgroup of O(3,1), called the point group K of the field in question:

$$K = \{L \mid \forall (t, L) \in G\} . \tag{2.7}$$

The subgroup U of G consisting of translations only ($U \stackrel{def}{=} T \cap G$) is normal in G. The proof uses the fact that T is normal in IO(3,1) [4]. For any $g \in G$:

$$g\, U\, g^{-1} = g\, (T \cap G)\, g^{-1} = g\, T\, g^{-1} \cap g\, G\, g^{-1} = T \cap G = U . \tag{2.8}$$

The elements of U are called primitive translations. The factor group G/U is isomorphic to the point group K. Note that for non-symmorphic G, the group G/U is not isomorphic to a subgroup of G. In fact the translational part t of an element of G is not, in general, a primitive translation. If $t \notin U$, then t can be written as:

$$t = a + u(L) , \qquad a \in U . \tag{2.9}$$

The translation $u(L)$ is called a non-primitive translation associated to L. The fundamental property of non-primitive translations is [2]:

$$u(L_1\, L_2) \equiv u(L_1) + L_1\, u(L_2) \pmod V , \qquad L_1 \text{ and } L_2 \in K , \tag{2.10}$$

so that it is sufficient to derive the non-primitive translations associated to a set of generators of K.

Let us from now on restrict our attention to fields that have a Fourier expansion:

$$F^{\alpha\beta}(x) = \sum_{k \in S} \hat{F}^{\alpha\beta}(k)\, e^{ikx}, \tag{2.11}$$

and let us call spectrum of the field $F^{\alpha\beta}(x)$ the set S of all vectors k occurring in the expansion (i.e. such that the corresponding Fourier coefficient $\hat{F}^{\alpha\beta}(k)$ does not vanish). Two fields are equal, if and only if they have the same spectrum and the same Fourier coefficients. It follows that:

$$K\, S = S , \tag{2.12}$$

a short-hand notation for expressing that if $k \in S$ and $L \in K$, then $L\, k \in S$.

Furthermore:

$$k a \equiv 0 \pmod{2\pi} , \qquad \forall\, k \in S ,\ \forall\, a \in U . \tag{2.13}$$

This last property is also a characterization of all the translations that are primitive:

$$U = \{a \in T \mid k a \equiv 0 \pmod{2\pi} ,\ \forall\, k \in S\} . \tag{2.14}$$

Finally:

$$\dot{L}^{\mu}_{\alpha}\, L^{\nu}_{\beta}\, \hat{F}^{\alpha\beta}(k) = F^{\mu\nu}(L\, k)\, e^{i(Lk)u(L)}. \tag{2.15}$$

Here L is any element of K and $u(L)$ is a non-primitive translation associated to L. Using matrix notation, the relation (2.15) can also be written in the very convenient form:

$$\dot{L}\, \hat{F}(k) = \hat{F}(L\, k)\, L^{*}\, e^{i(Lk)u(L)}, \tag{2.15a}$$

where L^{*} is the adjoint matrix of L, i.e. its transposed inverse.

It is useful to introduce the spectral group S defined as the largest subgroup of O(3,1) leaving the spectrum $\mathcal{S}$ invariant:

$$S = \{L \in O(3,1) \mid L\,k \in \mathcal{S},\ \forall\ k \in \mathcal{S}\}\ . \tag{2.16}$$

In particular, when $\mathcal{S}$ reduces to a single element k, the group S is also the little group relative to IO(3,1), T and the representation $\Delta(k)$. (See e.g. Ref. [5], pp. 230 and 328.)

A way of finding the relativistic symmetry group G of a given electromagnetic field can be formulated in the following set of rules:

(i) Determine the Fourier coefficients and the spectrum $\mathcal{S}$ of the field.
(ii) Using (2.14) find the group U of primitive translations.
(iii) Find the spectral group S by means of (2.16).
(iv) Find the point group K and a set of non- primitive translations $u(K)$ by looking for elements L of S that satisfy (2.15) for suitable chosen translations $u(L)$.

Note that, according to (2.10), if two elements of S satisfy (2.15), then so does their product. It is therefore convenient to find out first which generators of S belong to K and which not. One only needs to consider further those products which begin and end with generators of S not belonging to K.

(v) Finally, the group G is the set of all elements of IO(3,1) that are given by:

$$(a + u(L),\ L),\ \forall\ a \in U,\ \forall\ L \in K \text{ and } u(L) \in u(K)\ . \tag{2.17}$$

3. Spectral Group and Primitive Translations

The orthonormal basis considered above can be chosen in such a way that, in Gaussian units, a TEM wave with null vector h is given by:

$$\begin{aligned} E_3(x) &= A\cos(hx)\ , & E_1(x) &= B\sin(hx) \\ H_1(x) &= A\cos(hx)\ , & H_3(x) &= -B\sin(hx) \end{aligned} \tag{3.1}$$

where the contravariant components of h are:

$$h = \frac{2\pi}{\lambda}\,(1, 0, 1, 0)\ , \tag{3.2}$$

so that the wave propagates in the e_2-direction.

The corresponding field tensor is:

$$F^{\alpha\beta}(x) = A^{\alpha\beta}\cos(hx) + B^{\alpha\beta}\sin(hx) \tag{3.3}$$

where:

$$A^{\alpha\beta} = A\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \qquad B^{\alpha\beta} = B\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.4}$$

If $|A| \neq |B|$, the wave is elliptically polarized, and circularly polarized if $|A| = |B|$; right-hand polarized if A and B have the same sign, otherwise left-hand polarized. If B (or A) is zero, then the wave is linearly polarized.

The Fourier components of the field are:

$$\hat{F}^{\alpha\beta}(h) = \frac{1}{2}(A^{\alpha\beta} - i\,B^{\alpha\beta}) \text{ and } \hat{F}^{\alpha\beta}(-h) = \frac{1}{2}(A^{\alpha\beta} + i\,B^{\alpha\beta}), \tag{3.5}$$

so that the spectrum simply consists of two elements:

$$S = \{h, -h\}. \tag{3.6}$$

The group of primitive translations U^h does not depend on the wave polarization and follows immediately from (2.14) and (3.2):

$$U^h = \{a = (\varrho, \mu, \varrho + z\lambda, \nu) \mid \forall\, \varrho, \mu, \nu \in R \text{ and } \forall\, z \in Z\}, \tag{3.7}$$

so that

$$U^h \simeq R^3 \oplus Z.$$

The little group, $\mathcal{E}(2)$, of the vector h (future null) of IO(3,1) and T (see e.g. Ref. [5], p. 329) is a subgroup of the spectral group S of (3.6), and is generated by m_x, the mirror perpendicular to the x-axis (along e_1), by $R_y(\theta)$, any rotation of angle θ around the y-axis (along e_2) and by the Lorentz transformations $L(\sigma)$ and $\bar{L}(\varrho)$ (for any real σ and ϱ) given by:

$$L(\sigma) = \begin{pmatrix} 1+\frac{1}{2}\sigma^2 & \sigma & -\frac{1}{2}\sigma^2 & 0 \\ \sigma & 1 & -\sigma & 0 \\ \frac{1}{2}\sigma^2 & \sigma & 1-\frac{1}{2}\sigma^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \bar{L}(\varrho) = \begin{pmatrix} 1+\frac{1}{2}\varrho^2 & 0 & -\frac{1}{2}\varrho^2 & \varrho \\ 0 & 1 & 0 & 0 \\ \frac{1}{2}\varrho^2 & 0 & 1-\frac{1}{2}\varrho^2 & \varrho \\ \varrho & 0 & -\varrho & 1 \end{pmatrix} \tag{3.8}$$

Thus $\mathcal{E}(2)$, as well known, is isomorphic to the two-dimensional Euclidean group. (Note however that in Ref. [5] only the proper orthochronous inhomogeneous Lorentz group is considered, so that, there, the mirror m_x does not appear: but then of course the little group of a null vector is isomorphic not to the Euclidean group, but only to its connected component of the unity.) As $\mathcal{E}(2)$ is of index two in S, for generating the group S it is sufficient to add to the above set of generators of $\mathcal{E}(2)$ the total (space-time) inversion $\bar{1}'$, which transforms h into $-h$.

One then has:

$$S = \{\bar{1}', m_x, R_y(\theta), L(\sigma), \bar{L}(\varrho) \mid \forall\, \sigma, \varrho, \theta \in R\}. \tag{3.9}$$

4. Linearly Polarized Plane Wave

One obtains a linearly polarized plane wave by assuming in (3.3) that $B^{\alpha\beta} = 0$. Applying rule (iv) of Section 2, one verifies that the generators m_x, $L(\sigma)$ and $\bar{L}(\varrho)$ of S satisfy (2.15) for any real value of σ and ϱ, with vanishing associated non-primitive translations. These elements belong therefore to the generators of the point group K_l^h of a linearly polarized plane wave with wave-vector h. The total inversion $\bar{1}'$ is also an element of K_l^h but has an associated non-primitive translation that with the present choice of the origin has to satisfy the condition:

$$h\,u(\bar{1}') \equiv \pi \pmod{2\pi}\,. \tag{4.1}$$

A possible solution is $u(\bar{1}') = (0, 0, 1/2\,\lambda, 0)$, all other solutions differ from this one by primitive translations only, and represent equivalent choices.

Equation (4.1) thus takes into account that non-primitive translations are defined only modulo the primitive ones. According to the general theory [2] another source of arbitrariness is due to the possible changes of origin. Equations (2.11) and (2.15) show that a change of origin $x \to x + f$ induces a change $u(L) \to \bar{u}(L) = u(L) + f - L f$; the two systems, $u(K)$ and $\bar{u}(K)$, are however equivalent.

It is now sufficient to investigate the behaviour of the rotations around the y-axis. One obtains from (2.15) the condition $\sin\theta = 0$, which for $\theta = 0$ gives the identity and for $\theta = \pi$ another generator of K_l^h, namely 2_y, associated with a same non-primitive translation as the total inversion:

$$u(2_y) = \left(0, 0, \frac{1}{2}\,\lambda, 0\right). \tag{4.2}$$

It is convenient to consider instead of 2_y the generator $m_y' = \bar{1}'\,2_y$ (a mirror perpendicular to the y-axis, along e_2, followed by time inversion) because according to (2.10) and (4.2) the non-primitive translation associated to it is zero.

Conclusion: we have found that the point group K_l^h is generated by:

$$K_l^h = \{\bar{1}', m_x, m_y', L(\sigma), \bar{L}(\varrho) \mid \forall\, \sigma, \varrho \in R\} \tag{4.3}$$

and that the set:

$$u(m_x) = u(m_y') = u(L(\sigma)) = u(\bar{L}(\varrho)) = 0\,, \quad u(\bar{1}') = \left(0, 0, \frac{1}{2}\,\lambda, 0\right) \tag{4.4}$$

defines a system of non-primitive translations for the symmetry group G_l^h of a linearly polarized TEM wave with null vector h. One verifies that this system $u(K_l^h)$ is not equivalent to the trivial one. Therefore G_l^h is a non-symmorphic subgroup of the Poincaré group. This means that the point group K_l^h is not isomorphic to a subgroup of G_l^h, or, in other words, that G_l^h is a non split extension of U^h by K_l^h [2, 4].

There are elements of G_l^h which depend only on the direction of the null vector h, but not on the wave length λ. This is in particular the case for the elements of K_l^h, whose associated non-primitive translations are equivalent to zero and thus also are (homogeneous) elements of G_l^h.

In the limit of $\lambda \to \infty$, i.e. of $h \to 0$, $F^{\alpha\beta}(x)$ becomes the uniform field tensor $A^{\alpha\beta}$, whose symmetry group $G_\perp$ $(a^2 = 1)$ has already been determined [1]. We recall that $G_\perp$ $(a^2 = 1)$ is the semi-direct product of the group T of all translations with the point group $K_\perp$ $(a^2 = 1)$.

Comparison shows (see in particular (5.24) of Ref. [1]) that indeed all λ-independent symmetry elements of G_l^h also belong to $G_\perp$ $(a^2 = 1)$. In particular the only elements of K_l^h not belonging to $K_\perp$ $(a^2 = 1)$ are those whose associated non-primitive translations are not equivalent to zero. These latter become meaningless in the limit of an infinite wave length, and in fact the corresponding homogeneous parts are no more elements of $K_l^{h=0}$. One has:

$$U^{h=0} \subset U^h \subset T\,, \quad K_l^{h=0} = K_\perp\,(a^2 = 1) \subset K_l^h\,, \quad G_l^{h=0} \subset G_\perp\,(a^2 = 1)\,. \tag{4.5}$$

5. Circularly Polarized Plane Wave

It is now supposed that in (3.1) $A = B$. The TEM wave is then right-hand circularly polarized and the Fourier coefficients (3.5) become:

$$\hat{F}^{\alpha\beta}(h) = \frac{A}{2}\begin{pmatrix} 0 & -i & 0 & 1 \\ i & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \quad \hat{F}^{\alpha\beta}(-h) = \frac{A}{2}\begin{pmatrix} 0 & i & 0 & 1 \\ -i & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}. \tag{5.1}$$

Again rule (iv) of Section 2 is applied. One finds that $L(\sigma)$ and $\bar{L}(\varrho)$ satisfy (2.15) for any real value of σ and ϱ and zero non-primitive translations.

Furthermore $R_y(\theta)$ also belongs, for any value of θ, to the point group $K_\bigcirc^h$ of a circularly polarized plane wave with wave vector h. If the rotation angle is defined by:

$$R_y(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{5.2}$$

the associated non-primitive translation is

$$u(R_y(\theta)) = \left(0,\, 0,\, \frac{\theta\,\lambda}{2\,\pi},\, 0\right), \tag{5.3a}$$

whereas in the case of a left-hand circularly polarized wave ($A = -B$) one has:

$$u(R_y(\theta)) = \left(0,\, 0,\, -\frac{\theta\,\lambda}{2\,\pi},\, 0\right). \tag{5.3b}$$

The remaining generators of S, namely $\bar{1}'$, and m_x, do not satisfy (2.15). One has therefore to investigate the other elements of the group generated by $\bar{1}'$ and m_x, in fact only their product $2'_x$ (which is a rotation of angle π around the x-axis, followed by time inversion). This last element belongs to $K_\bigcirc^h$ and is associated to:

$$u(2'_x) = \left(0,\, 0,\, \frac{1}{2}\,\lambda,\, 0\right). \tag{5.4}$$

The result is that the point group $K_\bigcirc^h$ is generated by:

$$K_\bigcirc^h = \{2'_x,\, R_y(\theta),\, L(\sigma),\, \bar{L}(\varrho) \mid \forall\, \theta, \sigma, \varrho \in R\}\,. \tag{5.5}$$

A corresponding system of non-primitive translations is:

$$u(L(\sigma)) = u(\bar{L}(\varrho)) = 0\,,$$

$$u(2'_x) = \left(0,\, 0,\, \frac{1}{2}\,\lambda,\, 0\right), \quad u(R_y(\theta)) = \left(0,\, 0,\, \pm\,\frac{\theta\,\lambda}{2\,\pi},\, 0\right) \tag{5.6}$$

(the plus sign applies if the wave is right-hand polarized, the minus sign if it is left-hand polarized). In this case too the symmetry group $G_{\bigcirc}^h$ is non-symmorphic, i.e. a non-trivial extension of U^h by $K_{\bigcirc}^h$.

In the limit of an infinite wave length, one again gets a uniform field, and considerations like those made at the end of the previous Section lead to similar conclusions.

6. Elliptically Polarized Plane Wave

In this case, $|A| \neq |B|$. The Fourier components of the field are:

$$\hat{F}^{\alpha\beta}(h) = \frac{1}{2}\begin{pmatrix} 0 & -iB & 0 & A \\ iB & 0 & iB & 0 \\ 0 & -iB & 0 & A \\ -A & 0 & -A & 0 \end{pmatrix}, \quad \hat{F}^{\alpha\beta}(-h) = \frac{1}{2}\begin{pmatrix} 0 & iB & 0 & A \\ -iB & 0 & -iB & 0 \\ 0 & iB & 0 & A \\ -A & 0 & -A & 0 \end{pmatrix} \tag{6.1}$$

Calculation of the generators of the point group K_O^h and of the corresponding system of non-primitive translations is straightforward, so that we simply give the result:

$$K_O^h = \{2'_x, 2_y, L(\sigma), \bar{L}(\varrho) \mid \forall\, \sigma, \varrho \in R\}\,, \tag{6.2}$$

$$u(L(\sigma)) = u(\bar{L}(\varrho)) = 0\,,$$

$$u(2'_x) = u(2_y) = \left(0, 0, \frac{1}{2}\lambda, 0\right). \tag{6.3}$$

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