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The Unitarity Constraints for Multiple Resonances

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Abstract. The unitarity constraints for any number of overlapping resonances are derived. These are the necessary and sufficient conditions on the resonance masses, widths, and couplings for the S-matrix to satisfy unitarity. The unitarity constraints for an isolated resonance, two overlapping resonances, and the dipole are studied in greater detail. The eigenphase behavior for an isolated resonance and a degenerate dipole is also discussed.

I. Introduction

The recently discovered A_2 splitting [1–4] suggests the elementary particle resonances may be more complicated objects than hitherto thought. The improving energy resolution of scattering experiments may reveal splittings in other well established resonances. The most plausible explanation of such fine structure is that resonances like the A_2 are really superpositions of several resonances. A similar phenomenon is well known in nuclear physics [5]. The giant dipole and analogue resonance enhancements observed in low energy resolution experiments split up into many peaks at higher resolutions. The number of peaks involved far exceeds anything one would expect in the elementary particle case. An excellent fit to the A_2 data can be obtained assuming two overlapping resonances or the limiting case of this, a dipole [6, 7].

The A_2 splitting has stimulated considerable theoretical interest in overlapping resonances and multiple resonance poles [6–12]. Most of this work has dealt with the effect of doubled resonances and dipoles on scattering cross sections, something we will not discuss in this paper. The papers of particular interest to us are those dealing with the constraints on overlapping and multiple resonances implied by the unitarity condition. The unitarity constraints for two overlapping resonances were first derived by Durand and McVoy [13] who were motivated by the physical problem of K_L and K_S mixing in weak interactions. Rebbi and Slansky also have obtained the unitarity constraints for a dipole [7], however only for the special case that the background vanishes, which is considerably simpler mathematically.

In this paper we outline a procedure for finding the unitarity constraints for any number of resonances. Precisely, this means we find the conditions on the resonance masses, widths, and residues which are sufficient for the S-matrix to satisfy unitarity.

For elastic scattering the solution of this problem is simple. The simplest form for S satisfying elastic unitarity

$$S\,S^* = 1 \tag{1.1}$$

and having resonance poles at the complex energies

$$E_1 - i\,\Gamma_1, \ldots, E_n - i\,\Gamma_n$$

is

$$S = \sigma\,\frac{E - E_1 - i\,\Gamma_1}{E - E_1 + i\,\Gamma_1} \cdots \frac{E - E_n - i\,\Gamma_n}{E - E_n + i\,\Gamma_n}, \tag{1.2a}$$

where

$$\sigma\,\sigma^* = 1\,. \tag{1.2b}$$

(1.2a) can also be written as

$$S = \sigma + \frac{g_1}{E - E_1 + i\,\Gamma_1} + \cdots + \frac{g_n}{E - E_n + i\,\Gamma_n}, \tag{1.3}$$

where

$$g_l = -2\,i\,\Gamma_l\,\sigma \prod_{j \neq l} \frac{E_l - E_j - i\,(\Gamma_l + \Gamma_j)}{E_l - E_j - i\,(\Gamma_l - \Gamma_j)}\,. \tag{1.4}$$

Alternatively, equations (1.4) can be interpreted as the constraints imposed by unitarity on the resonance parameters. In multi-channel scattering no simple form such as (1.2a) exists for S. The appropriate Ansatz is (1.3) where the g_i are matrices; the derivation of the unitarity constraints is then far from trivial.

Sections II and III of this paper are devoted to the theory of isolated resonances in S-matrix theory, or, more precisely, in S-*sub*matrix theory. The S-submatrix is an irreducible block of the full S-matrix. Its elements link scattering states whose quantum numbers, both internal and external, are identical. These quantum numbers include the angular momentum. One commonly considers only two body states because of the theoretical difficulty of dealing with three or more body states. In this case the S-submatrix is a function of a single continuous variable, the energy, or equivalently, the Mandelstam invariant s.

It is analytic in s on a multisheeted Riemann surface with square root type branch points at values of s corresponding to the threshold energies of the individual scattering channels. For n distinct thresholds there are 2^n Riemann sheets. One is the 'physical sheet'. It is separated from the 'unphysical sheets' by the unitarity cut which extends upwards along the real axis from the lowest threshold s_1. Physical values of s are those on the physical sheet on the upper lip of the unitarity cut. This is shown is Figure 1. If two or more thresholds coincide there are fewer sheets. The sheets that could be reached by crossing the real axis between the thresholds become inaccessible. In these paper we shall ignore threshold effects and implicitly assume all thresholds coincide. In this case there are only two sheets.

Section II gives a popular, and incorrect, discussion of the eigenphases near an isolated resonance. It can be shown [14] by time reversal invariance that for the stong

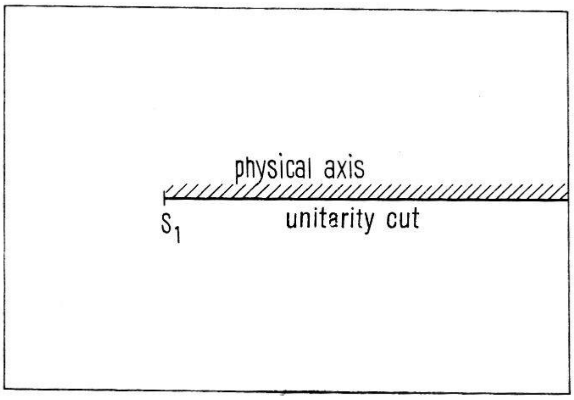

Figure 1
The complex s plane showing the unitarity cut and the physical axis (shaded).

interactions the S-submatrix is symmetric. Consequently it can be diagonalized by an orthogonal transformation. The eigenphases are those δ satisfying the equation

$$\mathrm{Det}\,(S(s) - e^{2i\delta}\, I) = 0\,. \tag{1.5}$$

It can be shown that at resonance (more precisely, at a pole of the K-matrix) one, and only one, eigenphase passes up through $\pi/2$ [15]. Yet it is incorrect to conclude, as was often done in the past, that only one eigenphase varies rapidly near a resonance. This was first pointed out by Weidenmueller [16] and then by Goebel and McVoy [17]. It can be rigorously shown that near a resonance the sum of the eigenphases increases by π [15]; if only one eigenphase were active it would necessarily increase by π and cross all the others (eigenphases are defined only modulo π). However, if we look for those s for which (1.5) has a double root we find it is very improbable that they are real, for the set of points on the real axis has measure zero in the complex plane. The points were two roots of (1.5) coincide are called 'crossing branch points' and the statement that they generally lie in the complex plane and not on the real axis is the 'no-crossing theorem'.

In Section III the discussion is more rigorous mathematically. We first derive the unitarity constraints for an isolated resonance. Using the unitarity constraints we can then find the eigenphases explicitly and verify that the no-crossing theorem is indeed satisfied. Lastly we demonstrate the existence of a crossing branch point near the resonance and show that a resonance pole 'attracts' a crossing branch point.

In Section IV we derive the unitarity constraints for any number of overlapping resonances. We find unitarity is equivalent to the orthogonality of a matrix which depends only on the resonance masses, widths, and residues. For two overlapping resonances this is used to express the resonance spacing and widths as functions of the residues. Lastly it is shown that the unitarity constraints for two overlapping resonances are satisfied in the case of elastic scattering.

Dipoles, which are the limiting cases of two overlapping resonances as the poles approach and the residues tend to infinity, are discussed in Section V. The limiting process is discussed in some detail to establish the factorization properties of the residue at the double pole. Then the unitarity constraints are derived and it is shown that they are a special case of the unitarity constraints for two overlapping resonances. The behavior of the eigenphases near a dipole is too complicated, and not of sufficient physical interest, to warrant a general discussion, and is only discussed for a particular case.

II. The Diagonalization of the S-submatrix

The symmetry of the S-submatrix on the Riemann surface for s implies it admits the diagonalization

$$S = 0\,\sigma\,0^T\,, \tag{2.1}$$

where 0 is a *complex* orthogonal matrix whose rows are the eigenvectors of S and σ is a diagonal matrix whose elements are the eigenvalues of S.

On the physical axis the unitarity of S

$$S\,S^* = I\,, \tag{2.2}$$

implies the diagonalization

$$S = 0\,\sigma\,0^+\,, \tag{2.3}$$

where 0 is a *unitary* matrix whose rows are again the eigenvectors of S and the elements of σ are the eigenvalues. Comparing (2.1) with (2.3) one sees that, on the physical axis, 0 is a *real* orthogonal matrix.

Equation (2.2) and the orthogonality of 0, implies on the physical axis,

$$\sigma\,\sigma^* = I\,. \tag{2.4}$$

The elements of σ have norm unity:

$$\sigma_k = e^{2\,i\,\delta_k}\,. \tag{2.5}$$

The δ_k are the eigenphases.

Since the elements of an orthogonal matrix are necessarily bounded, only σ in (2.1) can carry the resonance pole. The assumption that the pole residue is factorizable[1]) implies that but one of the elements of σ, say σ_j, is singular at the pole.

We now make the, what we will later see, incorrect assumption that σ_j has no other singularities near the resonance pole and the remaining elements of σ and the matrix 0 are essentially constant. Then the elements of S are

$$S_{kl} = \sigma_j\,0_{kj}\,0_{lj} + \text{background}\,. \tag{2.6}$$

If the pole is very near the real axis 0 is almost a real matrix; for the time being, at least, we assume that E

$$E_{kl} = 0_{kj}\,0_{lj} \tag{2.7}$$

is a *real* matrix.

Equation (2.4) implies that σ_j has a zero at the point conjugate to the resonance pole. The simplest parameterization consistent with unitarity is

$$\sigma_j = e^{2\,i\,\theta}\,\frac{s_0 + i\,\Gamma - s}{s_0 - i\,\Gamma - s}\,. \tag{2.8}$$

For the resonant eigenphase one finds, with (2.5),

$$\delta_j = \theta + \text{arc tan}\,\frac{\Gamma}{s_0 - s}\,. \tag{2.9}$$

[1]) The factorization of the pole residues can only be justified by appealing to quantum mechanics. It is a reflection of the statistical independence of the formation and decay of a resonant state.

For the T-matrix:

$$S = I + 2 i T , \tag{2.10}$$

and the K-matrix:

$$T^{-1} = K^{-1} - i , \tag{2.11}$$

one finds, using (2.1), (2.5), (2.6), (2.7) and (2.9),

$$T = \sin\delta_j \, e^{2 i \delta_j} E + \text{background} , \tag{2.12}$$

$$K = \tan\delta_j \, E + \text{background} . \tag{2.13}$$

From (2.9) it follows the eigenphase δ_j passes up through $\pi/2$ at

$$s_r = s_0 - \frac{\Gamma}{\tan(\pi/2 - \theta)} . \tag{2.14}$$

At this energy the K-matrix has a pole. From (2.12) it is evident the trajectory of T, or Argand diagram, describes a circle. The value of s corresponding to the top of the circle is s_r.

III. The True Eigenphase Behavior

In the last section we assumed that the eigenphase behavior for E *almost* real in (2.7) is the same as that for E *entirely* real. We shall see in this section that this is not true. For E real we found *one* eigenphase (2.9) increases by π and crosses all the others, which are constant (eigenphases are defined only modulo π). In this section we shall see that two or more and, in general, all the eigenphases are active if E is even slightly complex.

We proceed with more caution and greater mathematical rigor. In the last section we assumed that σ has no singularities other than the resonance pole. This is generally not true since σ and 0 in (2.1) may have singularities which are not present in S.

S itself has only the resonance pole which we assume is at $s_0 - i\,\Gamma$. We introduce the variable

$$X = s_0 - s . \tag{3.1}$$

The product

$$(X - i\,\Gamma)\, S$$

is nonsingular in a neighborhood of the resonance which we assume includes a segment of the physical axis. Expanding in a truncated Taylor series

$$(X - i\,\Gamma)\, S = S_0 + X\, S_1 + \ldots + X^n\, S_n \tag{3.2}$$

and inserting into the unitarity condition (2.2) one finds

$$S_n = S_{n-1} = \ldots = S_3 = 0 . \tag{3.3}$$

A workable model can be obtained only by truncating at the second term;

$$(X - i\,\Gamma)\,S = S_0 + X\,S_1\,. \tag{3.4}$$

This can be rearranged to read

$$S = B + \frac{2\,i\,R}{X - i\,\Gamma}\,, \tag{3.5}$$

where B and R are linear combinations of S_0 and S_1. R must be factorizable since it is proportional to the pole residue:

$$R_{ij} = r_i\,r_j\,, \tag{3.6}$$

or, in the notation to be used here:

$$R = r \cdot r^T\,. \tag{3.6a}$$

Both B and R are symmetric since S is.

Inserting (3.5) into the unitarity condition (2.2), multiplying by $X^2 + \Gamma^2$ and expanding in powers of X yields

$$X^2\,(B\,B^* - I) + 2\,i\,X\,(R\,B^* - B\,R^*) + \Gamma^2\,(B\,B^* - I) + 4\,R\,R^* \\ - 2\,\Gamma\,(R\,B^* + B\,R^*) = 0\,. \tag{3.7}$$

The coefficients of the separate powers of X must vanish. This implies

$$B\,B^* = I\,, \tag{3.8a}$$

$$R\,B^* = B\,R^*\,, \tag{3.8b}$$

$$R\,R^* = \frac{1}{2}\,\Gamma\,(R\,B^* + B\,R^*)\,. \tag{3.8c}$$

It follows from (3.6) and (3.8b) that

$$B\,r^* = \lambda\,r\,, \qquad B^*\,r = \lambda\,r^* \tag{3.9}$$

where λ is some constant; that it is real can be seen by taking the conjugate of the second of these relations and comparing with the first. Multiplying (3.8a) on the right by r and on the left by r^* one finds

$$\lambda^2 = 1\,, \quad \text{or} \quad \lambda = \pm 1\,.$$

Using (3.9) and (3.8c) one finds

$$\Gamma = \frac{1}{\lambda}\,r^*\,r = \frac{1}{\lambda}\sum r_j^*\,r_j\,.$$

Since the width must be positive:

$$\lambda = 1\,.$$

Summarizing, equations (3.8) imply the necessary and sufficient conditions for unitarity are

$$B\,B^* = I\,, \tag{3.10a}$$

$$B\,r^* = r\,, \tag{3.10b}$$

$$\Gamma = r^* \cdot r = \sum r_j^*\, r_j\,. \tag{3.10c}$$

Since B is a symmetric and unitary matrix it admits the diagonalization

$$B = 0\,\sigma^0\,0^T\,, \tag{3.11}$$

where 0 is a *real* orthogonal matrix. Introducing

$$\hat{S} = \sigma^0 + \frac{2\,i\,G}{X - i\,\Gamma}\,, \tag{3.12}$$

where

$$G = 0^T\,R\,0 = g\,g^T\,; \quad g = 0^T\,r\,, \tag{3.13}$$

we have

$$S = 0\,\hat{S}\,0^T\,. \tag{3.14}$$

Since $\hat{S}$ must itself satisfy unitarity we have, comparing with (3.10),

$$\sigma^0\,\sigma^{0*} = I\,, \tag{3.15a}$$

$$\sigma^0\,g^* = g\,, \tag{3.15b}$$

$$\Gamma = g^* \cdot g = \sum g_j^*\, g_j\,. \tag{3.15c}$$

Equation (3.15a) implies for the elements of σ^0:

$$\sigma_k^0 = e^{2\,i\,\delta_k^0}\,. \tag{3.16}$$

The δ_k^0 are the background eigenphases *below* the resonance. As we will demonstrate below the background eigenphases above the resonance do not coincide with those below. This is illustrated in Figure 2. There is one eigenphase in the set which is larger by π; in Figure 2 this is δ_1^0.

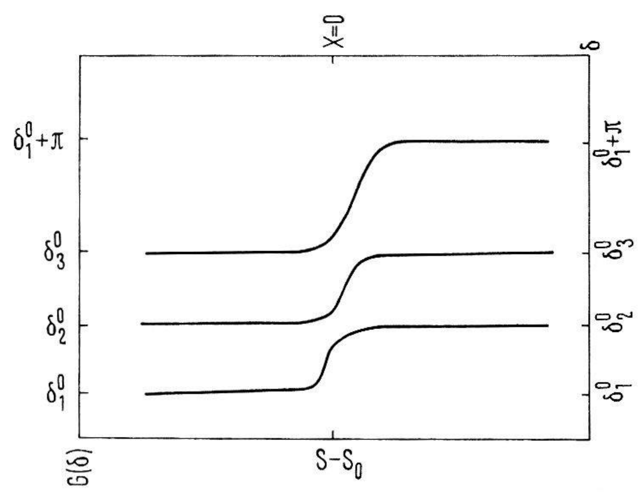

Figure 3
A plot of the function $G(\delta)$.

Figure 2
The eigenphases for an isolated resonance as function of s.

Since σ^0 is a diagonal matrix, (3.15b) is simply

$$\sigma_k^0 \, g_k^* = g_k \, . \tag{3.17}$$

We cannot yet draw the conclusion that σ_k^0 is equal the quotient g_k/g_k^*, for we cannot rule out the possibility that g_k vanishes. However the g's can be labeled so that the *first* m are nonvanishing;

$$g_k \neq 0 \, , \quad k = 1, \ldots, m ; \quad g_k = 0 \, , \quad k > m \, . \tag{3.18}$$

Then the first m components of σ^0 are

$$\sigma_k^0 = \frac{g_k}{g_k^*} \qquad k = 1, \ldots, m \, . \tag{3.19}$$

The remaining elements of σ^0 are arbitrary, they correspond to *constant* eigenphases. Let $\hat{\sigma}^0$ be the submatrix of σ^0 consisting the elements in the first m rows and columns. The main diagonal of $\hat{\sigma}^0$ contains the elements (3.19). Let $\tilde{\sigma}^0$ be the submatrix whose main diagonal consists of the remaining σ_k^0. Denote by $\hat{G}$ the submatrix of G containing the elements in the first m rows and columns; $\hat{G}$ contains all the nonvanishing elements of G. From (3.12) we have

$$\hat{S} = \begin{bmatrix} \hat{\sigma}^0 & \\ & \tilde{\sigma}^0 \end{bmatrix} + \frac{2\,i}{X - i\,\Gamma} \begin{bmatrix} \hat{G} & 0 \\ 0 & 0 \end{bmatrix} . \tag{3.20}$$

Evident eigenvalues of $\hat{S}$, and therefore also of S, are

$$\sigma_k^0 = e^{2\,i\delta_k^0} \, , \quad k > m \, .$$

The corresponding eigenphases are constant; we call them *passive* eigenphases. In the case considered in the previous section, that for which E in (2.7) is *entirely* real, all eigenphases but one are passive and G has but a single element. The reader can verify that in this case (2.1), (2.6) and (2.8) follow from (3.12), (3.14) and (3.17).

To find the active eigenphases we use the theorem.

Theorem I: Let Δ be a diagonal matrix and let $f \times g^T$ be factorizable;

$$\Delta_{ij} = \delta_{ij} \Delta_i ; \quad f \times g_{ij}^T = f_i \, g_j \, . \tag{3.21}$$

Then

$$\mathrm{Det}\,(\Delta + f \times g^T) = \left\{ \prod_i \Delta_i \right\} \left\{ 1 + \sum_j \frac{f_j \, g_j}{\Delta_j} \right\} . \tag{3.22}$$

Proof: We prove the theorem first for the special case that

$$\Delta = I \quad \text{and} \quad f = g = e \, .$$

Let $\hat{e}$ be the unit vector in the direction of e. There exist vectors $e^2, \ldots, e^n$ such that $e^1 = \hat{e}, e^2, \ldots, e^n$ form a complete orthonormal set. In this basis

$$I + e \times e^T = \begin{bmatrix} 1 + e \cdot e & 0 & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} ,$$

where

$$e \cdot e = \sum_i e_i^2 .$$

Evidently

$$\text{Det}\,(I + e \times e^T) = 1 + e \cdot e . \tag{3.23}$$

We now prove the theorem in general. Let R and L be the diagonal matrices

$$R_{ij} = \delta_{ij}\, R_i\,, \quad R_i = \left(\Delta_i\, \frac{g_i}{f_i}\right)^{1/2} \tag{3.24a}$$

$$L_{ij} = \delta_{ij}\, L_i\,, \quad L_i = \left(\Delta_i\, \frac{f_i}{g_i}\right)^{1/2}, \tag{3.24b}$$

and let e be the vector with components

$$e_i = \left(\frac{g_i\, f_i}{\Delta_i}\right)^{1/2} . \tag{3.25}$$

Then

$$\Delta + f \times g^T = L\,(I + e \times e^T)\, R . \tag{3.26}$$

Therefore

$$\text{Det}\,(\Delta + f \times g^T) = \text{Det}\,(L\,R)\ \text{Det}\,(I + e \times e^T) .$$

Since

$$\text{Det}\,(L\,R) = \prod_i \Delta_i$$

(3.22) follows from (3.23) and (3.25). Q.E.D.

. . .

The active eigenphases correspond to the eigenvalues of the submatrix

$$\hat{\sigma}^0 + \frac{2\,i}{X - i\,\Gamma}\,\hat{G}$$

of (3.20). They are the roots of the characteristic equation

$$\text{Det}\left(\hat{\sigma}^0 - \sigma\, I + \frac{2\,i}{X - i\,\Gamma}\,\hat{G}\right) = 0 . \tag{3.27}$$

Since $\hat{\sigma}^0 - \sigma\, I$ is a diagonal matrix Theorem I applies. The characteristic equation is

$$\left[\prod (\sigma_k^0 - \sigma)\right]\left(1 + \frac{2\,i}{X - i\,\Gamma} \sum_1^m \frac{g_j^2}{\sigma_j^0 - \sigma}\right) = 0 . \tag{3.28}$$

Using (3.15c) and (3.19) this can be rewritten as

$$\left[\prod (\sigma_k^0 - \sigma)\right]\left(X + \sum_1^m \left\{i\, \frac{\sigma_j^0 + \sigma}{\sigma_j^0 - \sigma}\right\} g_j^*\, g_j\right) = 0 . \tag{3.29}$$

The roots of this polynomial are the zeroes of the second factor.. Using

$$\sigma_j^0 = e^{2i\delta_j^0}, \quad \sigma = e^{2i\delta}$$

one finds the eigenphases are the δ's satisfying

$$X = G(\delta) = \sum_1^m g_j^* \, g_j \cot(\delta - \delta_j^0) . \tag{3.30}$$

This relation defines the function $G(\delta)$. For a *narrow* resonance each of the products $g_j \, g_j^*$ is very small; this can be seen from (3.15c).

The function $G(\delta)$ is then very small except at its singularities

$$\delta \cong \delta_j^0 + n\pi$$

where, approximately

$$G(\delta) = g_j^* \, g_j \cot(\delta_j^0 - \delta) .$$

In Figure 3 $G(\delta)$ is plotted for the case $m = 3$. There are singularities at δ_1^0, δ_2^0, δ_3^0 and $\delta_1^0 + \pi$. To solve (3.30) one simply draws a horizontal line indicating the value of X. Turning Figure 3 on its side gives Figure 2, a plot of the eigenphases a function of s! It is evident why the active eigenphases can't cross; every eigenphase is confined by two successive singularities of $G(\delta)$. Note, however, that an active eigenphase *may* cross a passive one.

From (3.29) it is easy to demonstrate there are crossing branch points near the resonance pole. We consider a root

$$\sigma_1 = e^{2i\delta_1}$$

which we analytically continue on the contour of the X, or s, plane shown in Figure 4. We assume the radius of the semi-circle is so small that the sum in (3.29) is negligable relative to X. On the semicircle the roots of (3.29) are nearly equal to the σ_k^0. At A, σ_1 has approximately the value σ_1^0. Analytically continuing along the real axis to B one arrives at σ_2^0, as can be seen from Figure 2. Now, continuing along the semi -circle back to A one obtains σ_2^0, not the original σ_1^0. This means there is a branch point within the continuation contour. At the branch point:

$$\sigma_1 = \sigma_2 .$$

We can also understand why 'a resonance attracts a crossing branch point'. For a narrow resonance it follows from (3.15c) that the $g_j^* \, g_j$ are very small. In this case the radius of the semi-circle in Figure 4 can be chosen to quite small.

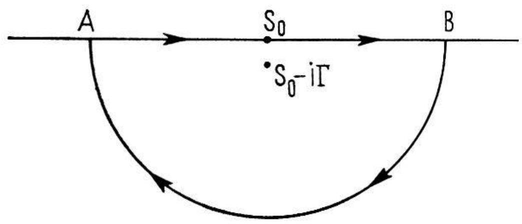

Figure 4
The analytic continuation contour in the s plane which must enclose a crossing branch point.

IV. Unitarity Constraints for Two Overlapping Resonances

In this section we discuss the unitarity constraints for n overlapping resonances and then, as a particular case, derive the unitarity constraints for $n = 2$.

We assume resonance poles at the complex s values $s_1, \ldots, s_n$. The appropriate Ansatz for S is

$$S = B + \sum_i \frac{f_i \times f_i^T}{s - s_i} . \tag{4.1}$$

B, since it is the asymptotic value of S, must itself satisfy unitarity;

$$B\,B^* = B^*\,B = I . \tag{4.2}$$

We assume, in analogy with (3.10b), that

$$B\,f_i^* = \sum_j U_{ij}\,f_j . \tag{4.3}$$

Furthermore we assume the f_i are independent vectors. This assumption is generally not justified, particularly when n exceeds the dimension of S. The results we obtain seem to be independent of this assumption[2]). For instance we shall show that the unitarity constraints we derive for two overlapping resonances are also satisfied in the elastic case.

From

$$B\,B^*\,f_i = f_i$$

and (4.3) there follows

$$U\,U^* = U^*\,U = I . \tag{4.4}$$

From

$$f_j\,B^*\,B\,f_i^* = f_j \cdot f_i^*$$

one also obtains

$$f_j \cdot f_i^* = \sum_{kl} U_{jk}^*\,U_{il}\,f_l \cdot f_k^* . \tag{4.5}$$

Inserting (4.3) into the unitarity condition

$$S\,S^* - I = 0 ,$$

and using (4.2) one obtains

$$\sum_{ji} \left\{ \frac{U_{ji}}{s - s_j^*} + \frac{U_{ij}^*}{s - s_i} + \frac{f_i \cdot f_j^*}{(s - s_j^*)\,(s - s_i)} \right\} f_i \times f_j^{*T} = 0 .$$

[2]) If the dimension of S equals or exceeds the number of resonances a linear dependence of the f_i is an accident; the f_i can be made independent by an infinitesimal perturbation. The validity of (4.9) follows from the continuity of these constraints under this perturbation. The matrix 0, given by (4.7) and (4.8), is nonsingular even when the f_i are linearly dependent. The case that the dimension of S is less than the number of resonances is dealt with by adding channels in which there is initially no scattering.

The coefficients of the independent matrices $f_i \times f_i^{*T}$ must vanish identically in s. This gives

$$U_{ij}^* = -U_{ji}, \tag{4.6}$$

and

$$U_{ji} = \frac{f_i \cdot f_j^*}{s_i - s_j^*}. \tag{4.7}$$

It is easily seen that (4.7) satisfies (4.6). The reader can also verify that (4.5) is satisfied by (4.7) if (4.4) holds. The unitarity constraints can be obtained by requiring that (4.7) satisfy (4.4). By (4.6) the latter equation can also be written as

$$U\,U^T = -I.$$

This is equivalent to the statement that

$$0 = i\,U \tag{4.8}$$

is an orthogonal matrix:

$$0\,0^T = I. \tag{4.9}$$

We now can derive the unitarity constraints for the special case of two overlapping resonances. We assume resonance poles at $s = m_1^2 - i\,\Gamma_1$ and $s = m_2^2 - i\,\Gamma_2$. It will convenient to use the variable

$$Y = s - \frac{m_1^2 + m_2^2}{2}. \tag{4.10}$$

Introducing

$$X = \frac{m_2^2 - m_1^2}{2}, \tag{4.11a}$$

$$\Gamma = \frac{\Gamma_1 + \Gamma_2}{2}, \tag{4.11b}$$

$$\Delta = \frac{\Gamma_1 - \Gamma_2}{2}, \tag{4.11c}$$

we note the matrix S has poles at $Y = X + i\,\Delta - i\,\Gamma$ and $Y = -X - i\,\Delta - i\,\Gamma$. Equation (4.1) can be written as

$$S = B + \frac{R}{Y + X + i\,\Delta + i\,\Gamma} + \frac{T}{Y - X - i\,\Delta + i\,\Gamma}, \tag{4.12}$$

where

$$R = r \times r^T, \quad T = t \times t^T. \tag{4.12a}$$

For the matrix 0 we have from (4.7) and (4.8)

$$0 = \frac{1}{2}\begin{bmatrix} -\dfrac{r \cdot r^*}{\Gamma + \Delta} & i\,\dfrac{t \cdot r^*}{X - i\,\Gamma} \\[2ex] -i\,\dfrac{r \cdot t^*}{X + i\,\Gamma} & -\dfrac{t \cdot t^*}{\Gamma - \Delta} \end{bmatrix}. \tag{4.13}$$

Since it must be orthogonal it must be of the form

$$0 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad \operatorname{Det} 0 = 1 \tag{4.14a}$$

or of the form

$$0 = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \quad \operatorname{Det} 0 = -1\,. \tag{4.14b}$$

Here θ is some complex angle.

The possibility (4.14b) can be ruled out. Indeed from the diagonal components we would obtain

$$\frac{\Gamma + \Delta}{\Gamma - \Delta} = -\frac{r \cdot r^*}{t \cdot t^*}\,.$$

The left hand side is the ratio of the resonance widths which must be positive since both poles are below the real axis. The left hand side, however, is negative definite.

From (4.14a) we see that diagonal components of (4.13) must be equal. This gives

$$\Delta = \Gamma \frac{r \cdot r^* - t \cdot t^*}{r \cdot r^* + t \cdot t^*}\,. \tag{4.15a}$$

From the off-diagonal elements we obtain

$$X = i\,\Gamma \frac{r \cdot t^* + t \cdot r^*}{r \cdot t^* - t \cdot r^*}\,. \tag{4.15b}$$

Evaluating the determinant of 0 we find

$$\frac{1}{4}\left\{\frac{r \cdot r^*\, t \cdot t^*}{\Gamma^2 - \Delta^2} - \frac{r \cdot t^*\, t \cdot r^*}{X^2 + \Gamma^2}\right\} = 1\,.$$

From (4.15) we find

$$\Gamma^2 - \Delta^2 = 4\,\Gamma^2 \frac{r \cdot r^*\, t \cdot t^*}{(r \cdot r^* + t \cdot t^*)^2}\,; \qquad X^2 + \Gamma^2 = -4\,\Gamma^2 \frac{r \cdot t^*\, t \cdot r^*}{(r \cdot t^* - t \cdot r^*)^2}\,;$$

so the above equation can be written also as

$$\Gamma = \frac{1}{4}\,((r \cdot r^* + t \cdot t^*)^2 + (r \cdot t^* - t \cdot r^*)^2)^{1/2}\,. \tag{4.16}$$

From (4.3), (4.8), (4.13) and (4.12a) we obtain

$$B \begin{bmatrix} r^* \\ t^* \end{bmatrix} = \frac{1}{2} \begin{bmatrix} i\,\dfrac{r \cdot r^*}{\Gamma + \Delta} & \dfrac{r \cdot t^*}{X + i\,\Gamma} \\ -\dfrac{t \cdot r^*}{X + i\,\Gamma} & i\,\dfrac{t \cdot t^*}{\Gamma - \Delta} \end{bmatrix} \begin{bmatrix} r \\ t \end{bmatrix} \tag{4.17}$$

Equations (4.15), (4.16), (4.17) are the necessary and sufficient conditions for the matrix S given by (4.12) to satisfy unitarity.

Example for Unitarity Constraints for Two Overlapping Resonances

The most general form for S in elastic scattering is

$$S = \sigma \frac{Y + X - i\Delta - i\Gamma}{Y + X + i\Delta + i\Gamma} \frac{Y - X + i\Delta - i\Gamma}{Y - X - i\Delta + i\Gamma}, \tag{4.18}$$

where

$$\sigma \sigma^* = 1 . \tag{4.19}$$

Evaluating the residues at the poles one finds

$$R = r^2 = -2 i \sigma (\Gamma + \Delta) \frac{X + i\Gamma}{X + i\Delta}, \tag{4.20a}$$

$$T = t^2 = -2 i \sigma (\Gamma - \Delta) \frac{X - i\Gamma}{X + i\Delta}. \tag{4.20b}$$

That the unitarity constraint (4.16) holds is easily verified. Since r and t are now just scalars this constraint can also be written as

$$\Gamma = \frac{1}{4} ((r^2 + t^2)(r^2 + t^2)^*)^{1/2} ,$$

whereas from (4.20) we find

$$r^2 + t^2 = -4 i \sigma \Gamma .$$

To verify (4.15a) we multiply equations (4.20) by their complex conjugates and take the square root. This gives

$$r r^* = 2 (\Gamma + \Delta) \left(\frac{X^2 + \Gamma^2}{X^2 + \Delta^2}\right)^{1/2}; \quad t t^* = 2 (\Gamma - \Delta) \left(\frac{X^2 + \Gamma^2}{X^2 + \Delta^2}\right)^{1/2};$$

whence:

$$\frac{r r^* - t t^*}{r r^* + t t^*} = \frac{\Delta}{\Gamma}.$$

Similarly, to verify (4.15b), we obtain

$$r t^* = 2 (X + i\Gamma) \left(\frac{\Gamma^2 - \Delta^2}{X^2 + \Delta^2}\right)^{1/2}; \quad t r^* = 2 (X - i\Gamma) \left(\frac{\Gamma^2 - \Delta^2}{X^2 + \Delta^2}\right)^{1/2};$$

whence:

$$\frac{r t^* + t r^*}{r t^* - t r^*} = \frac{X}{i\Gamma}.$$

It is left to the reader to verify that the unitarity constraints (4.17) are satisfied.

Unitarity Constraints for a Dipole

A dipole is the limiting case of two overlapping resonances as the poles approach each other. To investigate this limit we rewrite (4.12) as

$$S = B + \frac{(R+T)}{2}\left[\frac{1}{(Y+i\,\Gamma)+(X+i\,\Delta)} + \frac{1}{(Y+i\,\Gamma)-(X+i\,\Delta)}\right] + (T-R)\,\frac{(X+i\,\Delta)}{(Y+i\,\Gamma)^2+(X+i\,\Delta)^2}\,. \tag{5.1}$$

In the limit

$$X + i\,\Delta \to 0$$

this becomes

$$S = B + \frac{E}{Y+i\,\Gamma} + \frac{F}{(Y+i\,\Gamma)^2}\,, \tag{5.2}$$

where

$$E = \mathrm{Lim}\,(R+T)\,, \tag{5.3a}$$

$$F = \mathrm{Lim}\,(X+i\,\Delta)\,(T-R)\,. \tag{5.3b}$$

From (5.3a) one obtains

$$\mathrm{Lim}\,(X+i\,\Delta)\,(R+T) = \mathrm{Lim}\,(X+i\,\Delta)\ \mathrm{Lim}\,(R+T) = 0\,.$$

Then adding and subtracting (5.3b) one finds

$$F = 2\,\mathrm{Lim}\,(X+i\,\Delta)\,T\,, \tag{5.4a}$$

$$F = -2\,\mathrm{Lim}\,(X+i\,\Delta)\,R\,. \tag{5.4b}$$

We now prove F is factorizable. Equations (5.4) imply, in particular,

$$F_{ii} = 2\,\mathrm{Lim}\,((X+i\,\Delta)^{1/2}\,t_i)^2 = (2^{1/2}\,\mathrm{Lim}\,((X+i\,\Delta)^{1/2}\,t_i))^2$$
$$= -2\,\mathrm{Lim}\,((X+i\,\Delta)^{1/2}\,r_i)^2 = (2^{1/2}\,i\,\mathrm{Lim}\,((X+i\,\Delta)^{1/2}\,r_i))^2\,;$$

whence:

$$f_i = F_{ii}^{1/2} = \begin{cases} 2^{1/2}\,\mathrm{Lim}\,((X+i\,\Delta)^{1/2}\,t_i) & (5.5a)\\ 2^{1/2}\,i\,\mathrm{Lim}\,((X+i\,\Delta)^{1/2}\,r_i)\,. & (5.5b)\end{cases}$$

For the off-diagonal elements we find, from (5.4),

$$F_{ij} = 2\,\mathrm{Lim}\,((X+i\,\Delta)^{1/2}\,t_i\,(X+i\,\Delta)^{1/2}\,t_j)$$
$$= (2^{1/2}\,\mathrm{Lim}\,(X+i\,\Delta)^{1/2}\,t_i)\,(2^{1/2}\,\mathrm{Lim}\,(X+i\,\Delta)^{1/2}\,t_j)\,.$$

Equations (5.5) then give

$$F_{ij} = f_i\,f_j \tag{5.6a}$$

or

$$F = f \times f^T. \tag{5.6b}$$

E, in (5.2), is a sum of *two* factorizable matrices. First we note

$$R + T = r \times r^T + t \times t^T = \frac{1}{2} \left((t - i\, r) \times (t + i\, r)^T + (t + i\, r) \times (t - i\, r)^T\right). \quad (5.7)$$

By (5.5):

$$\operatorname{Lim} (X + i\, \Delta)^{1/2} (t + i\, r) = 2^{1/2} f . \quad (5.8)$$

The limit

$$g = \frac{1}{2^{1/2}} \operatorname{Lim} \frac{t - i\, r}{X + i\, \Delta} \quad (5.9)$$

is finite. Indeed, by (5.3a):

$$E = g \times f^T + f \times g^T . \quad (5.10)$$

For the diagonal components this reads

$$E_{i\,i} = 2\, g_i\, f_i .$$

If $f_i \neq 0$ we can conclude, since both $E_{i\,i}$ and f_i are finite, that g_i is finite. If $f_i = 0$ we can draw the same conclusion from

$$E_{i\,j} = g_i\, f_j .$$

We now discuss the unitarity constraints for a dipole. We could in principle derive them as a particular case of the constraints found for two overlapping resonances in the last section. Since the algebra is in this case considerably simpler we prefere to derive them separately and then to compare with the results of the last section.

Multiplying

$$S\, S^* - I = 0$$

by

$$(Y + i\, \Gamma)^2\, (Y - i\, \Gamma)^2 ,$$

and equating to zero the coefficients of the separate powers of Y one obtains, with (5.2), the equations

$$B\, B^* = I , \quad (5.11)$$

$$B\, E^* + E\, B^* = 0 , \quad (5.12)$$

$$E\, E^* + B\, (F^* + i\, \Gamma\, E^*) + (F - i\, \Gamma\, E)\, B^* = 0 , \quad (5.13a)$$

$$E\, F^* + F\, E^* + 2\, i\, \Gamma\, (B\, F^* - F\, B^*) = 0 , \quad (5.13b)$$

$$(F + i\, \Gamma\, E)\, (F^* - i\, \Gamma\, E^*) = \Gamma^2\, (B\, (F^* - i\, \Gamma\, E^*) + (F + i\, \Gamma\, E)\, B^*) . \quad (5.13c)$$

To deal with (5.12) we make the Ansatz

$$B\, g^* = \alpha\, g + \beta\, f , \qquad B\, f^* = \gamma\, g + \delta\, f .$$

In matrix notation this is

$$B \begin{bmatrix} g^* \\ f^* \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} g \\ f \end{bmatrix} . \tag{5.14}$$

The requirement

$$B\, B^* \begin{bmatrix} g \\ f \end{bmatrix} = \begin{bmatrix} g \\ f \end{bmatrix}$$

implies

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{5.15}$$

To satisfy (5.12) one finds from (5.14), its hermitean conjugate, and (5.10), the further restrictions

$$\delta^* = -\alpha ,$$

$$\beta + \beta^* = 0 \Rightarrow \beta = i\, b;\ b \text{ real} ,$$

$$\gamma + \gamma^* = 0 \Rightarrow \gamma = i\, c;\ c \text{ real} .$$

Inserting these relations into (5.15) one finds

$$\alpha\, \alpha^* + b\, c = 1 ,$$

$$c\, \alpha = b\, \alpha^* = 0 .$$

There are two possibilities. Either:

$$b = c = 0 , \quad \alpha\, \alpha^* = 1 ,$$

or

$$\alpha = 0 , \quad b\, c = 1 .$$

In the first case:

$$B\, g^* = \alpha\, g ; \quad B\, f^* = -\alpha^*\, f .$$

This possibility can be ruled out. Inserting these relations into equations (5.13) reduces them to linear combinations of the independent matrices

$$f \times f^{*T} , \quad f \times g^{*T} , \quad g \times f^{*T} , \quad g \times g^{*T} .$$

In particular, the coefficient of $g \times g^{*T}$ in the equation resulting from (5.13a) is $f \cdot f^*$. Consequently:

$$f \cdot f^* = 0 .$$

This is possible only if f itself vanishes, in which case we have a simple resonance pole; not a dipole.

For the second case we have

$$B\, g^* = i\, b\, f ; \quad B\, f^* = i\, c\, g . \tag{5.16}$$

Taking the hermitean conjugate of either equation, say

$$g^T\, B^* = -i\, b\, f^T\,,$$

and multiplying on the right by the original equation, one obtains

$$g^T\, B^*\, B\, g^* = b^2\, f^* \cdot f\,.$$

Consequently:

$$b = \left[\frac{g \cdot g^*}{f \cdot f^*}\right]^{1/2}; \quad c = \left[\frac{f \cdot f^*}{g \cdot g^*}\right]^{1/2}. \tag{5.17}$$

Inserting (5.16) into equations (5.13), making use of (5.6) and (5.10) one finds, from (5.13a)

$$\begin{aligned}(g \cdot g^* - 2\, b\, \Gamma)\, f \times f^{*T} + (g \cdot f^* - i\, c)\, f \times g^{*T}& \\ + (f \cdot g^* + i\, c)\, g \times f^{*T} + (f \cdot f^* - 2\, c\, \Gamma)\, g \times g^{*T} = 0\,,&\end{aligned} \tag{5.18a}$$

from (5.13b)

$$(g \cdot f^* + f \cdot g^*)\, f \times f^{*T} + (f \cdot f^* - 2\, c\, \Gamma)\, (g \times f^{*T} + f \times g^{*T}) = 0\,, \tag{5.18b}$$

and from (5.13c)

$$\begin{aligned}(\Gamma^2\, (g \cdot g^*) + i\, \Gamma\, (g \cdot f^* - f \cdot g^*) + (f \cdot f^*) - 2\, b\, \Gamma^3)\, f \times f^{*T} +& \\ + (\Gamma^2\, (g \cdot f^*) - i\, \Gamma\, (f \cdot f^*) + i\, c\, \Gamma^2)\, f \times g^{*T} +& \\ + (\Gamma^2\, (f \cdot g^*) + i\, \Gamma\, (f \cdot f^*) - i\, c\, \Gamma^2)\, g \times f^{*T} +& \\ + (\Gamma^2\, (f \cdot f^*) - 2\, c\, \Gamma^3)\, g \times g^{*T} = 0\,.&\end{aligned} \tag{5.18c}$$

The coefficients of the four independent matrices appearing in these equations must vanish. For (5.18a) this implies

$$g \cdot g^* = 2\, b\, \Gamma\,, \quad f \cdot f^* = 2\, c\, \Gamma\,,$$

or, by (5.17),

$$\Gamma = \frac{1}{2}\, ((g \cdot g^*)\, (f \cdot f^*))^{1/2} \tag{5.19a}$$

and

$$g \cdot f^* = -f \cdot g^* = i\, c\,. \tag{5.19b}$$

One easily verify that the coefficients in (5.18b) and (5.18c) also vanish if these conditions are satisfied. Of the three equations (5.18) only the first is independent!

The constraints (5.19) can also be derived as a special case of the unitarity constraints for two overlapping resonances studied in the previous section. We regard (5.8) and (5.9) not as limiting equations but as definitions of f and g for fixed t and r. Then, with some algebra, one finds

$$t \cdot t^* + r \cdot r^* = (X^2 + \Delta^2)^{-1/2}\, f \cdot f^* + (X^2 + \Delta^2)^{1/2}\, g \cdot g^*\,, \tag{5.20a}$$

$$r \cdot t^* - t \cdot r^* = -i\left((X^2 + \Delta^2)^{-1/2} f \cdot f^* - (X^2 + \Delta^2)^{1/2} g \cdot g^*\right), \tag{5.20b}$$

$$t \cdot t^* - r \cdot r^* = \left[\frac{X - i\Delta}{X + i\Delta}\right]^{1/2} f \cdot g^* + \left[\frac{X + i\Delta}{X - i\Delta}\right]^{1/2} g \cdot f^*, \tag{5.20c}$$

$$r \cdot t^* + t \cdot r^* = -i\left(\left[\frac{X - i\Delta}{X + i\Delta}\right]^{1/2} f \cdot g^* - \left[\frac{X + i\Delta}{X - i\Delta}\right]^{1/2} g \cdot f^*\right). \tag{5.20d}$$

For the width we find

$$\Gamma = \frac{1}{4}\left((t \cdot t^* + r \cdot r^*)^2 + (r \cdot t^* - t \cdot r^*)^2\right)^{1/2}$$

$$= \frac{1}{4}\left(2 f \cdot f^* g \cdot g^* + 2 f \cdot f^* g \cdot g^*\right)^{1/2} = \frac{1}{2}\left(g \cdot g^* f \cdot f^*\right)^{1/2};$$

in agreement with (5.19a). From (4.15a) we find, using (5.17), (5.20b) and (5.20d), and noting that for small X and Δ the second term in (5.20b) is small relative to the first,

$$X = i\Gamma \frac{\left[\frac{X - i\Delta}{X + i\Delta}\right]^{1/2} f \cdot g^* - \left[\frac{X + i\Delta}{X - i\Delta}\right]^{1/2} g \cdot f^*}{\frac{f \cdot f^*}{(X^2 + \Delta^2)^{1/2}}}$$

$$= \frac{i}{2c}\left((X - i\Delta) f \cdot g^* - (X + i\Delta) g \cdot f^*\right).$$

This is satisfied if (5.19b) is. Similarly, from (4.15b), (5.20a) and (5.20c) one obtains

$$\Delta = -\frac{1}{2c}\left((X - i\Delta) f \cdot g^* + (X + i\Delta) g \cdot f^*\right)$$

which is also satisfied if (5.19b) is.

The Eigenphase Behavior for a Degenerate Dipole

The eigenphase behavior for a dipole can be found using a generalization of Theorem I of Section III to matrices of the form (5.2). Since the derivation is considerably more involved and not particularly instructive, we consider here only the special case for which the vectors g and f are proportional:

$$f = \alpha g, \quad g = \frac{1}{\alpha} f. \tag{5.21}$$

From (5.17) and the unitarity constraints (5.19) there follows

$$\frac{1}{\alpha} f \cdot f^* = -\frac{1}{\alpha} f \cdot f^* = i\left[\frac{f \cdot f^*}{g \cdot g^*}\right]^{1/2};$$

whence:

$$\alpha = \frac{2\Gamma}{i}. \tag{5.22}$$

From (5.19a) we have also

$$\Gamma = \frac{1}{2}\, g \cdot g^* \,(\alpha\, \alpha^*)^{1/2} = \frac{1}{2}\, g \cdot g^* \,(4\, \Gamma^2)^{1/2} = \Gamma\, g \cdot g^*$$

whence:

$$g \cdot g^* = 1\,. \tag{5.23}$$

For a degenerate dipole we find from (5.6) and (5.10) that (5.2) can be written as

$$S = B + \frac{2\,\alpha\, G}{Y + i\,\Gamma} + \frac{\alpha^2\, G}{(Y + i\,\Gamma)^2} = B + \frac{\alpha}{(Y + i\,\Gamma)^2}\,(2\,Y + \alpha + 2\,i\,\Gamma)\,G\,;$$

or, by (5.22)

$$S = B - \frac{4\,i\,\Gamma\,Y}{(Y + i\,\Gamma)^2}\,G\,. \tag{5.24}$$

We may without loss of generality consider B a diagonal matrix, for, as shown in Section III, this may be achieved by an orthogonal transformation of S;

$$B_{i\,i} = \sigma_i^0 = e^{2\,i\,\delta_i^0}\,. \tag{5.25a}$$

From (5.16) and (5.17) we have also

$$B_{i\,i} = \frac{g_i}{g_i^*}\,. \tag{5.25b}$$

For diagonal B the matrix

$$S - \sigma\, I$$

is of the form (3.21) and we may apply Theorem I to evaluate its determinant. The result is

$$\mathrm{Det}\,(S - \sigma\, I) = \Big[\prod_1 (\sigma_1^0 - \sigma)\Big]\left(1 - \frac{4\,i\,\Gamma\,Y}{(Y + i\,\Gamma)^2}\sum_n \frac{g_n^2}{\sigma_n^0 - \sigma}\right). \tag{5.26}$$

The eigenphases are those δ for which

$$\sigma = e^{2\,i\,\delta}$$

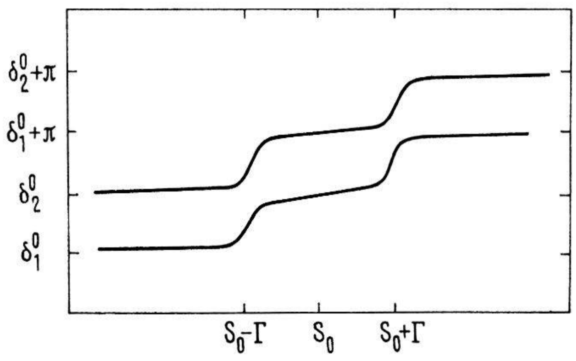

Figure 5
The eigenphases for a degenerate dipole.

are roots of (5.26). Using (5.23) and (5.25) this equation can be rewritten as

$$\frac{\Gamma^2 - Y^2}{2\,\Gamma\,Y} = G(\delta) = \sum_n g_n\, g_n^* \cot\,(\delta - \delta_n^0)\,; \tag{5.27a}$$

or, solving for Y,

$$Y = s - s_0 = -\Gamma\,(G \pm (G^2 + 1)^{1/2})\,. \tag{5.27b}$$

To obtain a plot of the eigenphases as functions of s one first plots the function appearing on the right-hand side of (5.27b) and then turns the figure on its side. For the two channel case the result is shown in Figure 5.

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