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On the Asymptotic Condition of Scattering Theory

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Abstract. We propose a new formulation of the asymptotic condition of scattering theory which applies to Coulomb interactions and other long range potentials and which generalizes the usual asymptotic condition. It consists of the requirement that the constants of the free motion be asymptotically stationary also under the real evolution. From this and two supplementary conditions we prove the existence of wave operators and of a scattering operator. It is then shown that the wave operators are strong operator limits as in standard scattering theory except that the free evolution may have to be replaced by a modified propagator. Finally, Dollard's method of proving the asymptotic convergence for the Coulomb interaction is extended to a more general class of potentials.

I. Introduction

In the quantum-mechanical description of a simple scattering process one deals with two groups of unitary operators $V_t = \exp(-i\,H\,t)$ and $U_t = \exp(-i\,H_0\,t)$. They describe the total evolution and the free evolution of the scattering system in question, and the difference $V = H - H_0$ of their infinitesimal generators is the interaction which produces the scattering. The *asymptotic condition* imposes an essential restriction between these two groups, namely that *in the remote past and in the distant future the scattering states evolving with V_t become free in some sense.* The precise meaning of the notion of 'becoming free' depends on the mathematical formulation of this condition.

A widely used form of the asymptotic condition for potential scattering in Hilbert space is due to Jauch [1]. It requires that the total evolution of any scattering state converges strongly to the free evolution of some other state for $t \to -\infty$, and vice versa that the free evolution of any vector of the Hilbert space $\mathcal{H}$ converges strongly to the total evolution of some scattering state as $t \to -\infty$, and likewise for $t \to +\infty$. An equivalent statement is the following: The operators $\Omega(t) = V_t^* U_t$ converge strongly on the entire Hilbert space for $t \to \pm\infty$; their limits $\Omega_\pm$ (called wave

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operators) are isometries which satisfy $\Omega_+^* \Omega_+ = \Omega_-^* \Omega_- = I$ and $\Omega_+ \Omega_+^* = P_+$, $\Omega_- \Omega_-^* = P_-$. Here I denotes the identity operator, and one usually requires that the ranges $P_+ \mathcal{H}$ and $P_- \mathcal{H}$ of $\Omega_\pm$ respectively be the same and coincide also with the subspace corresponding to the absolutely continuous part of the spectrum of H. The wave operators are intertwining operators for the two groups V_t and U_t, i.e. $V_t \Omega_\pm = \Omega_\pm U_t$.

The asymptotic condition in Jauch's version excludes important cases from non-relativistic scattering theory, in particular the scattering by a Coulomb potential and by other long range potentials. Its recent reformulation in a weaker topology by Jauch, Misra and Gibson [2] extends the class of admissible interactions but does not remedy the situation for the long range potentials. On the other hand, Dollard [3] and Mulherin and Zinnes [4] investigated the asymptotic condition for the Coulomb interaction. They established that, although the strong limits of $V_t^* U_t$ for $t \to \pm\infty$ do not exist in this case, one may construct isometries whose properties are such that their physical interpretation is the same as that of the wave operators for a short range potential. They satisfy in particular the usual intertwining relation.

In the present paper, we propose a new formulation of the asymptotic condition which will ensure the existence of wave operators for a more general class of potentials than those treated in [1–4]. For this, we remark that the prepared initial states and the detected final states in many scattering experiments are characterized by the momenta, polarizations and other internal quantum numbers of the constituents of the scattering system. Accordingly, one will obtain an adequate description of the scattering process if these observables become stationary at large negative and large positive times. In consideration of this we shall express the asymptotic condition by *imposing convergence of the time evolution* $V_t^* A V_t$ *of certain observables* A rather than convergence of the time evolution of states.

Mathematically we shall work with the von Neumann algebra $\mathcal{A}_0$ generated by the set of observables which serve to specify the asymptotic free movement. This set of observables which become asymptotically stationary may not be the same for all types of interactions. However, to make sure that $\mathcal{A}_0$ can give a complete characterization of the initial and final states, one should require that it contains at least one complete set of commuting observables. In this article we consider the case where $\mathcal{A}_0$ consists of all bounded operators which commute with H_0, i.e. we impose that all constants of the free movement U_t be also asymptotically constants of the real movement V_t. From this and some additional conditions on $V_t^* \mathcal{A}_0 V_t$ we deduce the existence of wave operators and of an isometric S-operator which is determined up to multiplication by unitary operators from the center of $\mathcal{A}_0$. The wave operators are shown to satisfy a generalized intertwining relation.

Section II contains the precise mathematical statement of the foregoing assertions for simple scattering systems. In Section III we prove that the wave operators $\Omega_\pm$ are expressible as strong operator limits in much the same way as in standard scattering theory: The asymptotic condition in our form is satisfied if and only if there exist two families $\{T_t^\pm\}$ of operators which are function of H_0 and asymptotically isometric and such that the strong limits of $V_t^* T_t^\pm$ as $t \to \pm\infty$ respectively exist (for the details, cf. Theorem 2). The wave operators $\Omega_\pm$ are then identical with these strong limits.

It follows from this that, with our choice of $\mathcal{A}_0$, the theory includes the non-relativistic scattering by all short range potentials. For the Coulomb interaction, Dollard [3] constructed explicitly two such families $\{T_t^{\pm}\}$ and proved the strong convergence of $V_t^* T_t^{\pm}$, which shows that our form of the asymptotic condition is satisfied also for Coulomb potentials. On the other hand, our results permit to situate Dollard's method in the framework of a general theory.

In Section IV we investigate the consequences of time reversal invariance and of spherical symmetry, and in Section V we establish that our theory is applicable also to potentials of the form $V(r) = r^{-\beta}$ with $3/4 \leqslant \beta < 1$. The convergence proof is carried through by a slight modification of the techniques employed by Dollard [3]. Finally, we should add that our formulation of the asymptotic condition can also be extended to multichannel processes. This will be the topic of a later report.

II. The Asymptotic Condition for Single-channel Scattering Systems

Let H_0 be the operator of the total kinetic energy of the participating particles, expressed in terms of their masses and their momenta in the usual way. The spectrum of H_0 is assumed to be absolutely continuous. Furthermore we assume that H_0 determines the unperturbed evolution of *all* vectors of the underlying Hilbert space $\mathcal{H}$, i.e. that the scattering system consists of only one channel. For non-relativistic potential scattering, H_0 is simply the Schrödinger operator $-\Delta/2\,m$ in the center-of-mass system.

Let $\mathcal{A}_0$ be the commutant of H_0:

$$\mathcal{A}_0 = \{H_0\}' \tag{1}$$

i.e. the set of all bounded operators on $\mathcal{H}$ which commute with all spectral projections of H_0. One remarks that the commutant $\mathcal{A}_0'$ of $\mathcal{A}_0$ is abelian and identical with the center $\mathcal{Z}_0$ of $\mathcal{A}_0$[3]):

$$\mathcal{Z}_0 = \mathcal{A}_0' = \{H_0\}'' \tag{2}$$

i.e. the algebra $\mathcal{A}_0'$ consists of all essentially bounded measurable functions of H_0. Let E be a projection operator in $\mathcal{A}_0$. Then we denote by $\mathcal{A}_{0E}$ the reduction of $\mathcal{A}_0$ to the subspace $E\,\mathcal{H}$: $\mathcal{A}_{0E}$ consists of all operators of the form $E\,A\,E$ with A in $\mathcal{A}_0$. This reduction $\mathcal{A}_{0E}$ is again a von Neumann algebra. Its commutant $(\mathcal{A}_{0E})'$ in the subspace $E\,\mathcal{H}$ is identical with $(\mathcal{A}_0')_E$. We shall henceforth denote it simply by $\mathcal{A}_{0E}'$. From these definitions it follows that $\mathcal{A}_{0E}'$ is again abelian and identical with the center of $\mathcal{A}_{0E}$ ([5], Section I.2.1).

We shall also need the notion of a *complete set of commuting observables.* A set Σ of commuting observables is said to be complete if the von Neumann algebra $\mathcal{B} = \Sigma''$ generated by them is maximal abelian [6], i.e. if

$$\mathcal{B} = \mathcal{B}'\,. \tag{3}$$

[3]) For the terminology and definitions concerning von Neumann algebras we refer to the book by Dixmier [5]. The *center* of a von Neumann algebra $\mathcal{A}$ is defined as the set of all those operators in $\mathcal{A}$ which commute with all operators of $\mathcal{A}$.

The requirement that $\mathcal{A}_0$ contains at least one maximal abelian subalgebra is fulfilled: In fact, every complete set of commuting observables containing H_0 generates a maximal abelian subalgebra of $\mathcal{A}_0$.

Let $\{V_t\}$, $-\infty < t < +\infty$, be the strongly continuous group of unitary operators on $\mathcal{H}$ which describes the total evolution of the scattering system. The free behaviour of a state will be described by means of the observables of $\mathcal{A}_0$. We shall consider only pure states. Such a state is said to become *asymptotically free* at $t \to \pm\infty$ if for all A in $\mathcal{A}_0$ the operators $V_t^* A V_t$ become stationary on this state in the strong operator topology as $t \to \pm\infty$. We then impose essentially the following three conditions on a scattering system:

(1) There should exist in $\mathcal{H}$ a set of vectors which are suitable for the description of the scattering process, i.e. a set of vectors which become asymptotically free for both $t \to -\infty$ and $t \to +\infty$. These vectors will span a subspace $P\mathcal{H}$ which we shall call the subspace of scattering states and which should be invariant under the group $\{V_t\}$.

(2) The algebra $\mathcal{A}_0$ should furnish a complete characterization of the scattering states in the limits $t \to \pm\infty$. This means that there must exist a complete set of commuting observables in $\mathcal{A}_0$ such that their asymptotic limits form a complete set of commuting observables on the scattering states.

(3) The set of scattering states should be sufficiently large so as to ensure that the initial states may be prepared arbitrarily: given an arbitrary vector f in $\mathcal{H}$ there must exist a scattering state g such that the expectation values of the observables of $\mathcal{A}_0$ in this state g converge to the corresponding expectation values in the initial state f as $t \to -\infty$.

We shall now express the preceding three conditions in mathematical terms:

(A1) There exists a projection operator P such that

(a) $$[P, V_t] = 0\,, \tag{4}$$

(b) for every operator A in $\mathcal{A}_0$ there exist two operators $A_\pm$ such that

$$s-\lim_{t\to\pm\infty} V_t^* A V_t P = A_\pm \tag{5}$$

and (c)

$$[P, A_\pm] = 0\,. \tag{6}$$

Let us denote by $\mu_\pm$ the two mappings defined by equation (5):

$$\mu_\pm(A) = A_\pm = s-\lim_{t\to\pm\infty} V_t^* A V_t P\,. \tag{7}$$

The requirement that P commute with their images $\mu_\pm(\mathcal{A}_0)$, equation (6), means essentially that $\mu_\pm$ preserve self-adjointness of observables (cf. the proof of Proposition 1 below). We do not assume that P be identical with F, the spectral projection of H onto the absolutely continuous part of its spectrum. Equations (4) and (6) and many later assertions about P are trivially true if $P = F$.

(A2) There exists a maximal abelian von Neumann algebra $\mathcal{B}$ in $\mathcal{A}_0$ whose asymptotic images $\mu_\pm(\mathcal{B})$ generate maximal abelian subalgebras of the algebra of all bounded operators on $P\mathcal{H}$.

(A3) For every vector f in $\mathcal{H}$ there exists a vector g in $P\,\mathcal{H}$ such that for all A in $\mathcal{A}_0$:

$$\lim_{t\to-\infty} (V_t\, g, A\, V_t\, g) = (f, A\, f)\,.$$

By successively applying these three conditions we shall deduce the existence of wave operators and of an isometric S-operator. S may be non-unitary, which follows from the fact that (A3) is not symmetric in the sign of the time t. However this last possibility is excluded if the scattering system is time reversal invariant (cf. Section IV). It will also be shown that (A2) does not distinguish any particular maximal abelian subalgebra of $\mathcal{A}_0$; i.e. if the property (A2) is true for one maximal abelian subalgebra of $\mathcal{A}_0$, then it is true for every maximal abelian subalgebra of $\mathcal{A}_0$.

In the sequel we shall frequently omit the indices $\pm$ and the designation $t\to\pm\infty$ for the limits. Wherever this occurs, it is understood that all statements hold true for the $+$ sign in the limit $t\to+\infty$ as well as for the $-$ sign in the limit $t\to-\infty$. On the other hand, in an equation or a statement where the double sign appears several times, we mean that the equation or the statement are valid separately for the upper and for the lower signs. We shall also use the notation

$$A(t) = V_t^*\, A\, V_t\,.$$

Before we proceed to discuss the properties of the mappings $\mu_\pm$, we introduce another von Neumann algebra $\mathcal{A}_P$ which will be needed for the following proposition. Let $\mathcal{A}$ be the commutant of the total Hamiltonian H, i.e. $\mathcal{A} = \{H\}'$. Then $\mathcal{A}_P$ is its restriction to the subspace $P\,\mathcal{H}$ of the scattering states, i.e.

$$\mathcal{A}_P = P\{H\}'\, P\,. \tag{8}$$

We may now assert:

Proposition 1: Let (A1) be satisfied. Then the mappings $\mu_\pm$ defined by equation (7) are homomorphisms from $\mathcal{A}_0$ onto a von Neumann algebra $\mu(\mathcal{A}_0) \subseteq \mathcal{A}_P$. Moreover there exist two maximal projection operators $E_\pm$ in the center of $\mathcal{A}_0$ such that the restrictions of μ_+ and μ_- to $\mathcal{A}_{0E_+}$ and $\mathcal{A}_{0E_-}$ respectively are injective.

The maximality of E is understood in the following sense: if E' is any projection operator in $\mathcal{A}_0'$ such that the homomorphism μ restricted to $\mathcal{A}_{0E'}$ is injective, then $E' \leqslant E$.

Proof:

(i) $\mu(A)$ is bounded for all $A \in \mathcal{A}_0$, since for any $f \in \mathcal{H}$

$$\|\mu(A)\, f\| = \lim \|V_t^*\, A\, V_t\, P f\| \leqslant \lim \|A\|\, \|V_t\, P f\| \leqslant \|A\|\, \|f\|\,.$$

From equation (4) and the multiplication law of the group V_t one deduces for all real τ

$$V_\tau^*\, \mu(A)\, V_\tau = V_\tau^*\, s - \lim V_t^*\, A\, V_t\, P\, V_\tau = s - \lim V_{t+\tau}^*\, A\, V_{t+\tau}\, P = \mu(A)\,.$$

Hence $\mu(A)$ commutes with all the spectral projections of H, i.e. $\mu(\mathcal{A}_0) \subseteq \mathcal{A}$. Since $\mu(A)\, P = \mu(A)$, one gets from equation (6) that

$$P\,\mu(A)\, P = P\,\mu(A) = \mu(A)\, P = \mu(A) \tag{9}$$

which proves that $\mu(\mathcal{A}_0) \subseteq \mathcal{A}_P$.

(ii) It is clear that μ is linear. Using (9) one finds

$$\begin{aligned}&\| [A(t)\; B(t)\; P - \mu(A)\; \mu(B)] f \| = \\ &= \| A(t)\; [B(t)\; P - \mu(B)] f + [A(t)\; P - \mu(A)]\; \mu(B) f \| \leqslant \\ &\leqslant \| A \| \; \| (B(t)\; P - \mu(B)) f \| + \| (A(t)\; P - \mu(A))\; \mu(B) f \| \,.\end{aligned}$$

Hence, if $A, B \in \mathcal{A}_0$:

$$\mu(A)\; \mu(B) = s - \lim A(t)\; B(t)\; P = s - \lim\; (A\; B)\; (t)\; P = \mu(A\; B) \tag{10}$$

which shows that μ preserves the product.

It follows from (9) that

$$\mu(A) = s - \lim P\; A(t)\; P\,. \tag{11}$$

Therefore the adjoint sequence $(P\; A(t)\; P)^* = P\; A^*(t)\; P$ converges weakly to $\mu(A)^*$. On the other hand, since A^* belongs also to $\mathcal{A}_0$, $P\; A^*(t)\; P$ converges strongly and hence weakly to $\mu(A^*)$. It follows from the uniqueness of the weak limit that $\mu(A)^* = \mu(A^*)$, i.e. μ is also adjoint preserving and therefore a homomorphism.

(iii) These algebraic properties of μ show that $\mu(\mathcal{A}_0)$ is a *-algebra. To proceed with the proof one has to examine the topological properties of $\mu(\mathcal{A}_0)$.

The absolute continuity of the spectrum of H_0 ensures that there is no projection operator with finite-dimensional range in the center of $\mathcal{A}_0$. Furthermore the underlying Hilbert space is separable. According to a result by Feldman and Fell ([7], Theorem 1 and Corollary on page 241), these two conditions imply that the homomorphisms $\mu_\pm$ are ultraweakly continuous[4]).

Let $\mathcal{N}$ be the kernel of μ, i.e. the set of all $A \in \mathcal{A}_0$ for which $\mu(A) = 0$. $\mathcal{N}$ is a two-sided ideal of $\mathcal{A}_0$, and since μ is ultraweakly continuous, $\mathcal{N}$ is ultraweakly closed. This implies that there exists one and only one projection operator Q in $\mathcal{A}_0'$ such that $\mathcal{N}$ is the set of all those $A \in \mathcal{A}_0$ that satisfy $A = A\; Q$ ([5], Section I.3.4, Corollary 3). Define $E = I - Q$. Then $\mathcal{A}_{0_E} \cap \mathcal{N} = \{0\}$, i.e. μ is injective on $\mathcal{A}_{0_E}$. The maximality of E is a consequence of the uniqueness of Q and of the fact that all projection operators in $\mathcal{A}_0'$ commute with one another.

(iv) One may now prove that $\mu_\pm(\mathcal{A}_0)$ are von Neumann algebras. First, since μ is an isomorphism between $\mathcal{A}_{0_E}$ and $\mu(\mathcal{A}_{0_E}) = \mu(\mathcal{A}_0)$, it follows that the restriction of μ to $\mathcal{A}_{0_E}$ is norm preserving ([5], Section I.1.5, Proposition 8). Let us denote by $\mathcal{A}^1_{0_E}$ and $\mu(\mathcal{A}_0)^1$ the unit spheres of $\mathcal{A}_{0_E}$ and $\mu(\mathcal{A}_0)$ respectively, i.e. the set of those operators in $\mathcal{A}_{0_E}$ and $\mu(\mathcal{A}_0)$ respectively having norm 1. Then $\mu(\mathcal{A}^1_{0_E}) = \mu(\mathcal{A}_0)^1$. Furthermore the unit sphere of a von Neumann algebra is compact in the weak topology. Since μ is ultraweakly continuous, its restriction to $\mathcal{A}^1_{0_E}$ is also weakly continuous. It follows that $\mu(\mathcal{A}_0)^1$, the image of $\mathcal{A}^1_{0_E}$, is weakly compact, and hence weakly closed. This in turn means that $\mu(\mathcal{A}_0)$ is a von Neumann algebra ([5], Section I.3.4, Corollary 2). ∎

Proposition 1 exhibits the properties of a scattering system which satisfies (A 1). The requirement (A 2) represents an additional condition on some maximal abelian

[4]) For the definition and the properties of the various topologies on von Neumann algebras we refer to [5], Section I.3.1. In [7] the ultraweak topology is called σ-weak.

subalgebra of $\mathcal{A}_0$. To see the mathematical significance of (A2), it is useful to consider first the consequences of (A1) for maximal abelian subalgebras of $\mathcal{A}_0$.

For this, let E be the projection operator in $\mathcal{A}_0'$ determined by Proposition 1, and let $\mathcal{B}$ be a maximal abelian subalgebra of $\mathcal{A}_0$, i.e. $\mathcal{B}' = \mathcal{B} \subset \mathcal{A}_0$.

This implies that $\mathcal{B}_E' = \mathcal{B}_E \subset \mathcal{A}_{0_E}$, which may also be written as

$$\mathcal{B}_E' \cap \mathcal{A}_{0_E} = \mathcal{B}_E \, . \tag{12}$$

Using the fact that μ is an isomorphism from $\mathcal{A}_{0E}$ onto $\mu(\mathcal{A}_0)$, one easily deduces that the relation (12) is preserved by μ, i.e.

$$\mu(\mathcal{B})_P' \cap \mu(\mathcal{A}_0) = \mu(\mathcal{B}) \tag{13}$$

where the commutant is restricted to bounded operators acting on the subspace $P\,\mathcal{H}$. Equation (13) means that $\mu(\mathcal{B})$ is maximal abelian in $\mu(\mathcal{A}_0)$ ([5], Section I.7.1, Proposition 13).

(A2) requires that there exists a particular maximal abelian subalgebra $\mathcal{B}_0$ of $\mathcal{A}_0$ such that $\mu(\mathcal{B}_0)$ is maximal abelian in the algebra of *all* bounded operators on $P\,\mathcal{H}$, i.e. satisfying

$$\mu(\mathcal{B}_0)_P' = \mu(\mathcal{B}_0) \subset \mu(\mathcal{A}_0) \, . \tag{14}$$

It is obvious that (14) is a stronger statement than (13).

Proposition 2: Let (A1) and (A2) be satisfied. Then there exist two partial isometries $\Omega_\pm$ on $\mathcal{H}$ such that for all A in $\mathcal{A}_0$

$$\mu_\pm(A) = \Omega_\pm\, A\, \Omega_\pm^* \tag{15}$$

and

$$\Omega_+\, \Omega_+^* = \Omega_-\, \Omega_-^* = P \, , \tag{16}$$

$$\Omega_+^*\, \Omega_+ = E_+ \qquad \Omega_-^*\, \Omega_- = E_- \, . \tag{17}$$

$\Omega_\pm$ are determined up to multiplication from the right by a unitary function of $H_0\, E_\pm$.

Proof: (14) implies for the commutants of the involved algebras that

$$\mu(\mathcal{A}_0)_P' \subseteq \mu(\mathcal{B}_0)_P' = \mu(\mathcal{B}_0) \, . \tag{18}$$

Combining (18) and (14), one obtains

$$\mu(\mathcal{A}_0)_P' \subset \mu(\mathcal{A}_0) \, . \tag{19}$$

(18) implies in particular that $\mu(\mathcal{A}_0)_P'$ is abelian.

Since the commutant of $\mathcal{A}_{0_E}$ is also abelian, and since μ is an isomorphism between the von Neumann algebras $\mathcal{A}_{0_E}$ and $\mu(\mathcal{A}_0)$, it follows that μ is spatial ([5], Section III.3.1, Corollary 1). This means that there exists a unitary operator Ω from $E\,\mathcal{H}$ onto $P\,\mathcal{H}$ such that $\mu(A) = \Omega\, A\, \Omega^*$ for all $A \in \mathcal{A}_{0_E}$. We extend Ω to a partial isometry on $\mathcal{H}$ by setting $\Omega f = 0$ for all vectors f in the orthogonal complement of $E\,\mathcal{H}$. Furthermore, if A belongs to the kernel of μ, then $E\,A\,E = 0$, and hence $\Omega\, A\, \Omega^* = \Omega\, E\, A\, E\, \Omega^* = 0$, which proves equation (15) for all $A \in \mathcal{A}_0$.

Let Ω_1 be another partial isometry satisfying (15)–(17). Then for all $A \in \mathcal{A}_0$

$\mu(A) = \Omega\, A\, \Omega^* = \Omega_1\, A\, \Omega_1^*$.

This implies together with (17) that

$$E\, A\, \Omega^*\, \Omega_1 = \Omega^*\, \Omega_1\, A\, E$$

which shows that $U_0 = \Omega^*\, \Omega_1$ belongs to the center of $\mathcal{A}_{0E}$, and $\Omega_1 = \Omega\, U_0$. The unitarity of U_0 on the subspace $E\, \mathcal{H}$ is an easy consequence of (16) and (17). This proves the last assertion of the proposition. ∎

We shall now verify that (A2) does not distinguish any particular maximal abelian subalgebra of $\mathcal{A}_0$. For this, let $\mathcal{B}$ be any such subalgebra of $\mathcal{A}_0$. According to (13), $\mu(\mathcal{B})$ is maximal abelian in $\mu(\mathcal{A}_0)$. $\mu(\mathcal{B})$ must therefore contain the center of $\mu(\mathcal{A}_0)$ ([5], Section I.1.7). If (A2) is satisfied, the center of $\mu(\mathcal{A}_0)$ coincides with $\mu(\mathcal{A}_0)'_P$ (cf. equation (19)). Therefore

$$\mu(\mathcal{A}_0)'_P \subseteq \mu(\mathcal{B}) .$$

This means for the commutants that

$$\mu(\mathcal{B})'_P \subseteq \mu(\mathcal{A}_0)''_P = \mu(\mathcal{A}_0) . \tag{20}$$

Combining (20) and (13) one finds

$$\mu(\mathcal{B})'_P = \mu(\mathcal{B})$$

which shows that $\mu(\mathcal{B})$ is also maximal abelian on $P\, \mathcal{H}$. This result could also easily be deduced from the fact that the isomorphism μ is spatial.

According to Proposition 2, the requirements (A1) and (A2) guarantee the existence of wave operators. An important property of the wave operators in standard scattering theory is the *intertwining relation* $V_t\, \Omega_\pm = \Omega_\pm\, U_t$ for all t. This means essentially that $f(H)\, \Omega_\pm = \Omega_\pm f(H_0)$ for bounded functions f. In the following proposition we establish a similar relation as a consequence of (A1) and (A2).

Proposition 3: For every essentially bounded measurable[5]) function f on the spectrum of $H\, P = P\, H\, P$ there exist two essentially bounded measurable functions $g_\pm$ on the spectrum of $H_0\, E_+$ and $H_0\, E_-$ respectively such that

$$f(H\, P)\, \Omega_\pm = \Omega_\pm\, g_\pm(H_0\, E_\pm) . \tag{21}$$

Proof: From Proposition 1 we know that $\mu(\mathcal{A}_0) \subseteq \mathcal{A}_P$. Using also (19), this implies

$$\mathcal{A}'_P \subseteq \mu(\mathcal{A}_0)'_P \subset \mu(\mathcal{A}_0) . \tag{22}$$

Consequently $\mathcal{A}'_P$ lies in the range of μ. Denote by μ_E the restriction of μ to $\mathcal{A}_{0E}$. μ_E is an isomorphism between $\mathcal{A}_{0E}$ and $\mu(\mathcal{A}_0)$. Hence its restriction to the center $\mathcal{A}'_{0E}$ of $\mathcal{A}_{0E}$ is an isomorphism from $\mathcal{A}'_{0E}$ onto the center $\mu(\mathcal{A}_0)'_P$ of its range. Therefore (22) leads to

$$\mu_E^{-1}(\mathcal{A}'_P) \subseteq \mu_E^{-1}(\mu(\mathcal{A}_0)'_P) = \mathcal{A}'_{0E} . \tag{23}$$

[5]) For more details about the spectral representation and the functional calculus we refer to [8] and [9]. Cf. also the proof of Theorem 2 in Section III below.

Now the set of all essentially bounded measurable functions of $H\,P$ coincides with the center $\mathcal{A}'_P$ of $\mathcal{A}_P$ and the set of all essentially bounded measurable functions of $H_0\,E$ with $\mathcal{A}'_{0E}$. Therefore, according to (23), given any essentially bounded measurable function f on the spectrum of $H\,P$, there exists such a function g on the spectrum of $H_0\,E$ such that

$$\mu_E^{-1}(f(H\,P)) = g(H_0\,E) \tag{24}$$

or

$$f(H\,P) = \mu(g(H_0\,E)) = \Omega\, g(H_0\,E)\,\Omega^* . \tag{25}$$

The assertion of the proposition follows by multiplying (25) from the right by Ω. ∎

The functions f and $g_\pm$ in (21) are not necessarily identical, and neither are g_+ and g_-. However, it is obvious from (24) that the relation between f and $g_\pm$ is determined solely by the isomorphisms $\mu_\pm$, i.e. it does not depend on the unfixed unitary function of $H_0\,E_\pm$ in the wave operators.

We shall now introduce two important families of operators in the center of $\mathcal{A}_0$. They are obtained by choosing a particular family of functions f_t in equation (24), namely $f_t(H\,P) = \exp(-i\,H\,P\,t)\,P = V_t\,P$. This defines for every real t two operators $W_t^\pm$ in $\mathcal{A}'_{0E_+}$ and $\mathcal{A}'_{0E_-}$ respectively:

$$W_t^\pm = \mu_{E_\pm}^{-1}(V_t\,P) = \Omega_\pm^*\,V_t\,\Omega_\pm . \tag{26}$$

Each of the families $\{W_t^+\}$ and $\{W_t^-\}$ forms a group in t:

$$W_{t_1+t_2} = \Omega^*\,V_{t_1+t_2}\,P\,\Omega = \Omega^*\,V_{t_1}\,P\,V_{t_2}\,\Omega = \Omega^*\,V_{t_1}\,\Omega\,\Omega^*\,V_{t_2}\,\Omega = W_{t_1}\,W_{t_2} .$$

Furthermore they are strongly continuous in t and satisfy $W_t^\pm\,E_\pm = W_t^\pm$, and the operators $W_t^\pm$ are unitary on the subspace $E_+\,\mathcal{H}$ or $E_-\,\mathcal{H}$ respectively.

Let $K^\pm$ be the infinitesimal generators of the groups $\{W_t^\pm\}$ respectively, i.e.

$$W_t^\pm = \exp(-i\,K^\pm\,t)\,E_\pm \tag{27}$$

$K^\pm$ are self-adjoint operators defined on $E_\pm\,\mathcal{H}$ respectively. It follows from equation (26) that their spectra coincide with that of $H\,P$.

The groups $\{W_t^\pm\}$ may be considered as the asymptotic free evolutions of the scattering system at large positive and negative times respectively. One arrives at this interpretation by rewriting equation (26) as

$$V_\tau\,P = \mu_\pm(W_\tau^\pm) = s - \lim_{t\to\pm\infty} V_t^*\,W_\tau^\pm\,V_t\,P \tag{28}$$

which is equivalent to

$$\lim_{t\to\pm\infty} \|V_{t+\tau}\,g - W_\tau^\pm\,V_t\,g\| = 0 \tag{29}$$

for every scattering state g in $P\,\mathcal{H}$ and every τ. The convergence in (29) may or may not be uniform in τ, a property which will be of importance in the discussion of the different types of scattering systems at the end of Section III. In equation (29), $V_{t+\tau}\,g$ represents the scattering state g at the time $t+\tau$, whereas in $W_\tau^+\,V_t\,g$ the total evolution V_t is replaced by the asymptotic free evolution W_t^+ in the time interval

$(t, t+\tau)$. Equation (29) asserts that these two states differ in norm by an arbitrarily small number if t is sufficiently large. In other words, in any given time interval of length τ the asymptotic free evolution W_t^+ approximates the real evolution V_t with an arbitrarily prescribed accuracy if only the given time interval lies sufficiently far in the future, and likewise for W_t^- and $t \to -\infty$. One may therefore say that V_t is asymptotically comparable to $W_t^\pm$ on any finite interval of time in the distant future or the remote past respectively.

We shall indicate in Section III the asymptotic free evolutions $W_t^\pm$ corresponding to different classes of interactions. For short range potentials $W_t^\pm = U_t = \exp(-i H_0 t)$, and the limit in (29) is uniform in τ ([10], Section 3). In more general cases $W_t^\pm$ may differ from U_t, in which event there will be a renormalization of the free energy. It could even occur that W_t^+ and W_t^- are not identical if the scattering system is not time reversal invariant.

If $W_t \neq U_t E$, the von Neumann algebra $\{K\}''$ generated by K on $E\,\mathcal{H}$ may differ from the center $\mathcal{A}'_{0E} = \{H_0 E\}''$ of $\mathcal{A}_{0E}$. It follows from (26) and (23) that

$$\mu_E^{-1}(\mathcal{A}'_P) = \{K\}'' \subseteq \mathcal{A}'_{0E}\,. \tag{30}$$

In those cases where the inclusion in this relation is an equality, the converse of Proposition 3 is true:

Proposition 4: Assume that $\{K^-\}'' = \{H_0 E_-\}''$ on $E_-\,\mathcal{H}$. Then for every essentially bounded measurable function g on the spectrum of $H_0 E_-$ there exists an essentially bounded measurable function f_- on the spectrum of $H P$ such that

$$f_-(H P)\,\Omega_- = \Omega_-\,g(H_0 E_-)\,. \tag{31}$$

Furthermore

$$\mu_-(\mathcal{A}_0) = \mathcal{A}_P\,. \tag{32}$$

A similar statement holds for the $+$ sign.

Proof: (30) combined with the assumption of the Proposition gives

$$\mu(\mathcal{A}'_{0E}) = \mathcal{A}'_P \tag{33}$$

which implies (31). Using (33) and the fact that $\mu(\mathcal{A}'_{0E}) = \mu(\mathcal{A}_0)'_P$ (cf. the remark leading to equation (23)), it follows that $\mathcal{A}'_P = \mu(\mathcal{A}_0)'_P$, and hence for the commutants that $\mathcal{A}''_P = \mu(\mathcal{A}_0)''_P$. Since both $\mathcal{A}_P$ and $\mu(\mathcal{A}_0)$ are von Neumann algebras acting on $P\,\mathcal{H}$, this implies (32). ∎

It is now possible to define a *scattering operator* S by the usual prescription

$$S = \Omega_+^*\,\Omega_-\,. \tag{34}$$

Its properties are enunciated in the following theorem. In particular we shall derive from the postulate (A3) that S is an *isometric* operator on $\mathcal{H}$.

Theorem 1: The requirements (A1), (A2) and (A3) are sufficient for the existence of wave operators $\Omega_\pm$ having the properties of Proposition 2 with $E_- = I$ and of an isometric S-operator which is determined up to multiplication by unitary functions of H_0 from the left or from the right and which preserves the asymptotic free energy, viz.

$$K^+ S = S K^- . \tag{35}$$

Proof: (i) According to Proposition 2, (A1) and (A2) ensure the existence of wave operators. It remains to show that (A3) implies $E_- = I$. For this, it suffices to establish that the kernel $\mathcal{N}_-$ of μ_- consists of the zero operator only (cf. part (iii) of the proof of Proposition 1). For this, let $A \in \mathcal{N}_-$. If $A \neq 0$, there exists a vector $f \in \mathcal{H}$ such that $(f, A f) \neq 0$. By (A3) there exists $g \in P \mathcal{H}$ such that

$$(f, A f) = (g, \mu(A) g) \neq 0$$

which contradicts the assumption that $\mu(A) = 0$. Hence $A = 0$.

(ii) The S-operator was defined in equation (34). Its isometric property follows from the corresponding properties of $\Omega_\pm$, equations (16) and (17). One finds

$$S^* S = I , \qquad S S^* = E_+ . \tag{36}$$

The indeterminateness of S which is expressed in the theorem is a consequence of the indeterminateness of $\Omega_\pm$. From (26) one deduces

$$W_t^+ S = W_t^+ \Omega_+^* \Omega_- = \Omega_+^* V_t \Omega_- = \Omega_+^* \Omega_- W_t^- = S W_t^- .$$

This implies (35) by the usual arguments (cf. [15], pp. 530/531). ∎

The statement that S preserves the asymptotic free energy may be reexpressed by considering the matrix elements of S. Let $f_{in} \in \mathcal{H}$ be an incoming state whose asymptotic free energy lies in an interval I_- (i.e. the support of f in the spectral representation of K^- is the interval I_-.) Let $f_{out} \in E_+ \mathcal{H}$ be an outgoing state whose support in the spectral representation of K^+ is the interval I_+. Whenever I_- and I_+ are disjoint, then $(f_{out}, S f_{in}) = 0$. Hence there are no transitions between states of different asymptotic free energy. This follows easily from equation (35) (cf. also [11], pp. 330/331).

The properties of the S-operator which we obtained as a consequence of (A1), (A2) and (A3) are slightly more general than what one usually requires of this operator: S may be only isometric and need not commute with the kinetic energy operator H_0. We shall see later that for time reversal invariant scattering systems S is unitary and $K^+ = K^- = K$. According to equation (35), S then commutes with the asymptotic free energy K (i.e. with the renormalized free energy.) Moreover, in most cases the hypothesis of Proposition 4 will be true, i.e. K and H_0 will generate the same von Neumann algebra. Then S will commute also with $\mathcal{A}_0'$ and in particular with H_0.

III. An Equivalent Formulation of the Asymptotic Condition

In the usual formulation of the asymptotic condition [1], the wave operators $\Omega_\pm$ are defined as the strong limits of $V_t^* U_t = \exp(i H t) \exp(-i H_0 t)$ as $t \to \pm\infty$. These limits exist if the potential has a sufficiently short range. In this section we wish to show that the wave operators which we obtained in our more general formulation of the asymptotic condition may still be constructed as strong limits of the form

$$\Omega = s - \lim V_t^* T_t .$$

Here $\{T_t\}$ denotes a family of operators which are functions of H_0 but which may differ from $U_t = \exp(-i H_0 t)$ as well as from the asymptotic free evolution $W_t \equiv \Omega^* V_t \Omega$. These operators T_t do not in general form a group and may be only asymptotically isometric. They are determined by the asymptotic behaviour of V_t at large times and incorporate the residual effects at large distances from the scattering center due to the long range of the interaction (e.g. the logarithmic phase distortion for the Coulomb interaction [3].) They are those functions of H_0 that govern the asymptotic behaviour of a state f which has the property of approaching some scattering state g in the usual sense, i.e. for which $\lim \| V_t g - T_t f \| = 0$.

By means of such a family $\{T_t\}$ of operators one obtains a formulation of the asymptotic condition which is equivalent to that of the preceding section and at the same time similar to that of standard scattering theory. For the sake of simplifying the notation, we shall assume that both projection operators $E_\pm$ obtained in Proposition 2 are the identity operator. By slight and obvious modifications, one may adapt all statements and proofs to the more general case where $E_+ \neq I$.

Theorem 2: Let $\mathcal{A}_0$ be the von Neumann algebra defined by equation (1) and $\{V_t\}$, $-\infty < t < +\infty$, a group of unitary operators. Suppose that (A1), (A2) and (A3) are satisfied and that $E_+ = I$, and let $\Omega_\pm$ be the corresponding wave operators. Then there exist two families $\{T_t^\pm\}$, $-\infty < t < \infty$, of closed linear operators acting on $\mathcal{H}$ with the following properties:

(a) The operators of each family have a common domain $\mathcal{D}^\pm$ which is dense in $\mathcal{H}$ and invariant under $\mathcal{A}_0$ and on which they commute with $\mathcal{A}_0$.

(b) The families $\{T_t^\pm\}$ are asymptotically isometric on $\mathcal{D}^\pm$ as $t \to \pm\infty$, i.e.

$$\lim_{t \to +\infty} \| T_t^+ h \| = \| h \| \qquad \text{for all } h \in \mathcal{D}^+ \tag{37a}$$

and

$$\lim_{t \to -\infty} \| T_t^- h \| = \| h \| \qquad \text{for all } h \in \mathcal{D}^- . \tag{37b}$$

(c) The strong limits of $V_t^* T_t$ as $t \to \pm\infty$ exist on $\mathcal{D}^\pm$.

(d) $$s - \lim_{t \to +\infty} V_t^* T_t^+ h = \Omega_+ h \qquad \text{for all } h \in \mathcal{D}^+ \tag{38a}$$

and

$$s - \lim_{t \to -\infty} V_t^* T_t^- h = \Omega_- h \qquad \text{for all } h \in \mathcal{D}^- . \tag{38b}$$

Conversely, suppose that there exist two families $\{T_t^\pm\}$ of linear operators satisfying (a), (b) and (c). Then the strong limits of $V_t^* T_t^\pm$ as $t \to \pm\infty$ define two isometries $\Omega_\pm$. Assume in addition that the ranges of these two isometries are identical and reduce the group $\{V_t\}$. Then (A1), (A2) and (A3) are satisfied with $E_+ = I$, and $\Omega_\pm$ are the wave operators determined by (A1), (A2) and (A3) (up to multiplication by a unitary function of H_0 from the right.)

We may add that, in the terminology of von Neumann algebras, the operators $T_t^\pm$ are affiliated with the center $\mathcal{A}_0'$ of the von Neumann algebra $\mathcal{A}_0$.

Proof of Theorem 2: The proof is the same for the two limits $t \to \pm\infty$. We shall therefore again omit the double index $\pm$.

(I) Let (A1), (A2) and (A3) be satisfied, and let Ω be one of the wave operators given by Theorem 1.

(i) Define for all real t

$$Y_t \equiv V_t \Omega . \tag{39}$$

It follows from the unitarity of V_t, from $[P, V_t] = 0$ and from the isometry of Ω that Y_t is isometric:

$$Y_t^* Y_t = I , \qquad Y_t Y_t^* = P . \tag{40}$$

Moreover, if $g \in \mathcal{H}$ and $f = \Omega^* g$:

$$\| (V_t^* A V_t P - \Omega A \Omega^*) g \| = \| (A V_t \Omega - V_t \Omega A) f \| = \| [A, Y_t] f \| . \tag{41}$$

If $A \in \mathcal{A}_0$, condition (A1) states that the left-hand member of (41) converges to zero. Since the range of Ω^* is the entire Hilbert space $\mathcal{H}$, this implies that

$$s - \lim [A, Y_t] = 0 \text{ for all } A \in \mathcal{A}_0 . \tag{42}$$

(ii) One may write the given Hilbert space as a direct integral over the spectrum of H_0 [8]:

$$\mathcal{H} = \int^{\oplus} \mathcal{H}_\lambda \, d\lambda . \tag{43}$$

Since H_0 has absolutely continuous spectrum, the measure in this direct integral decomposition is equivalent to the Lebesgue measure on the spectrum $\sigma(H_0)$ of H_0. For every $\lambda \in \sigma(H_0)$, $\mathcal{H}_\lambda$ is a Hilbert space which describes the degeneracy of the spectral point λ. We denote by $(g(\lambda), f(\lambda))_\lambda$ the scalar product between two vectors $g(\lambda), f(\lambda) \in \mathcal{H}_\lambda$ and by $\| f(\lambda) \|_\lambda$ the norm in $\mathcal{H}_\lambda$. An element f of $\mathcal{H}$ is given by a family of vectors $\{f(\lambda)\}$, $f(\lambda) \in \mathcal{H}_\lambda$, satisfying the following conditions: $\| f(\lambda) \|_\lambda$ is measurable and square-integrable, and for any $g \in \mathcal{H}$, the function $(g(\lambda), f(\lambda))_\lambda$ is measurable, and

$$(g, f) = \int d\lambda (g(\lambda), f(\lambda))_\lambda .$$

The operators of $\mathcal{A}_0$ are decomposable in the representation (43) of $\mathcal{H}$, i.e. if $A \in \mathcal{A}_0$, then

$$A = \int^{\oplus} A_\lambda \, d\lambda$$

where A_λ acts in $\mathcal{H}_\lambda$, and the norms $\| A_\lambda \|_\lambda$ are essentially bounded. $\mathcal{A}_0'$ is that subset of operators of $\mathcal{A}_0$ which can be diagonalised ([5], Section II.2): if $B \in \mathcal{A}_0'$ then

$$B = \int^{\oplus} B(\lambda) \, I_\lambda \, d\lambda \tag{44}$$

where I_λ is the identity operator in $\mathcal{H}_\lambda$ and $B(\lambda)$ an essentially bounded measurable function on $\sigma(H_0)$.

(iii) We shall now construct a particular operator C in $\mathcal{A}_0$. For this, we remark that there exists a vector e in $\mathcal{H}$ which is cyclic for $\mathcal{A}_0$, i.e. such that the set $\{\mathcal{A}_0 e\}$ is dense in $\mathcal{H}$ ([5], Section I.1.4, Def. 3 and Prop. 5, Section I.2.1, Corollary to Prop. 3). This vector e has the property that $\| e(\lambda) \|_\lambda \neq 0$ for almost all $\lambda \in \sigma(H_0)$. We define C_λ as the projection operator onto the vector $e(\lambda)$ in $\mathcal{H}_\lambda$ and C as the direct integral of C_λ: if $f = \{f(\lambda)\} \in \mathcal{H}$, then

$$C f = \left\{ \frac{(e(\lambda), f(\lambda))_\lambda}{\| e(\lambda) \|_\lambda^2} e(\lambda) \right\} . \tag{45}$$

Clearly C belongs to $\mathcal{A}_0$ and is a projection operator on $\mathcal{H}$. In particular

$$C e = e , \tag{46}$$

$$C Y_t e = \{\xi_t(\lambda) e(\lambda)\} \tag{47}$$

with the coefficients

$$\xi_t(\lambda) = \frac{(e(\lambda), (Y_t e)(\lambda))_\lambda}{\| e(\lambda) \|_\lambda^2} . \tag{48}$$

These coefficients $\xi_t(\lambda)$ define for each t a measurable function on $\sigma(H_0)$, since both the numerator and the denominator of (48) are measurable and the denominator vanishes at most on a set of measure zero.

Consider the domain $\mathcal{D}_t$

$$\mathcal{D}_t = \left\{ f \in \mathcal{H} : \int | \xi_t(\lambda) |^2 \| f(\lambda) \|_\lambda^2 \, d\lambda < \infty \right\} \tag{49}$$

and the operator T_t defined on $\mathcal{D}_t$ as follows

$$T_t f = \{\xi_t(\lambda) f(\lambda)\} , \quad f \in \mathcal{D}_t . \tag{50}$$

It follows from Theorem 6.4 of [12] that $\mathcal{D}_t$ is dense in $\mathcal{H}$ and T_t is a closed linear operator in $\mathcal{H}$ with domain $\mathcal{D}_t$.

(iv) It remains to show that this family $\{T_t\}$ of operators has the properties (a)–(d) of the Theorem. For this, let $\mathcal{D}$ be the dense set $\{\mathcal{A}_0 e\}$. The vectors h in $\mathcal{D}$ have the form

$$h = A e = \{A_\lambda e(\lambda)\} \text{ with } A \in \mathcal{A}_0 . \tag{51}$$

Clearly $\mathcal{D}$ is invariant under $\mathcal{A}_0$. Furthermore, for $h \in \mathcal{D}$ one gets from (47):

$$T_t h = \{\xi_t(\lambda) A_\lambda e(\lambda)\} = \{A_\lambda \xi_t(\lambda) e(\lambda)\} = A C Y_t e . \tag{52}$$

Since $\| Y_t \| = 1$ and $\| C \| = 1$, it follows that

$$\| T_t h \| \leqslant \| A \| \| C \| \| Y_t \| \| e \| = \| A \| \| e \| . \tag{53}$$

Hence $\| T_t h \|$ is uniformly bounded in t for any $h \in \mathcal{D}$, i.e. $\mathcal{D} \subseteq \mathcal{D}_t$ for all t. It is evident from the above construction that $[\mathcal{A}_0, T_t] = 0$ on $\mathcal{D}$. This proves (a).

Next, applying successively (39), (51), (46) and (52), we get for $h \in \mathcal{D}$

$$\| (\Omega - V_t^* T_t) h \| = \| (Y_t - T_t) h \| = \| Y_t A C e - A C Y_t e \|$$

which converges to zero according to (42). This proves (c) and (d).

Finally, for $h \in \mathcal{D}$,

$$|(T_t h, T_t h) - (h, h)| = |((T_t - Y_t) h, Y_t h) + (T_t h, (T_t - Y_t) h)| \leqslant \leqslant \|(T_t - Y_t) h\| (\|Y_t h\| + \|T_t h\|) \tag{54}$$

Since $\|Y_t h\|$ and $\|T_t h\|$ are uniformly bounded in t, the last member of this inequality converges to zero. This proves (b).

(II) To prove the converse, let $h \in \mathcal{D}$. According to assumption (c), there exists $g = s - \lim V_t^* T_t h$. Using (b), one then obtains

$$\|g\|^2 = \lim (V_t^* T_t h, V_t^* T_t h) = \lim (T_t h, T_t h) = \|h\|^2 .$$

This shows that the linear operator Ω defined by $g = \Omega h$ for $h \in \mathcal{D}$ is isometric on $\mathcal{D}$. Therefore it may be extended in a unique way by continuity to an isometric operator on all of $\mathcal{H}$. This extension will also be denoted by Ω, and we define again $Y_t = V_t \Omega$ as in (39).

We shall now show that Y_t commutes asymptotically with $\mathcal{A}_0$. For this, let f be an arbitrary vector in $\mathcal{H}$. Given $\varepsilon > 0$ and $A \in \mathcal{A}_0$, there exists a vector $h \in \mathcal{D}$ such that $\|f - h\| < \varepsilon/(4 \|A\|)$. Then

$$\|[Y_t, A] f\| \leqslant \|[Y_t, A] h\| + 2 \|A\| \|f - h\| < \|[Y_t, A] h\| + \frac{\varepsilon}{2} . \tag{55}$$

Since by assumption (a) T_t commutes with $\mathcal{A}_0$ on $\mathcal{D}$, one has

$$\|(Y_t A - A Y_t) h\| = \|(Y_t - T_t) A h - A (Y_t - T_t) h\| \leqslant \leqslant \|(Y_t - T_t) A h\| + \|A\| \|(Y_t - T_t) h\| . \tag{56}$$

Since $\mathcal{D}$ is invariant under $\mathcal{A}_0$, Ah lies also in $\mathcal{D}$. Hypothesis (c) then implies that the right-hand side of (56) can be made smaller than $\varepsilon/2$ if only t is sufficiently large. Inserting this in (55), one finds that

$$s - \lim [Y_t, A] = 0 \text{ for all } A \in \mathcal{A}_0 . \tag{57}$$

To complete the proof, it is necessary to distinguish between the two limits $t \to \pm\infty$. Let P denote the projection operator onto the common range of $\Omega_\pm$. Using the identity (41), one deduces from (57) that

$$s - \lim_{t \to \pm\infty} V_t^* A V_t P = \Omega_\pm A \Omega_\pm^* \qquad \text{for all } A \in \mathcal{A}_0 . \tag{58}$$

By hypothesis $[P, V_t] = 0$, and it is obvious from (58) that $[P, \mu_\pm(A)] = 0$. This proves that (A1) is satisfied. (A2) is an immediate consequence of the fact that the isomorphisms $\mu_\pm$: $A \to \Omega_\pm A \Omega_\pm^*$ are spatial.

Finally, let $f \in \mathcal{H}$ and define $g = \Omega_- f$. For $A \in \mathcal{A}_0$, (58) implies

$$(f, A f) = (g, \Omega_- A \Omega_-^* g) = \lim_{t \to -\infty} (g, V_t^* A V_t g)$$

which proves (A3). ∎

One concludes from Theorem 2 that only the asymptotic properties of $\{T_t\}$ are of importance. For finite times, T_t could be changed arbitrarily. However, from the construction of $\{T_t\}$ given in the proof it does not follow that the convergence of $V_t^* T_t$ could be extended from $\mathcal{D}$ to all of $\mathcal{H}$. This convergence will be true on all of $\mathcal{H}$ in those cases where it is possible to find a family $\{T_t\}$ belonging to $\mathcal{A}_0'$, i.e. a family of *bounded* operators T_t. Furthermore, since the wave operators are determined only up to multiplication by a unitary function of H_0 (from the right), it is also clear that the families $\{T_t\}$ exhibit the same indeterminateness asymptotically.

We have seen that our formulation of the asymptotic condition involves (for each of the limits $t \to \pm\infty$) four families of operators depending on time as a parameter, namely the total evolution V_t and three families which are function of the kinetic energy operator H_0: The unperturbed (or free) evolution $U_t = \exp(-i H_0 t)$, the asymptotic (or renormalized) free evolution $W_t = \exp(-i K t)$ and the operators T_t constructed in Theorem 2. These last three families may all be different from one another. U_t was used for defining the algebra of observables $\mathcal{A}_0$. W_t and T_t were determined only after introducing the interaction. We shall now establish some simple additional properties of the operators T_t and give the connection between T_t and W_t. To conclude this section, we shall then indicate the different possibilities for the asymptotic behaviour of a scattering system which arise when two or all of the families $\{U_t\}$, $\{W_t\}$ and $\{T_t\}$ are the same. Throughout the remainder of this section we assume (A1), (A2) and (A3) to be satisfied.

The difference between T_t and W_t is effectively described by $\hat{W}_t \equiv W_t^* T_t$ defined on $\mathcal{D}$. This modification of W_t is necessary in order to obtain strong convergence of $V_t^* T_t$, as it may happen that neither of the strong limits of $V_t^* U_t$ and $V_t^* W_t$ exists. In Corollary 2 we shall establish that there is no modification (i.e. $\hat{W}_t = I$ for all t) if and only if the family $\{T_t\}$ forms a group in t. In the general case where $\hat{W}_t \neq I$, these modifications are restricted to a particular class of operators: for any $h \in \mathcal{D}$, the family $\{\hat{W}_t h\}$ of vectors is *feebly oscillating* ([13], page 505), i.e. for any real τ

$$\lim_{t \to \pm\infty} \| (\hat{W}_{t+\tau}^{\pm} - \hat{W}_t^{\pm})\, h \| = 0 \,. \tag{59}$$

From the definition of $\hat{W}_t$ and the group property of W_t, this assertion (59) is seen to be equivalent to the following statement:

Corollary 1: For any real number τ one has

$$\lim_{t \to +\infty} \| (T_{t+\tau}^{+} - W_\tau^{+} T_t^{+})\, h \| = 0 \qquad \text{for all } h \in \mathcal{D}^{+} \tag{60a}$$

$$\lim_{t \to -\infty} \| (T_{t+\tau}^{-} - W_\tau^{-} T_t^{-})\, h \| = 0 \qquad \text{for all } h \in \mathcal{D}^{-} \,. \tag{60b}$$

Proof: Let $h \in \mathcal{D}$ and $\Omega h = s - \lim V_t^* T_t h$. Then

$$\begin{aligned} \| (T_{t+\tau} - W_\tau T_t)\, h \| &= \| V_{t+\tau}^* T_{t+\tau} h - V_{t+\tau}^* W_\tau T_t h \| = \| (V_{t+\tau}^* T_{t+\tau} - \Omega)\, h \\ &+ (\Omega - V_{t+\tau}^* W_\tau V_t \Omega)\, h + (V_{t+\tau}^* W_\tau V_t \Omega - V_{t+\tau}^* W_\tau V_t V_t^* T_t)\, h \| \\ &\leqslant \| (V_{t+\tau}^* T_{t+\tau} - \Omega)\, h \| + \| (V_{t+\tau} - W_\tau V_t)\, \Omega h \| + \| (\Omega - V_t^* T_t)\, h \| \,. \end{aligned} \tag{61}$$

The first and the third term in the last member of this inequality converge to zero by Theorem 2. The second term converges to zero as a consequence of equation (29). ∎

One may remark that the property of feeble oscillation of $\{\hat{W}_t h\}$ for $h \in \mathcal{D}$ allows an immediate rederivation of the intertwining relation, equation (26). In fact, if (60) is true, one obtains for $h \in \mathcal{D}$

$$V_\tau \Omega h = V_\tau (s - \lim V_t^* T_t h) = s - \lim_t V_{t-\tau}^* T_t h$$
$$= s - \lim_t V_t^* T_{t+\tau} h = s - \lim_t V_t^* T_t W_\tau h = \Omega W_\tau h .$$

This shows that $V_\tau \Omega = \Omega W_\tau$ on $\mathcal{D}$. Since only bounded operators are involved in this identity, it holds true on all of $\mathcal{H}$.

Corollary 2: Suppose that the operators $T_t^\pm$ leave their common domain $\mathcal{D}^\pm$ invariant. Then $T_t^\pm$ form a group in t on $\mathcal{D}^\pm$ if and only if $T_t^\pm \subseteq W_t^\pm$.

Proof: The 'if' part is trivial. For the 'only if' part, one uses successively Corollary 1, the group property of $\{T_t\}$ and the fact that $\{T_t\}$ commutes with $\{W_t\}$ on $\mathcal{D}$, and obtains for $h \in \mathcal{D}$ and any real τ

$$0 = \lim_t \| (T_{t+\tau} - W_\tau T_t) h \| = \lim_t \| T_t (T_\tau - W_\tau) h \| . \tag{62}$$

Now $W_\tau h \in \mathcal{D}$ according to (a) of Theorem 2, and $T_\tau h \in \mathcal{D}$ by hypothesis. Applying the isometric property of $\{T_t\}$ on $\mathcal{D}$ (cf. (b) of Theorem 2), one deduces from (62) that for any real τ and $h \in \mathcal{D}$

$$\| (T_\tau - W_\tau) h \| = 0 .$$

Therefore $T_\tau = W_\tau$ on $\mathcal{D}$. Since W_τ is bounded, it is an extension of T_τ onto all of $\mathcal{H}$. ∎

We showed in Section II, equation (29), that the total evolution V_t is asymptotically comparable to the asymptotic free evolution W_t on any finite time interval. According to Corollary 1, this property of V_t is shared by the operators T_t. Moreover, one may classify the different types of scattering systems according to whether the assertion of equation (29), namely that $s - \lim_{t \to \pm\infty} V_{t+\tau} P = W_\tau^\pm V_t P$, holds uniformly in τ or not. We shall now verify that this convergence is uniform in τ if and only if one possible choice for the family $\{T_t\}$ of Theorem 2 is the operators $\{W_t\}$:

Proposition 5: In equation (29), the strong convergence as $t \to +\infty$ of $V_{t+\tau} P$ to $W_\tau^+ V_t P$ is uniform in $\tau \geqslant 0$ if and only if the operators $V_t^* W_t^+$ converge strongly on $\mathcal{H}$ for $t \to +\infty$, and similarly for the limit $t \to -\infty$ (with $\tau \leqslant 0$).

Proof:

(i) Suppose that the convergence in (29) for $t \to +\infty$ is uniform in $\tau \geqslant 0$. Let Ω_+ be one of the possible wave operators for $t \to +\infty$ (Prop. 2), and let $\{T_t^+\}$ be the corresponding family of operators such that $\Omega_+ = s - \lim_{t \to +\infty} V_t^* T_t^+$ on some dense domain $\mathcal{D}$. From the inequality (61) it follows that the convergence in (60a) is uniform in $\tau \geqslant 0$. This in turn implies uniform convergence as $t \to +\infty$ in (59):

$$\lim_{t \to +\infty} \| (\hat{W}_{t+\tau}^+ - \hat{W}_t^+) h \| = 0 \qquad \text{uniformly in } \tau \geqslant 0 .$$

Consequently the operators $\hat{W}_t^+$ form a strong Cauchy sequence on $\mathcal{D}$. Denote by X its limit. Since $\{\hat{W}_t^+\}$ is asymptotically isometric on $\mathcal{D}$ (property (b) of Theorem 2),

X may be extended to an isometric operator on $\mathcal{H}$. Since $\{\hat{W}_t^+\}$ commutes with $\mathcal{A}_0$ on $\mathcal{D}$, this extension belongs to the center of $\mathcal{A}_0$. Therefore it commutes with its adjoint, which means that it is unitary.

Let now $h \in \mathcal{D}$. Then

$$\| (V_t^* W_t^+ - \Omega_+ X^*) h \| \leqslant \| (V_t^* W_t^+ - V_t^* T_t^+ X^*) h \|$$
$$+ \| (V_t^* T_t^+ X^* - \Omega_+ X^*) h \| = \| (X - \hat{W}_t^+) h \| + \| (V_t^* T_t^+ - \Omega_+) X^* h \| .$$

In the last member of this inequality, the first term converges to zero as $t \to +\infty$ by the definition of X and the second term by Theorem 2. This shows the strong convergence of $V_t^* W_t^+$ on $\mathcal{D}$. Since these operators are unitary, the convergence holds on all of $\mathcal{H}$.

(ii) Conversely, suppose that $V_t^* W_t^+$ converges strongly on $\mathcal{H}$ as $t \to +\infty$. According to the converse of Theorem 2, $s - \lim_{t \to +\infty} V_t^* W_t^+ = \Omega_+$. Since V_t and W_t^+ are unitary, it follows that $s - \lim_{t \to +\infty} (W_t^+)^* V_t P = \Omega_+^*$. Hence $\{(W_t^+)^* V_t P\}$ forms a strong Cauchy sequence in $\mathcal{H}$: Given $\varepsilon > 0$ and $g \in P\mathcal{H}$, there exists a number T such that

$$\| ((W_s^+)^* V_s - (W_t^+)^* V_t) g \| < \varepsilon \qquad \text{if } s, t \geqslant T . \tag{63}$$

Given $\tau \geqslant 0$ and $t > T$, set $s = t + \tau$. Then $s \geqslant T$. Therefore (63) implies together with the multiplication law of the group $\{W_t^+\}$ that

$$\| (V_{t+\tau} - W_\tau^+ V_t) g \| < \varepsilon \qquad t \geqslant T .$$

This shows that the convergence of (29) is uniform in $\tau \geqslant 0$. ∎

One may now divide the asymptotic behaviour of admissible scattering systems into four classes according to whether the convergence in equation (29) is uniform in τ or not and whether there is a renormalization of the free energy or not:

Case 1: The strong limit of $V_t^* U_t$ exists. This is the asymptotic condition of standard scattering theory which is satisfied for the short range potentials. One has in particular $W_t = U_t = T_t$.

If the strong limit of $V_t^* U_t$ does not exist, i.e. if $U_t \neq T_t$, we obtain one of the following three situations:

Case 2: $W_t = U_t \neq T_t$. This case will be investigated in more detail in Section IV with $H_0 = -\Delta/2m$. It includes in particular the scattering by a Coulomb potential (cf. Dollard [3]) and by potentials of the form $V(r) = r^{-\beta}$ with $3/4 \leqslant \beta < 1$ (cf. Section V). More generally, if (A1), (A2) and (A3) are satisfied for a local interaction $V(\boldsymbol{x})$ which tends to zero as $|\boldsymbol{x}| \to \infty$, the scattering system will belong to this class.

Case 3: $W_t \neq U_t$ but the strong limit of $V_t^* W_t$ exists (i.e. $W_t = T_t$). In this case the asymptotic free energy operator K is different from the kinetic energy operator H_0. One obtains a simple model for this renormalization of the free energy if the interaction is a sum $V = V_0 + V_1$ in such a way that V_0 is a function of H_0 and V_1 satisfies the usual asymptotic condition with respect to the modified unperturbed operator $K = H_0 + V_0$. Then $W_t = \exp(-i(H_0 + V_0)t)$ and $\Omega = s - \lim V_t^* W_t$. A trivial example is furnished by a constant interaction $V = \alpha I$. In this case $W_t = \exp(-i\alpha t) U_t$, and hence $\Omega_\pm = I$ and $S = I$.

Case 4: $W_t \neq U_t$ and neither of the strong limits of $V_t^* U_t$ and $V_t^* W_t$ exists, i.e. the families $\{U_t\}$, $\{W_t\}$ and $\{T_t\}$ are pairwise different from each other. This is the most general case. The corresponding interactions combine the features of the interactions belonging to Case 2 and those belonging to Case 3.

IV. Consequences of Symmetry

In this part we examine some properties of scattering systems which are invariant under time reversal or under rotations of three-dimensional space. In the latter case we shall obtain sufficient conditions for (A 2) and (A 3) to hold.

We first treat time reversal invariance. We say that the scattering system is *time reversal invariant* if there exists on $\mathcal{H}$ an antiunitary operator θ which leaves the subspace $P\,\mathcal{H}$ of scattering states invariant and which commutes with H_0 and with H. We shall use none of the other properties of the time reversal operator, i.e. we only need to require that for any complex numbers α_1, α_2 and for f_1, f_2 in $\mathcal{H}$

$$\theta\,(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1^*\,\theta f_1 + \alpha_2^*\,\theta f_2\,, \tag{64}$$

$$(\theta f_1, \theta f_2) = (f_2, f_1) \tag{65}$$

and that

$$[\theta, H_0] = [\theta, H] = [\theta, P] = 0\,. \tag{66}$$

This suffices to establish the following assertion:

Proposition 6: Suppose that the conditions (A 1), (A 2) and (A 3) are satisfied in the limit $t \to -\infty$, and let Ω_- be one of the corresponding wave operators. Suppose in addition that the scattering system is time reversal invariant with $P = \Omega_- \Omega_-^*$. Then (A 1), (A 2) and (A 3) are satisfied also in the limit $t \to +\infty$, the S-operator is unitary, and $W_t^+ = W_t^-$. Moreover, the two families of operators $\{T_t^\pm\}$ of Theorem 2 may be chosen in such a way that they possess a common dense domain $\mathcal{D}$ which is invariant under θ and such that $T_t^+ = (T_{-t}^-)^*$ on $\mathcal{D}$.

Proof: (i) We first apply the hypothesis of time reversal invariance to deduce that

$$\theta^{-1} B\,\theta = B^* \text{ for } B \in \mathcal{A}_0' \text{ or } \mathcal{A}_P'\,. \tag{67}$$

The method for deriving (67) from (66) is similar for $B \in \mathcal{A}_0'$ and for $B \in \mathcal{A}_P'$, so that we shall give it only for the former case.

We shall construct a unitary operator U in $\mathcal{A}_0$ and a conjugation J which commutes with H_0 such that $\theta = U\,J$ (J is an antiunitary operator such that $J^2 = I$). We again use the direct integral representation (43) of $\mathcal{H}$ with respect to H_0. For every λ in the spectrum of H_0, let J_λ be an arbitrary conjugation in $\mathcal{H}_\lambda$. (A particular J_λ may be constructed by choosing an orthonormal basis of vectors in $\mathcal{H}_\lambda$ and defining J_λ as that antilinear operator which leaves this basis invariant.) We then define J on $\mathcal{H}$ by $J f = \{J_\lambda f(\lambda)\}$.

Next, let B belong to the center of $\mathcal{A}_0$. In the spectral representation of H_0, it is given by an essentially bounded measurable function $B(\lambda)$ on $\sigma(H_0)$, equation (44).

To its adjoint B^* corresponds the complex-conjugate function $B^*(\lambda)$. Hence, for any $f \in \mathcal{H}$

$$J\, B f = \{J_\lambda\, B(\lambda)\, f(\lambda)\} = \{B^*(\lambda)\; J_\lambda f(\lambda)\} = B^*\; J f\,. \tag{68}$$

From (68) it follows that J commutes with all self-adjoint operators in $\mathcal{A}_0'$ and with H_0. Now $U \equiv \theta\, J$ is unitary and belongs to $\mathcal{A}_0$. Hence $\theta^{-1}\, B\, \theta = J\, U^{-1}\, B\, U\, J = J\, B\, J = B^*$, which proves (67) for $B \in \mathcal{A}_0'$.

(ii) We now use the hypothesis that (A1), (A2) and (A3) are satisfied for $t \to -\infty$. Given a wave operator Ω_-, there exists $\{T_t^-\}$ having the properties (a)–(d) of Theorem 2 in the limit $t \to -\infty$. Its explicit construction as given in the proof of Theorem 2, equations (48) and (50), was based on the choice of an arbitrary cyclic vector e for $\mathcal{A}_0$. We shall now fix upon cyclic vectors of a particular form:

Let $\varepsilon(\lambda)$ be a real-valued function on $\sigma(H_0)$ which is different from zero almost everywhere and square-integrable over $\sigma(H_0)$ with respect to the Lebesgue measure. Further, for every $\lambda \in \sigma(H_0)$, let $j(\lambda)$ be a unit vector in $\mathcal{H}_\lambda$ which is invariant under the conjugation J_λ. Then $e = \{\varepsilon(\lambda)\, j(\lambda)\}$ is a cyclic vector for $\mathcal{A}_0$, and $J\, e = e$. Let $\mathcal{D} = \{\mathcal{A}_0\, e\}$ be the common dense domain of the family $\{T_t^-\}$, and denote by $\xi_t(\lambda)$ the corresponding function (48) obtained by setting $Y_t = V_t\, \Omega_-$ in (48). Let $h \in \mathcal{D}$, i.e. $h = A\, e$ for some $A \in \mathcal{A}_0$. Then

$$\theta\, h = \theta\, A\, e = \theta\, A\, \theta^{-1}\, U\, J\, e = \theta\, A\, \theta^{-1}\, U\, e\,. \tag{69}$$

Since $[\theta, H_0] = 0$, the correspondence $A \to \theta\, A\, \theta^{-1}$ is an automorphism of $\mathcal{A}_0$. Hence $\theta\, A\, \theta^{-1}\, U \in \mathcal{A}_0$, and (69) implies that $\theta\, h \in \mathcal{D}$. This shows the invariance of $\mathcal{D}$ under θ.

(iii) We now define a family $\{T_t^+\}$ with the same common domain $\mathcal{D}$ as above by $T_t^+ = (T_{-t}^-)^*$. This definition makes sense, since it follows from Theorem 6.4 of Ref. [12] that $(T_t^-)^*$ exists, has the same domain $\mathcal{D}_t$ as T_t^- and is given by

$$(T_t^-)^* f = \{\xi_t^*(\lambda)\, f(\lambda)\} \qquad f \in \mathcal{D}_t\,. \tag{70}$$

We shall verify that this family $\{T_t^+\}$ satisfies the hypothesis of the converse of Theorem 2 in the limit $t \to +\infty$, which then implies that (A1), (A2) and (A3) are satisfied for $t \to +\infty$.

First, since $\mathcal{D}$ is the same for $\{T_t^+\}$ and for $\{T_t^-\}$, condition (a) of Theorem 2 is trivially true for $\{T_t^+\}$. Next, one sees from (50) and (70) that $\|\, T_t^-\, h\| = \|\,(T_t^-)^*\, h\|$ for $h \in \mathcal{D}$. Since the left-hand member of this equality converges to $\|\,h\|$ as $t \to -\infty$, it follows that $\|\, T_t^+\, h\| \to \|\,h\|$ as $t \to +\infty$. Hence the family $\{T_t^+\}$ satisfies (b) of Theorem 2.

It follows from (67) that

$$\theta^{-1}\, V_t\, \theta = V_{-t}\,. \tag{71}$$

By reasoning similarly as in (i), one may also deduce that

$$\theta^{-1}\, T_t^-\, \theta\, h = (T_t^-)^*\, h \text{ for } h \in \mathcal{D}\,. \tag{72}$$

Hence, for $h \in \mathcal{D}$

$$s - \lim_{t \to +\infty} V_t^*\, T_t^+\, h = s - \lim_{t \to -\infty} V_{-t}^*(T_t^-)^*\, h = \theta^{-1}\, s - \lim_{t \to -\infty} V_t^*\, T_t^-\, \theta\, h\,.$$

Since $\theta\, h \in \mathcal{D}$, the last limit exists by the construction of $\{T_t^-\}$. This defines Ω_+ and proves that $\{T_t^+\}$ satisfies also the requirement (c) of Theorem 2, and that $\Omega_+ = \theta^{-1}\,\Omega_-\,\theta$. Since $[\theta, P] = 0$, one has $\Omega_+\,\Omega_+^* = P$. Therefore the ranges of Ω_+ and of Ω_- are the same, and all the hypothesis of the converse of Theorem 2 are true.

(iv) The unitarity of S follows from $\Omega_+^*\,\Omega_+ = I$. Using (26), (66), (71) and (67), one has also

$$W_t^+ = \Omega_+^*\,V_t\,\Omega_+ = \theta^{-1}\,\Omega_-^*\,\theta\,V_t\,\theta^{-1}\,\Omega_-\,\theta$$
$$= \theta^{-1}\,\Omega_-^*\,V_{-t}\,\Omega_-\,\theta = \theta^{-1}\,W_{-t}^-\,\theta = W_t^- . \quad \blacksquare$$

We now consider rotational invariance. A scattering system is said to be *invariant under rotations* if there exists in $\mathcal{H}$ a representation $U(R)$ of the group 0(3) of rotations of three-dimensional Euclidean space in such a way that $U(R)$ leaves the subspace $P\,\mathcal{H}$ of scattering states invariant and commutes with H_0 and with H, i.e.

$$[U(R), H_0] = [U(R), H] = [U(R), P] = 0 . \tag{73}$$

We denote by L_i $(i = 1, 2, 3)$ the infinitesimal generators of the rotations about the i-th coordinate axis and define $L^2 = \sum_{i=1}^{3} L_i^2$. It will be assumed that $\{H_0, L^2, L_3\}$ form a complete set of commuting observables. (This means that we consider only particles with spin zero.)

Let F denote the projection operator onto the subspace corresponding to the absolutely continuous part of the spectrum of H, and let R_z^0 and R_z be the resolvent operators associated with the groups $\{U_t\}$ and $\{V_t\}$ respectively. One may then assert the following consequences of rotational invariance:

Proposition 7: Suppose that (A1) is satisfied with $P \leqslant F$ and that the scattering system is invariant under rotations.

(a) If the von Neumann algebra generated by $\{H\,P, L^2, L_3\}$ is maximal abelian in $P\,\mathcal{H}$ and if $(R_z - R_z^0)\,P$ is compact for at least one non-real z, then (A2) is true.

(b) If in addition $P = F$ and the absolutely continuous part of the spectrum of H is the same as the spectrum of H_0, then (A3) is true.

(c) If the hypothesis of (a) are satisfied and (A3) is true, then S is unitary and $W_t^+ = W_t^- = \exp(-i\,H_0\,t)$.

The content of this proposition is analogous to results obtained by Lavine [14]. Part (b) is equivalent to the affirmation that the homomorphisms $\mu_\pm$ defined by (A1) are invertible, which corresponds to Theorem 2.9 of [14], and part (a) ensures the existence of wave operators (cf. Theorem 3.9 of [14]).

Proof: (i) Let the hypothesis of (a) be true. Since the spectrum of H is absolutely continuous on $P\,\mathcal{H}$, $V_t\,P$ converges weakly to zero by the Riemann-Lebesgue lemma. Let ξ $(Im\,\xi \neq 0)$ be such that $(R_\xi^0 - R_\xi)\,P$ is compact. Since a compact operator transforms each weakly convergent sequence of vectors into a strongly convergent sequence, $(R_\xi - R_\xi^0)\,V_t\,P$ converges strongly to zero. Hence, for any $f \in \mathcal{H}$

$$\lim_t \|(V_t^*\,R_\xi^0\,V_t\,P - R_\xi\,P)\,f\| = \lim_t \|(R_\xi^0 - R_\xi)\,V_t\,P\,f\| = 0 .$$

This shows that $\mu(R_\xi^0) = R_\xi\,P$.

Denote by Q_{lm} $(l = 0, 1, 2, \ldots, |m| \leqslant l)$ the projection operators onto the common eigenvectors of L^2 and L_3, and let $\mathcal{B}$ be the maximal abelian von Neumann algebra generated by $\{H_0, L^2, L_3\}$.

The rotational invariance, equation (73), implies that $\mu(Q_{lm}) = Q_{lm} P$. Hence

$$\{R_\xi P, Q_{lm} P\} \subset \mu(\mathcal{B}) . \tag{74}$$

The restriction of μ to $\mathcal{B}$ is ultraweakly continuous (cf. the proof of Prop. 1, part (iii)), and hence $\mu(\mathcal{B})$ is a von Neumann algebra. Using also the fact that $\{H\}_P'' = \{R_\xi\}_P''$ ([15], Theorem III.6.5), it follows from (74) that

$$\{H, L^2, L_3\}_P'' = \{R_\xi P, Q_{lm} P\}_P'' \subseteq \mu(\mathcal{B})_P'' = \mu(\mathcal{B}) . \tag{75}$$

Since $\{H, L^2, L_3\}_P''$ is maximal abelian in $P\,\mathcal{H}$ by hypothesis and $\mu(\mathcal{B})$ is abelian, equation (75) implies that $\{H, L^2, L_3\}_P'' = \mu(\mathcal{B})$. Therefore $\mu(\mathcal{B})$ is maximal abelian in $P\,\mathcal{H}$, which shows that (A2) is satisfied and proves (a).

(ii) According to Proposition 2, the hypothesis of (a) imply the existence of wave operators Ω satisfying $\Omega^* \Omega = E$. We showed above that there exists a non-real number ξ such that

$$\Omega R_\xi^0 \Omega^* = R_\xi P . \tag{76}$$

For all non-real z, define $\Gamma_z = \Omega R_z^0 \Omega^*$. The operators R_z^0 satisfy the resolvent equation

$$R_z^0 - R_{z'}^0 = (z - z')\, R_z^0 R_{z'}^0 \quad \text{for non-real } z, z' . \tag{77}$$

Using $[R_z^0, E] = 0$, one may deduce from (77) that the operators Γ_z satisfy this same equation in the subspace $P\,\mathcal{H}$. As $\Gamma_\xi = R_\xi P$ according to (76), this implies that there exists a unique linear transformation T on $P\,\mathcal{H}$ whose resolvent exists and coincides with Γ_z for non-real z (cf. Theorem 4.10 of [12]). Since $\Gamma_\xi = R_\xi P$, if follows that $T = H P$, and hence $\Gamma_z = R_z P$ for all non-real z. Hence equation (76) holds true for all non-real ξ.

Let F_λ and $E_\lambda^\pm$ be the spectral families of the operators H and $H_0 E_\pm$ respectively. By the usual construction of the spectral family of a self-adjoint operator from its resolvent (cf. for instance [16], Section 65), it follows from equation (76) that

$$\Omega_\pm E_\lambda^\pm \Omega_\pm^* = F_\lambda P .$$

Therefore, for every essentially bounded measurable function f on $\sigma(H P)$, the hypothesis of (a) imply that

$$\Omega_\pm f(H_0 E_\pm)\, \Omega_\pm^* = f(H P) . \tag{78}$$

Therefore $\sigma(H_0 E_+) = \sigma(H P) = \sigma(H_0 E_-)$. Since $E_\pm \in \mathcal{A}_0'$, they are spectral projections of H_0. Hence $E_+ = E_-$.

(iii) Under the additional hypothesis of (b), the spectrum of $H_0 E_\pm$ coincides with that of H_0, which implies $E_\pm = I$. Hence (A3) is true (cf. the last paragraph of the proof of Theorem 2). This proves (b).

(iv) If the hypothesis of (a) and (A3) are true, one has $E_- = I$ (cf. the proof of Theorem 1). Then also $E_+ = I$ from the last remark of (ii), and hence $S S^* = S^* S = I$. Setting in (78) $f(\lambda) = \exp(-i \lambda t)$, one obtains

$$\Omega_\pm U_t \Omega_\pm^* = V_t P \tag{79}$$

which shows that $W_t^+ = W_t^- = U_t$ and proves (c). ∎

In order to establish that a given scattering system satisfies (A 1), (A 2) and (A 3), it is useful to apply the converse of Theorem 2. For this, one will have to exhibit two families of operators $\{T_t^{\pm}\}$ satisfying (a), (b) and (c) of Theorem 2 and verify that the ranges $P_{\pm}$ of the corresponding wave operators reduce V_t and are identical. This scheme was realized for the non-relativistic scattering by a Coulomb potential by Dollard [3], who also proved the completeness of the wave operators (i.e. that the ranges of $\Omega_{\pm}$ coincide with the projection operator F onto the subspace determined by the absolutely continuous part of the spectrum of H). In Section V we shall work out this same program for a more general class of potentials in non-relativistic scattering theory. Before doing this, we give sufficient conditions for $P_+ = P_-$ to hold and for the completeness of the wave operators.

Proposition 8: Suppose that the scattering system is invariant under rotations and that there exist two families $\{T_t^{\pm}\}$ of operators satisfying (a), (b) and (c) of Theorem 2. Denote by $P_{\pm}$ the projection operators onto the ranges of $\Omega_{\pm}$ and by Q_{lm} those onto the common eigenvectors of L^2 and L_3, and suppose $P_{\pm} \leqslant F$. If the von Neumann algebra generated by $\{H\,F, L^2, L_3\}$ is maximal abelian in $F\,\mathcal{H}$ and if $(R_z - R_z^0)\,F$ is compact for at least one non-real z, then $P_{\pm}$ reduce V_t, and $P_+ = P_-$. If in addition the absolutely continuous part of the spectrum of H is the same as the spectrum of H_0 and the spectrum of $H_0\,Q_{lm}$ is independent of the values of l and m, then the wave operators are complete, i.e. $P_+ = P_- = F$.

Proof: (i) The hypothesis of invariance under rotations implies that $T_t^{\pm}$ as well as V_t commute with all rotations, and hence so do $\Omega_{\pm} = s - \lim V_t^*\,T_t^{\pm}$ and $P_{\pm} = \Omega_{\pm}\,\Omega_{\pm}^*$. One may write $\mathcal{H} = \sum_{lm}^{\oplus} Q_{lm}\,\mathcal{H}$, and every subspace $Q_{lm}\,\mathcal{H}$ reduces H_0, H, F, $\Omega_{\pm}$ and $P_{\pm}$. If A is any of these operators, we denote by $(A)_{lm}$ its restriction to the subspace $Q_{lm}\,\mathcal{H}$.

Using the hypothesis that $(R_z - R_z^0)\,F$ is compact for at least one non-real z and that $P_{\pm} \leqslant F$, one may deduce in the same way as in the preceding proof that

$$\Omega_{\pm}\,U_t\,\Omega_{\pm}^* = V_t\,P_{\pm} = P_{\pm}\,V_t\,P_{\pm}\,. \tag{80}$$

It follows from (80) that $[V_t, P_{\pm}] = 0$. Since Q_{lm} commutes with all the operators occuring in (80), this equation holds true in every subspace $Q_{lm}\,\mathcal{H}$ individually. For every pair $\{l, m\}$ the spectra of $(H_0)_{lm}$, $(H\,P_+)_{lm}$ and $(H\,P_-)_{lm}$ are therefore identical.

From the hypothesis that $\{H, L^2, L_3\}_F''$ be maximal abelian in $F\,\mathcal{H}$, it follows that $(H\,F)_{lm}$ generates a maximal abelian algebra in $Q_{lm}\,F\,\mathcal{H}$. This means that the spectrum of $(H\,F)_{lm}$ is simple ([9], Section 3). Since $(P_{\pm})_{lm} \leqslant (F)_{lm}$, the operator $(H\,F)_{lm}$ is reduced by the subspace $(P_{\pm})_{lm}\,\mathcal{H}$. These properties of $(H\,F)_{lm}$ imply that $(P_{\pm})_{lm}$ are spectral projections of $(H\,F)_{lm}$ ([12], Theorem 7.16). Since $\sigma((H\,P_+)_{lm}) = \sigma((H\,P_-)_{lm})$, one has $(P_+)_{lm} = (P_-)_{lm}$ and ergo $P_+ = P_-$ (cf. also Theorem 3.3 of [17]). This proves the first part of the proposition.

(ii) Under the additional hypothesis that $\sigma((H_0)_{lm})$ is independent of the values of l and m, i.e. $\sigma((H_0)_{lm}) = \sigma(H_0)$ for all pairs $\{l, m\}$, one has $\sigma((H\,P_{\pm})_{lm}) = \sigma(H_0)$ for all pairs $\{l, m\}$. If in addition $\sigma(H\,F) = \sigma(H_0)$, this implies that $\sigma((H\,P_{\pm})_{lm}) = \sigma(H\,F)$. Since $\sigma((H\,P_{\pm})_{lm}) \subseteq \sigma((H\,F)_{lm}) \subseteq \sigma(H\,F)$, this implies $\sigma((H\,P_{\pm})_{lm}) = \sigma((H\,F)_{lm})$.

Since $(P_\pm)_{lm}$ are spectral projections of $(H\,F)_{lm}$, one has $(P_\pm)_{lm} = (F)_{lm}$ and ergo $P_\pm = F$. ∎

To conclude this part, we apply some results of Rejto [18] and of Weidmann [19] to indicate a class of interactions in non-relativistic potential scattering which fulfil the hypothesis of (a) and (b) of Proposition 7 and some of the hypothesis of Proposition 8. Let $H_0 = -\Delta/2\,m$ be the Schrödinger operator and $H = H_0 + V$ where $V = V(r)$ is a spherically symmetric real potential defined in Euclidean three-space which has the following properties:

There exist three real numbers $C > 0$, $R > 0$ and $s < 3/2$ such that

(P1) for $r < R$: $\quad r^s\,|V(r)| \leqslant C\,,$

(P2) for $r \geqslant R$: $\quad |V(r)| \leqslant C\,, \qquad \lim_{r\to\infty} V(r) = 0$

and $V(r) = V_1(r) + V_2(r)$ where $V_1(r)$ is of bounded variation and $V_2(r)$ belongs to $L^1(R, \infty)$.

For such potentials it follows from Theorem V.5.4 of [15] that H is essentially self-adjoint with domain $\mathcal{D}_H = \mathcal{D}_{H_0}$. Furthermore these potentials satisfy the conditions of Theorem 2.1 of [18] and of Corollary 6.3 of [19]. The conclusions of these two theorems will now be applied to verify the hypothesis of our Propositions 7 and 8 (apart from the asymptotic convergence).

Theorem 2.1 of [18] states that $V\,R^0_{z=i}$ is compact. Since $R_i - R_i^0 = -R_i\,V\,R_i^0$ and R_i is bounded, it follows that $(R_i - R_i^0)\,P$ is compact for any projection operator P on $\mathcal{H}$ (in particular for $P = F$). Theorem 2.1 of [18] asserts also that the essential spectrum of H is the same as that of H_0 and coincides with the interval $[0, \infty)$. Corollary 6.3 of [19] affirms that the spectrum of H (as well as that of H_0) is absolutely continuous on $(0, \infty)$. Combining these last two statements, one sees that the spectrum of H_0 as well as the absolutely continuous part of the spectrum of H coincide with the interval $[0, \infty)$.

Finally, the operator $(H)_{lm}$ is unitarily equivalent to the differential operator $-d^2/dr^2 + l\,(l+1)\,r^{-2} + V(r)$ acting in $L^2(0, \infty)$ (cf. [19], Section 6). If $V(r)$ satisfies the conditions (P1) and (P2), then the spectrum of this differential operator is simple on the interval $(0, \infty)$ ([19], Theorem 5.1 and Corollary 6.3). Hence the von Neumann algebra generated by $(H\,F)_{lm}$ on $Q_{lm}\,F\,\mathcal{H}$ is maximal abelian. From this it follows that the von Neumann algebra $\{H, L^2, L_3\}''_F$ is maximal abelian in $F\,\mathcal{H}$. Furthermore, for $P \leqslant F$ and $P \in \{H, L^2, L_3\}''_F$, the reduced algebra $\{H, L^2, L_3\}''_P$ is also maximal abelian. This shows that the above-mentioned potentials satisfy the hypothesis of (a) and (b) of Proposition 7.

V. Long-range Potentials

In this section we establish the existence of a class of long-range potentials which decrease more slowly than the Coulomb potential at large distances from the scattering center and which satisfy the asymptotic condition (A1)–(A3). The main result is the following:

Proposition 9: Let $H_0 = -\Delta/2\,m$ and $H = H_0 + \gamma\,x^{-\beta}$, where γ is real, $3/4 \leqslant \beta < 1$ and $x = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. Then (A1), (A2) and (A3) are satisfied with $P = F$. Moreover $W_t^+ = W_t^- = U_t$.

Proof: To show that (A1), (A2) and (A3) are satisfied, we shall verify that the hypothesis of the converse of Theorem 2 are true. For this, it suffices to construct a family $\{T_t^+\}$ satisfying (a), (b) and (c) of Theorem 2 in the limit $t \to +\infty$ and such that $P_+ \leqslant F$. Since the potentials considered in the Proposition satisfy the conditions (P1) and (P2) at the end of Section IV, it then follows from Proposition 8 that $P_+ = F$ and hence $[P_+, V_t] = 0$. The converse of Theorem 2 then implies that the statements of (A1), (A2) and (A3) are true in the limit $t \to +\infty$. Furthermore, if we choose for θ the operator of complex conjugation of the wave functions, the scattering system is time reversal invariant. According to Proposition 6 the statements of (A1), (A2) and (A3) are then verified also in the limit $t \to -\infty$ with $P_- = P_+ = F$.

We now define for every $t > 0$ a self-adjoint operator which is a function of H_0, namely

$$H_0(t) = H_0\, t + \gamma\, m^\beta\, (1-\beta)^{-1}\, (2\, m\, H_0)^{-\beta/2}\, t^{1-\beta} \tag{81}$$

and we set

$$T_t \equiv T_t^+ = \exp\left(-i\, H_0(t)\right) . \tag{82}$$

Since T_t is unitary and belongs to $\mathcal{A}_0'$, the conditions (a) and (b) of Theorem 2 are trivially verified. The asymptotic convergence (condition (c) of Theorem 2) will be established through a sequence of lemmas the proofs of which are collected in the Appendix.

The underlying Hilbert space may be represented as $\mathcal{H} = L^2(R^3)$. We shall write $f(\boldsymbol{x})$ for the function in $L^2(R^3)$ corresponding to a vector f in the $\boldsymbol{x}$-representation and $\tilde{f}(\boldsymbol{\varkappa})$ for its Fourier transform. We also define a dense set $\mathcal{C}$ in $\mathcal{H}$ as follows: f belongs to $\mathcal{C}$ if $\tilde{f}(\boldsymbol{\varkappa}) \in \mathcal{S}$ (the set of all infinitely differentiable functions of rapid decrease) and vanishes in some neighbourhood of the origin. (The same set $\mathcal{C}$ of test functions was used by Dollard [3].) $\mathcal{C}$ is invariant under $H_0(t)$ and T_t. For f in $\mathcal{C}$, we define $f_t \equiv \hat{U}_t f \equiv U_t^*\, T_t f$, or explicitly in the $\boldsymbol{\varkappa}$-representation:

$$\tilde{f}_t(\boldsymbol{\varkappa}) = \exp\left(-i\, \mu\, \varkappa^{-\beta}\, t^\alpha\right) \tilde{f}(\boldsymbol{\varkappa}) \tag{83}$$

where $\varkappa = |\boldsymbol{\varkappa}|$, $\alpha = 1 - \beta$ and $\mu = \gamma\, m^\beta\, (1-\beta)^{-1}$. According to Corollary 1 of Theorem 2, this family $\{f_t\}$ should be feebly oscillating. This is indeed the case:

Lemma 1: Let $f \in \mathcal{C}$. Then the family $\{f_t\}$ of vectors defined by equation (83) is feebly oscillating.

Since $\hat{U}_t$ is unitary, the convergence in Lemma 1 can be extended to all of $\mathcal{H}$, i.e. $\lim_{t\to+\infty} \| (\hat{U}_{t+\tau} - \hat{U}_t) f \| = 0$ for all vectors f in $\mathcal{H}$. Hence equation (60a) is true on all of $\mathcal{H}$ with $W_\tau^+ = U_\tau$. Assuming the asymptotic convergence, this implies the intertwining relation $V_t\, \Omega_+ = \Omega_+\, U_t$ (cf. the remark after Corollary 1 in Section III). The intertwining relation implies that $[P_+, V_t] = [\Omega_+\, \Omega_+^*, V_t] = 0$ and also that $P_+ \leqslant F$, since the spectrum of H_0 is absolutely continuous. The property $W_t^+ = W_t^-$ $(= U_t)$ follows from time reversal invariance.

It is useful to associate with each f in $\mathcal{C}$ another sequence of vectors F_t whose $\boldsymbol{x}$-representation is defined as

$$F_t(\boldsymbol{x}) \equiv \int d^3y\, \exp\left(-i\, \boldsymbol{x} \cdot \boldsymbol{y}\right) \left[\exp\left(\frac{i\, m\, y^2}{2\, t}\right) - 1\right] f_t(\boldsymbol{y}) . \tag{84}$$

The following lemma gives some estimates concerning these functions for large times t:

Lemma 2: Let $\eta > 1 - \beta$. Then

$$\text{(a)} \quad |F_t(\boldsymbol{x})| \leqslant C_1\, t^{7\eta/2-1} + o(t^{-K}) \,, \tag{85}$$

$$\text{(b)} \quad \|F_t\| \leqslant C_2\, t^{2\eta-1} + o(t^{-K}) \tag{86}$$

where C_1 and C_2 are constants and K is an arbitrary positive number.

We shall now give the convergence proof for $V_t^* T_t$. It is based on the following standard result ([15], Section X, Theorem 3.7):

Lemma 3: Let $\{\Omega(t)\}$ be a family of operators and f a vector in $\mathcal{H}$ such that the derivatives $\partial/\partial t(\Omega(t) f)$ exist and are strongly continuous in t. In order that the sequence $\{\Omega(t) f\}$ converge strongly as $t \to +\infty$, it suffices that the integral

$$\int_{t_0}^{\infty} \left\| \frac{\partial}{\partial t}\, \Omega(t)\, f \right\| dt$$

exist for some finite value of t_0.

We shall verify the hypothesis of this lemma for $\Omega(t) = V_t^* T_t$ and f in $\mathcal{C}$. Since V_t and T_t are unitary, the convergence of $\Omega(t)$ can then be extended to all of $\mathcal{H}$, and the requirement (c) of Theorem 2 will be satisfied with $\mathcal{D} = \mathcal{H}$.

For f in $\mathcal{C}$ one has

$$\left\| \frac{\partial}{\partial t}\, V_t^*\, T_t f \right\| = \gamma\, \| (x^{-\beta} - m^\beta\, t^{-\beta}\, (2\, m\, H_0)^{-\beta/2})\, T_t f \| \,. \tag{87}$$

We now introduce the explicit representation of the free evolution $U_t = \exp(-i\, H_0\, t)$ as an integral operator acting on the functions $h(\boldsymbol{x})$ in $L^2(R^3)$:

$$(U_t\, h)\,(\boldsymbol{x}) = \left(\frac{m}{2\,\pi\, i\, t}\right)^{3/2} \int d^3y \exp\left(\frac{i\, m\, |\boldsymbol{x} - \boldsymbol{y}|^2}{2\, t}\right) h(\boldsymbol{y}) \,.$$

Applying this representation to the function $f_t(\boldsymbol{x}) = (\hat{U}_t f)\,(\boldsymbol{x})$, we decompose $T_t f$ into a sum of two vectors

$$T_t f = U_t\, \hat{U}_t f = U_t f_t = f_t^{(1)} + f_t^{(2)} \tag{88}$$

where

$$f_t^{(1)}(\boldsymbol{x}) = \left(\frac{m}{2\,\pi\, i\, t}\right)^{3/2} \exp\left(\frac{i\, m\, x^2}{2\, t}\right) \int d^3y \exp\left(-\frac{i\, m\, \boldsymbol{x} \cdot \boldsymbol{y}}{t}\right) f_t(\boldsymbol{y}) \,, \tag{89}$$

$$f_t^{(2)}(\boldsymbol{x}) = \left(\frac{m}{2\,\pi\, i\, t}\right)^{3/2} \exp\left(\frac{i\, m\, x^2}{2\, t}\right) \int d^3y \exp\left(-\frac{i\, m\, \boldsymbol{x} \,.\, \boldsymbol{y}}{t}\right) \times \left[\exp\left(\frac{i\, m\, y^2}{2\, t}\right) - 1\right] f_t(\boldsymbol{y}) \,. \tag{90}$$

Introducing the Fourier transform $\tilde{f}_t(\boldsymbol{\varkappa})$ of $f_t(\boldsymbol{x})$, and in view of the definition (84), one may express these two functions as

$$f_t^{(1)}(\boldsymbol{x}) = \left(\frac{m}{i\,t}\right)^{3/2} \exp\left(\frac{i\,m\,x^2}{2\,t}\right) \tilde{f}_t\left(\frac{m\,\boldsymbol{x}}{t}\right) \tag{91}$$

$$f_t^{(2)}(\boldsymbol{x}) = \left(\frac{m}{2\,\pi\,i\,t}\right)^{3/2} \exp\left(\frac{i\,m\,x^2}{2\,t}\right) F_t\left(\frac{m\,\boldsymbol{x}}{t}\right) \tag{92}$$

We define also the vector g in C by

$$g = (2\,m\,H_0)^{-\beta/2} f \tag{93}$$

and consider the decomposition of $T_t\,g$ defined by equations (88)–(90):

$$T_t\,g = g_t^{(1)} + g_t^{(2)}\,. \tag{94}$$

It follows immediately from (91) and (93) that

$$g_t^{(1)}(\boldsymbol{x}) = \left(\frac{m\,x}{t}\right)^{-\beta} f_t^{(1)}(\boldsymbol{x})\,. \tag{95}$$

Introducing (88) and (94) into (87), one finds that the contributions from $f_t^{(1)}$ and $g_t^{(1)}$ cancel as a consequence of (95), and one is left with

$$\left\| \frac{\partial}{\partial t} V_t^*\,T_t f \right\| = \gamma\,\| x^{-\beta} f_t^{(2)} - m^\beta\,t^{-\beta}\,g_t^{(2)} \| \leqslant \gamma\,\| x^{-\beta} f_t^{(2)} \| + \gamma\,m^\beta\,t^{-\beta}\,\| g_t^{(2)} \|\,. \tag{96}$$

We now give estimates for the two norms of the last member of (96) by applying Lemma 2. For the first term, we use the explicit form (92) for $f_t^{(2)}$:

$$\| x^{-\beta} f_t^{(2)} \| = \left(\frac{m}{2\,\pi\,t}\right)^{3/2} \left[\int d^3x\;x^{-2\beta} \left| F_t\left(\frac{m\,\boldsymbol{x}}{t}\right)\right|^2\right]^{1/2}. \tag{97}$$

For fixed and large t, we divide the integration over R^3 into two parts: an integral over the sphere $S_t \equiv \{\boldsymbol{x} \mid x \leqslant t^\delta\}$ and an integral over the exterior of this sphere, where δ is some positive constant. In the integral over S_t, we make the substitution $y_i = t^{-\delta}\,x_i$ and then apply the inequality (85). Since $x^{-2\beta}$ is integrable at $\boldsymbol{x} = 0$ for $\beta < 1$, this gives

$$\left(\frac{m}{2\,\pi\,t}\right)^{3/2} \left[\int_{x \leqslant t^\delta} d^3x\;x^{-2\beta} \left| F_t\left(\frac{m\,\boldsymbol{x}}{t}\right)\right|^2\right]^{1/2} = \left(\frac{m}{2\,\pi}\right)^{3/2} t^{3(\delta-1)/2}\,t^{-\beta\delta}$$
$$\times \left[\int_{y \leqslant 1} d^3y\;y^{-2\beta}\,|\,F_t(m\,t^{\delta-1}\,\boldsymbol{y})\,|^2\right]^{1/2} \leqslant (\text{const.})\;t^{3(\delta-1)/2-\beta\delta+7\eta/2-1}\,. \tag{98}$$

For $x > t^\delta$, we use $x^{-2\beta} < t^{-2\beta\delta}$ and the inequality (86):

$$\left(\frac{m}{2\,\pi\,t}\right)^{3/2} \left[\int_{x > t^\delta} d^3x\;x^{-2\beta} \left| F_t\left(\frac{m\,\boldsymbol{x}}{t}\right)\right|^2\right]^{1/2} \leqslant t^{-\beta\delta} \left(\frac{m}{2\,\pi\,t}\right)^{3/2}$$
$$\times \left[\int_{R^3} d^3x \left| F_t\left(\frac{m\,\boldsymbol{x}}{t}\right)\right|^2\right]^{1/2} = \left(\frac{1}{2\,\pi}\right)^{3/2} t^{-\beta\delta}\,\|F_t\| \leqslant (\text{const.})\;t^{2\eta-\beta\delta-1}\,. \tag{99}$$

It is not difficult to verify that, if $3/4 \leqslant \beta < 1$, one can find $\eta > 1 - \beta$ and $\delta > 0$ such that the two contributions (98) and (99) are integrable at $t = \infty$ (e.g. $\eta = 0.26$ and $\delta = 0.74$).

For the second term in the right-hand member of (96), one finds from (92)

$$\gamma\, m^\beta\, t^{-\beta}\, \| g_t^{(2)} \| = \gamma\, m^\beta\, t^{-\beta} \left(\frac{m}{2\pi t}\right)^{3/2} \left[\int d^3x \left| G_t\left(\frac{m\,\boldsymbol{x}}{t}\right)\right|^2\right]^{1/2}$$

$$= \frac{m^\beta}{(2\pi)^{3/2}}\, t^{-\beta}\, \| G_t \| \qquad (100)$$

where G_t is obtained from g by the definition (84). Using again (86), one sees that this contribution is smaller than (const.) $t^{2\eta-1-\beta}$ with $\eta > 1 - \beta$, a function which is again integrable at $t = +\infty$ for $3/4 \leqslant \beta < 1$. This completes the proof of the proposition. ∎

To conclude this Section, we may add that the convergence proof is essentially the same if the potential is of the form $V(\boldsymbol{x}) = \gamma\, x^{-\beta} + V_1(\boldsymbol{x})$, where $V_1(\boldsymbol{x})$ is square-integrable and real, and $3/4 \leqslant \beta < 1$. The operators T_t^+ will still be defined by (81) and (82), and the only modification in the proof consists of two additional contributions in the last member of (96), namely $\| V_1 f_t^{(1)} \|$ and $\| V_1 f_t^{(2)} \|$. By means of convenient estimates one may verify that these quantities are also integrable at $t = \infty$. If $V_1(\boldsymbol{x})$ is also spherically symmetric and satisfies the conditions (P1) and (P2) at the end of Section IV, one concludes that the wave operators are also complete (i.e. $\Omega_\pm\, \Omega_\pm^* = F$).

The value $\beta = 3/4$ does not appear as a limit which is inherent in (A1)–(A3), and we presume that the statement of Proposition 9 could be extended to values $\beta < 3/4$ by means of more refined estimates[6]).

VI. Concluding Remarks

In conclusion we add a few comments upon the possibility of an energy renormalization and upon the indeterminateness of the S-operator as expressed in Theorem 1.

We are inclined to thinking that local potentials which go to zero as $|\boldsymbol{x}| \to \infty$ do not give rise to an energy renormalization[7]) and that the only examples of such a renormalization are of the type which we considered under Case 3 at the end of Section III. In models of interactions between quantized fields, the energy renormalization is usually accompanied by an amplitude renormalization, and this latter seems incompatible with the existence of strong limits such as they were formulated in the present paper (for an investigation of asymptotic limits in a simple field-theoretic model, cf. [21]).

The indeterminateness of the S-operator arises from the fact that we considered the asymptotic behaviour of a limited number of observables only. Our theory gives an equivalence class of S-operators. The members of this class differ by a multipli-

[6]) Prof. L. Faddeev has communicated to us that Buslaev and Matveev have proved asymptotic convergence by different techniques for all $0 \leqslant \beta \leqslant 1$ and that their results will be published in the new Soviet journal 'Theoretical and Mathematical Physics'.

[7]) Such interactions exhibit no diagonal singularities in the sense of Van Hove [20].

cative unitary function of H_0 (cf. Theorem 1), and two different S-operators are equivalent in the sense that they predict identical results for the measurements of the observables of $\mathcal{A}_0$. If one imposed asymptotic convergence for a smaller algebra than $\mathcal{A}_0$, one would arrive at a correspondingly larger class of S-operators. One may assure oneself that there does not exist any larger von Neumann algebra containing $\mathcal{A}_0$ for which the form (A1)–(A3) of the asymptotic condition is physically meaningful and mathematically possible. The choice of $\mathcal{A}_0$ therefore gives the best possible determination of S. For short range potentials and also for the interactions of Case 3 of Section III there exists a distinguished S-operator, namely the one obtained from $\Omega_\pm = s - \lim_{t \to \pm\infty} V_t^* W_t^\pm$ (one remembers that the operators $W_t^\pm$ are uniquely determined by the asymptotic condition).

A quantity which is measured in scattering experiments is the probability of finding the scattered particle in a cone C whose apex coincides with the position of the scattering center. (In the sequel the scattering center is assumed to be at the origin of coordinates.) For the non-relativistic scattering by any short range potential and by Coulomb potentials, it was shown by Dollard [22] that this probability $P(f, C)$ for the initial state f is given by

$$P(f, C) = \int_C |(\widetilde{Sf})(\varkappa)|^2 \, d^3\varkappa \tag{101}$$

where $(\widetilde{Sf})(\varkappa)$ is the Fourier transform of the wave function corresponding to the final state $S f$. Following the lines of [22], it is not difficult to verify that this formula is correct also for the potentials considered in Proposition 9. It is clear from (101) that this probability $P(f, C)$ is also independent of the choice of S from the equivalence class found in Theorem 1.

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Appendix

In this appendix we collect the proofs of the technical lemmas used in Section V.

Proof of Lemma 1: Let $f \in \mathbb{C}$ and $\tau > 0$. It follows from (83) that

$$\| (\hat{U}_{t+\tau} - \hat{U}_t) f \|^2 = \int d^3\varkappa \, |\exp[-i \mu \varkappa^{-\beta}((t+\tau)^\alpha - t^\alpha)] - 1|^2 \, |\tilde{f}(\varkappa)|^2 .$$

Using the binomial expansion of $(t+\tau)^\alpha$ for $t > \tau$ and $\alpha < 1$, one sees that the integrand converges pointwise to zero as $t \to +\infty$. On the other hand it is bounded above by the integrable function $2\,|\tilde{f}(\varkappa)|^2$. By the Lebesgue dominated convergence theorem, the integral converges to zero. ∎

In the proof of Lemma 2 we shall need some estimates for the functions $F_t(\boldsymbol{x})$ defined in equation (83). We first state and prove these estimates:

Let f be in C and $x = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

(a) there exists a number $M < \infty$ such that for all $\boldsymbol{x}$ and t

$$|f_t(\boldsymbol{x})| \leqslant M . \tag{102}$$

(b) for every positive integer N, there exists a constant M_N such that for $t > 0$ and $x \neq 0$

$$|f_t(\boldsymbol{x})| \leqslant x^{-2N} (M_N t^{2\alpha N} + o(t^{2\alpha N})) . \tag{103}$$

Proof:

(a)

$$|f_t(\boldsymbol{x})| = (2\pi)^{-3/2} \left| \int d^3\varkappa \exp(i \boldsymbol{\varkappa} \cdot \boldsymbol{x}) \exp(-i \mu \varkappa^{-\beta} t^\alpha) \tilde{f}(\boldsymbol{\varkappa}) \right|$$
$$\leqslant (2\pi)^{-3/2} \int d^3\varkappa\, |\tilde{f}(\boldsymbol{\varkappa})| \equiv M < \infty .$$

(b) One obtains by integrating by parts

$$- x^2 f_t(\boldsymbol{x}) = (2\pi)^{-3/2} \int d^3\varkappa \exp(i \boldsymbol{\varkappa} \cdot \boldsymbol{x}) \Delta_\varkappa [\exp(-i \mu \varkappa^{-\beta} t^\alpha) \tilde{f}(\boldsymbol{\varkappa})] .$$

The Laplacian of the square bracket gives a sum of three terms

$$[t^{2\alpha} \tilde{f}^{(1)}(\boldsymbol{\varkappa}) + t^\alpha \tilde{f}^{(2)}(\boldsymbol{\varkappa}) + \tilde{f}^{(3)}(\boldsymbol{\varkappa})] \exp(-i \mu \varkappa^{-\beta} t^\alpha).$$

The powers of t arise from the derivatives of the exponential term, and the functions $\tilde{f}^{(j)}(\boldsymbol{\varkappa})$, $j = 1, 2, 3$, belong to C. Therefore, for $x \neq 0$, $f_t(\boldsymbol{x})$ is of the form

$$f_t(\boldsymbol{x}) = x^{-2} [t^{2\alpha} f_t^{(1)}(\boldsymbol{x}) + t^\alpha f_t^{(2)}(\boldsymbol{x}) + f_t^{(3)}(\boldsymbol{x})] .$$

Applying the same formula to the functions $f_t^{(j)}(\boldsymbol{x})$, one obtains another factor x^{-2} and new powers of t. Carrying through this operation N times, one finds for $x \neq 0$

$$f_t(\boldsymbol{x}) = x^{-2N} \sum_{j=1}^{2N} t^{j\alpha} f_t^{(j)}(\boldsymbol{x})$$

with $f^{(j)} \in \mathsf{C}$, $j = 1, 2, \dots 2N$. The inequality (103) now results from the fact that $t^{2\alpha N}$ is the leading power and that all of the $f_t^{(j)}(\boldsymbol{x})$ are uniformly bounded in $\boldsymbol{x}$ and in t according to (102). ∎

Proof of Lemma 2: Let f be in C and $\eta > 1 - \beta$.

(a) One has from (84)

$$|F_t(\boldsymbol{x})| \leqslant \int d^3y \left| \exp\left(\frac{i m y^2}{2t} - 1\right) \right| |f_t(\boldsymbol{y})| . \tag{104}$$

For fixed $t > 0$, we integrate in (104) separately over the sphere $S_t = \{\boldsymbol{y} \mid y \leqslant t^\eta\}$ and over the exterior of this sphere. To the first integral we apply Schwartz's inequality and get

$$\int_{y \leqslant t^\eta} d^3y \left| \exp\left(\frac{i\,m\,y^2}{2\,t}\right) - 1 \right| |f_t(\boldsymbol{y})| \leqslant \left[\int_{y \leqslant t^\eta} d^3y \left| \exp\left(\frac{i\,m\,y^2}{2\,t}\right) - 1 \right|^2 \right]^{1/2} \|f_t\| \,.$$

But $\|f_t\| = \|\tilde{U}_t f\| = \|f\|$. Using the fact that for $y \leqslant t^\eta$

$$\left| \exp\left(\frac{i\,m\,y^2}{2\,t} - 1\right) \right| \leqslant \frac{m\,y^2}{2\,t} \leqslant \frac{m}{2}\, t^{2\eta-1} \tag{105}$$

one gets

$$\int_{y \leqslant t^\eta} d^3y \left| \exp\left(\frac{i\,m\,y^2}{2\,t}\right) - 1 \right| |f_t(\boldsymbol{y})| \leqslant \frac{m}{2}\, \|f\|\, t^{2\eta-1} \left[\int_{y \leqslant t^\eta} d^3y \right]^{1/2} = C_1\, t^{7\eta/2-1} \,.$$

In the integral over the exterior of the sphere S_t, we use the fact that $|\exp\,(i\,m\,y^2/2\,t) - 1\,| \leqslant 2$ and apply (103) with N sufficiently large:

$$\int_{y > t^\eta} d^3y \left| \exp\left(\frac{i\,m\,y^2}{2\,t}\right) - 1 \right| |f_t(\boldsymbol{y})| \leqslant 2\,[M_N\, t^{2\alpha N} + o(t^{2\alpha N})] \int_{y > t^\eta} d^3y\, y^{-2N} \,.$$

Setting $y_i = t^\eta\, y_i'$ in the integral of the right-hand member, one sees that this integral behaves as $t^{(3-2N)\eta}$. The dominating power in the right-hand member is therefore $t^{3\eta+2N(\alpha-\eta)}$. Since $\alpha - \eta = 1 - \beta - \eta < 0$ and N is arbitrary, the integral over the exterior of S_t converges to zero for $t \to \infty$ faster than t^{-K} for any positive number K.

$$\text{(b)}\ \|F_t\| = \left[\int d^3x \left| \exp\left(\frac{i\,m\,x^2}{2\,t}\right) - 1 \right|^2 |f_t(\boldsymbol{x})|^2 \right]^{1/2} .$$

We divide the integration over R^3 into two parts in the same way as above. The contribution from the exterior of the sphere S_t again converges to zero faster than any power of $1/t$ for $t \to \infty$. For the integral over S_t one finds by using (105)

$$\left[\int_{x \leqslant t^\eta} d^3x \left| \exp\left(\frac{i\,m\,x^2}{2\,t}\right) - 1 \right|^2 |f_t(\boldsymbol{x})|^2 \right]^{1/2} \leqslant \frac{m}{2}\, t^{2\eta-1} \left[\int_{x \leqslant t^\eta} d^3x\, |f_t(\boldsymbol{x})|^2 \right]^{1/2}$$

$$\leqslant \frac{m}{2}\, t^{2\eta-1}\, \|f_t\| = \frac{m}{2}\, \|f\|\, t^{2\eta-1} = C_2\, t^{2\eta-1} \,. \qquad \blacksquare$$

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