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Autor(en): **Ingleby, M.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **44 (1971)**

Heft 3

PDF erstellt am: **13.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-114283

Some Criticisms of Quantum Logic

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(11. VII. 70)

Abstract. We argue that the division ring (sfield) obtained by C. Piron after embedding a coherent lattice of propositions (questions) in a projective geometry is a lower bound on the division ring of coefficients over which a quantum-mechanical Hilbert Space must be constructed. Using the first four of G. W. Mackey's six axioms preceding his adoption of von Neumann's quantum mechanics in his seventh axiom, a Hilbert Space over any valuated sfield is constructed. Then observables are represented as projection-valued measures and certain states are represented as rays.

1. Introduction

Kolmogorov [1] laid the foundations of an axiomatic theory of probability operating with a triple $(\Omega, \mathfrak{J}, P)$ in which Ω is a set of *trials*, $\mathfrak{J}$ a collection of subsets (called *events*) of Ω, forming a σ-algebra, and P a map or probability measure assigning a probability $0 \leqslant P(F) \leqslant 1$ to each event F in $\mathfrak{J}$. Birkoff and von Neumann [2] observed, however, that the state of a quantum mechanical system is a density matrix assigning a probability to each closed subspace of some complex Hilbert space H: the set $Q(H)$ of closed subspaces is a complete, orthocomplemented but non-distributive lattice, the last property signifying that, unlike Kolmogorov's $\mathfrak{J}$, $Q(H)$ is not a σ-algebra. The temptation to conclude that Kolmogorov's framework must be broadened if it is to include the intuitive content of quantum mechanics proved too strong to be resisted.

Mackey [3], borrowing heavily from Khinchin's [4] application of the Kolmogorov approach to Gibbsian classical statistical mechanics, synthesised several attempts to obtain a replacement for Kolmogorov's 'logic of events' $\mathfrak{J}$. Mackey constructed a set Q_0 of *questions* of the form 'Does observable A have a value in the Borel set B of the real line$\mathsf{E}$' from sets S and O of states and observables respectively. Such a Q_0 carries a partial ordering but, as Gunson [5] has observed, it is not in general a lattice. Set Q_0 can, however, be embedded in a smallest lattice Q_1 or in a smallest σ-lattice Q_2 or even in a smallest complete lattice Q_3 as we shall show. In recent work Varadarajan [6] works in Q_1 and Jauch and Piron [7] in Q_3.

The fact that embedding is necessary is not usually stressed, though it seems to be quite important: clearly Q_0 may be embedded in any lattice (resp. σ-lattice, resp. complete lattice) which contains Q_1 (resp. Q_2, resp. Q_3) and some caution is called for in attempting to draw profound conclusions from embedding in a smallest hull.

Work such as Varadarajan's [8] on simultaneous observability requires a hull at least as big as Q_2 since it is concerned with Boolean sub-σ-lattices of a hull. The crux of this work is the invocation of a theorem on these sub-lattices which Varadarajan attributes exclusively to Loomis [9]. (The attribution is unjust to Sikorski [10] whose book contains a history of the theorem and a particularly transparent proof by construction.)

In Q_3, the addition of an extra property, supposed to be equivalent to the unrestricted superposition principle of quantum physics, allowed Piron [11] to embedd Q_3 in a smallest projective geometry. The circle of ideas surrounding the extra property is discussed in the book of Jauch [12] (and also in Gunson's paper [5]), under the names 'coherence' and 'simplicity'. Piron invoked a fundamental theorem of projective geometry to show that Q_3 is isomorphic to a lattice of closed subspaces of some Hilbert Space over a sfield D_0 determined by the projective geometry in which Q_3 sits. A slightly improved version of Piron's proof is presented in Varadarajan's book [6], and a considerable addition has been made by Eckmann and Zabey [13] who show that D_0 cannot be finite.

The significance of these results becomes clearer if one considers what they *do not say* about quantum theory. Because one is dealing with some smallest projective geometry containing Q_0, one is free to choose a bigger hull and hence extend D_0 to some $D \supset D_0$. The Eckmann-Zabey result is thus seen to push up the lower bound D_0 beyond 'quaint' finite sfields, but there is no a priori reason for working over the minimal sfield D_0.

The developments reported here take place over any valuated sfield C. The space $H(S, C)$ calls for no embedding but nevertheless carries Q_0 as a set of projection operators. Only the first four of Mackey's [3] axioms are employed, and the principal object of interest is the lattice $\mathcal{K}(S)$ of *convex* subsets of the set of states. The study of convexity follows a manuscript circulated by the author at Sussex and Geneva.

2. The Axioms

The symbol $\mathcal{B}$ denotes the σ-algebra of Borel sets of the real line A general information system is a map p: $S \times O \times \mathcal{B} \to [0, 1]$, where S is a set of states, O is a set of observables and, for each state α and observable A, $p(\alpha, A, .) : \mathcal{B} \to [0, 1]$ is a probability measure, $p(\alpha, A, B)$ being the probability in state α that observable A has a value in the Borel set $B \in \mathcal{B}$. This covers the ground of Mackey's first axiom. Additional axioms are:

2.1. (Separation) (a) $p(\alpha, A, B) = p(\alpha', A, B) \quad \forall A, B$
$\Rightarrow \alpha = \alpha'$

(b) $p(\alpha, A, B) = p(\alpha, A', B) \quad \forall A, B$
$\Rightarrow A = A'$;

2.2. (Convexity) $\forall (\alpha_i)_{i=1}^{n \leqslant \infty} \; (0 \leqslant t_i \leqslant {}_1)_{i=1}^{n \leqslant \infty} \; \Sigma_i t_i = 1$

$\exists \, \alpha$ denoted $\alpha = \Sigma_i t_i \alpha_i$ such that

$p(\alpha, A, B) = \Sigma_i t_i p(\alpha_i, A, B) \quad \forall A, B$;

2.3. (Functional closure) $\forall$ measureable functions $f: \mathsf{R} \to \mathsf{R}$, $\forall A \in \mathcal{O}$

$\exists f(A) \in \mathcal{O}$ such that

$p(\alpha, f(A), B) = p(\alpha, A, f^{-1}(B))$, $\mathsf{A}\, \alpha, B$.

Property 2.2 merely says that $\mathcal{S}$ has to be large enough to accommodate mixtures of states with any weights $(t_i)_{i=1}^{n \leqslant \infty}$. Property 2.3 says that $\mathcal{O}$ has to be large enough to accommodate measureable functions of a single observable; this property is crucial for constructing $\mathcal{Q}_0$.

The measureable function $\Phi_E: \mathsf{R} \to \mathsf{R}$, the Heaviside step function on $E \in \mathcal{B}$, has an inverse defined by $\Phi_E^{-1}(B) = \mathsf{R}$ if $\{0, 1\} \subset B$, $\Phi_E^{-1}(B) = E$ if $1 \in B$ and $0 \notin B$, $\Phi_E^{-1}(B) = -E$ if $1 \notin B$ and $0 \in B$ and $\Phi_E^{-1}(B) = \phi$ otherwise. For any A and any $E \in \mathcal{B}$, $\Phi_E(A)$ is a question and $p(\alpha, \Phi_E(A), B) = p(\alpha, A, \Phi_E^{-1}(B)) = 1$ if $\{0, 1\} \subset B$, $= p(\alpha, A, E)$ if $1 \in B$ and $0 \notin B$, $= 1 - p(\alpha, A, E)$ if $1 \notin B$ and $0 \in B$, and $= 0$ if $1 \notin B$ and $0 \notin B$. Set $\mathcal{Q}_0$ is the set of all questions of this form, each one being characterised by the numbers $M_\alpha(\Phi_E(A)) = p(\alpha, A, E)$. The complement of $\Phi_E(A)$ is Φ_E (A), the partial ordering of $\mathcal{Q}_0$ is defined by $Q_1 \leqslant Q_2$ if $M_\alpha(Q_1) \leqslant M_\alpha(Q_2)$ for all α. A question Q_3 is said to be the *meet* of Q_1 and Q_2 if $M_\alpha(Q_3) = \inf\{M_\alpha(Q_1), M_\alpha(Q_2)\}$; clearly not all pairs have meets but we may use the set $\{M_\alpha(Q_3)\}$ as α runs over as a *definition* of $Q_3 = Q_1 \wedge Q_2$. Joins are defined by taking complements $Q_1 \wedge Q_2 = Q_1 \wedge Q_2$ and this gives us the lattice $\mathcal{Q}_1$ alluded to in the introduction. The *definition* of countable (or even continuous) meets in a similar way $M_\alpha(\bigwedge_{i\varepsilon I} Q_i) = \inf_{i\varepsilon I}\{M_\alpha(Q_i)\}$ where the set of indices I is countable (or continuous) defines the hulls $\mathcal{Q}_2$ (and $\mathcal{Q}_3$) alluded to in the introduction.

A subset $K \subset \mathcal{S}$ is called *convex* if any countable family $(\alpha_i)_{i=1}^{n \leqslant \infty}$ in K has all its mixtures with any weights $(t_i)_{i=1}^{n \leqslant \infty}$, $\alpha = \sum_i t_i \alpha_i$ in K. The set of all convex subsets of $\mathcal{S}$ will be denoted $\mathcal{K}(\mathcal{S})$, which is not empty since $\mathcal{S}$ and ϕ belong. For future reference, the state α *affirms* Q_1 if $M_\alpha(Q_1) = 1$; α negates Q_2 if $M_\alpha(Q_2) = 0$ or if $M_\alpha(Q_2') = 1$. A set of states which affirms (or a set which negates) a question is convex. More generally $\mathcal{S}_t(Q) = \{\alpha \in \mathcal{S} \mid M_\alpha(Q) = t\}$ is convex for all t in $[0, 1]$.

3. Convex Sets of States

The aim of the following list of definitions is to set up a dimension theoty of convex sets of states which resembles as closely as possible the dimension theory of finite linear subspaces of a vector space or module.

3.1. *Definition*

The *convex hull* $\overline{R}$ of a subset $R \subset \mathcal{S}$ is the set $\{\alpha = \sum_{i=1}^{i<n\leqslant\infty} t_i \alpha_i$; where $\forall i = 1, \ldots, n$, $0 < t_i < 1$, $\alpha_i \in R$ and $\sum_{i=1}^{n} t_i = 1\}$. Clearly if $R \in \mathcal{K}(\mathcal{S})$ $R = \overline{R}$, and in general $R \subset \overline{R}$.

3.2. *Definition*

A set $R \subset \mathcal{S}$ is convexly independent if $\forall \alpha \in R\; \alpha \notin \overline{R - \{\alpha\}}$, i.e. α is not a mixture of other states in R.

3.3. *Definition*

A convexly independent set $R \subset S$ is maximally convexly independent (is a *convexity basis*) if $\overline{R} = S$.

3.4. *Definition*

The dimension of a convex set $K \in \mathcal{K}(S)$ is the cardinal number $\mu(K) = |K \cap R|$ for some convexity basis R.

The next developments are necessary to show that 3.1–3.3 are precisely analogous to corresponding notions in linear algebra and to show that μ defined in 3.4 does not depend on choice of basis.

Given two general information systems p_1: $S_1 \times O_1 \times \mathcal{B} \to [0, 1]$ and p_2: $S_2 \times O_2 \times \mathcal{B} \to [0, 1]$ satisfying 2.1–2.3, a pair of maps η: $S_1 \to S_2$ and η^+: $O_1 \to O_2$ is a *morphism* if

3.5. *Definition*

$$p_1(\alpha, A, B) = p_2(\eta(\alpha), \eta^+(A), B) \qquad \forall\, \alpha, A, B.$$

3.6. *Proposition*

The maps η and η^+ which define a morphism between p_1: $S_1 \times O_1 \times \mathcal{B} \to [0, 1]$ and p_2: $S_2 \times O_2 \times \mathcal{B} \to [0, 1]$ are set monomorphisms.

Proof: (a) $\eta^+(A) = \eta^+(A') \to A = A'$,

$$\begin{aligned} \text{for } p_1(\alpha, A, .) &= p_2(\eta(\alpha), \eta^+(A), .) && \forall\, \alpha \in S_1 \\ &= p_2(\eta(\alpha), \eta^+(A'), .) && \forall\, \alpha \in S_1 \\ &= p_1(\alpha, A', .) && \forall\, \alpha \in S_1 \end{aligned}$$

$\Rightarrow A = A'$ by separation 2.1 (b) applied to p_1;

(b) $\eta(\alpha) = \eta(\alpha') \Rightarrow \alpha = \alpha'$,

$$\begin{aligned} \text{for } p_1(\alpha, A, .) &= p_2(\eta(\alpha), \eta^+(A), .) && \forall\, A \in O_1 \\ &= p_2(\eta(\alpha'), \eta^+(A), .) && \forall\, A \in O_1 \\ &= p_1(\alpha', A, .) && \forall\, A \in O_1 \end{aligned}$$

$\to \alpha = \alpha'$ by separation 2.1 (a) applied to p_1.

Convexity and functional closure are not invoked, so proposition 3.6 holds even when 2.2 and 2.3 do not. The same is true of the following

3.7. *Proposition*

For any separated general information system p_1: $S_1 \times O_1 \times \mathcal{B} \to [0, 1]$ and any 1:1 map η: $S_1 \to S_2$ onto a set S_2, there is a set O_2 a map η^+: $O_1 \to O_2$ and a separated general information system p_2: $S_2 \times O_2 \times \mathcal{B} \to [0, 1]$ isomorphic under the pair (η, η^+) to p_1.

Proof: Because of the separation property, O_2 is specified by giving an S_2-indexed family of probability measures. Thus, let $O_2 = \{A';\, A'_\alpha = p_1(\eta^{-1}(\alpha), A, .)\ \forall\, \alpha \in S_2\}$ and define p_2: $S_2 \times O_2 \times \mathcal{B} \to [0, 1]$ by $p_2(\alpha, A', .) = p_1(\eta^{-1}(\alpha), A, .)$, ${\eta^+}^{-1}$: $O_2 \to O_1$ by ${\eta^+}^{-1}(A') = A$. Then η^{-1}, η^+ form an isomorphism p_2 to p_1, so they are invertible to (η, η^+): $p_1 \to p_2$.

Corollary 1. The convex hull in S_2 of $\eta(R)$, where $R \subset S_1$, is the image under η of the convex hull $\bar{R}$ in S_1.

Proof: Let $\alpha = \sum_1 t_i \alpha_i$ where each $\alpha_i \in R$ and observe that

$$p_1(\alpha, A, .) = \sum_i t_i\, p_1(\alpha_i, A, .) \qquad \forall A \in O_1$$
$$= \sum_i t_i\, p_2(\eta(\alpha_i), \eta^+(A) = A', .) \qquad \forall A' \in O_2 \text{ in 3.7.}$$

Hence $\eta(\alpha) = \sum_i t_i\, \eta(\alpha_i)$ is contained in $\overline{\eta(R)}$ or $\eta(\bar{R}) \subseteq \overline{\eta(R)}$. The reverse inclusion is proved likewise.

Corollary 2. The image in S_2 of a convexly independent set in S_1 is convexly independent in S_2; the image in S_2 of a convexity basis in S_1 is a convexity basis. The proofs follow immediately from the definitions 3.2–3.3 and the application of corollary 1 to the convex hulls entering into these definitions.

The stage is now set for the main

3.8. *Theorem*

In an isomorphism class of general information systems satisfying 2.1 and 2.2 the cardinal numbers of any pair of convexity bases are equal to each other and to a number, characteristic of the class, called the *convex dimension* of the class.

Proof: Let p_1: $S_1 \times O_1 \times \mathcal{B} \to [0, 1]$ and p_2: $S_2 \times O_2 \times \mathcal{B} \to [0, 1]$ be isomorphic under maps η, η^+; and let $R_1 \subset S_1$ and $R_2 \subset S_2$ be convexity bases. The proof proceeds by showing that $|R_1| < |R_2|$ leads to a contradiction.

By 3.7 corollary 2, $\eta(R_1)$ is a convexity basis for p_2 and either (i) $\eta(R_1) \subset R_2$ properly or (ii) $\eta(R_1) \cup R_2 = R$ ($\neq \eta(R_1)$). In case (i) $\exists\, \alpha \in R_2 - \eta(R_1)$ nonzero and $\alpha \in \overline{\eta(R_1)}$, contradicting the convex independence of R_2. The case (ii) must be reduced to (i) by a suitable isomorphism.

Let $\eta(R_1) = R \cup R_1'$ and $R_2 = R \cap R_2'$, $R_1' \cap R_2' = R \cap R_1' = R \cap R_2' = \phi$, and suppose $|R_1'| < |R_2'|$ so there is a 1:1 map φ_{00}: $R_1' \to R_2'$ for which $R_2' = \operatorname{Im}\varphi_{00}$ is properly contained in R_2'. Define φ_0: $\eta(R_1) \to R_1$ by $\varphi_0(\alpha) = \begin{cases} \eta^{-1}(\alpha) \text{ if } \alpha \in R \\ \eta^{-1}(\varphi_{00}(\alpha)) \text{ if } \alpha \in R_1', \end{cases}$

whose image in S_1 is a convexity base for the new system defined by constructing φ_0^+, $O_3 = \varphi^+(O_2)$ and p_3 as in 3.7. By construction $\eta(\varphi_0^{-1}(\eta(R_1))) = R \,\breve{\cup}\, R_2''$ properly contained in $R \cup R_2' = R_2$, reducing case (ii) to case (i).

Corollary. The cardinal numbers $\mu(\mathcal{K})$ of definition 3.4 are independent of convexity base.

Proof: For any convex K consider the non-separated general information system p_1: $K \times O_1 \times \mathcal{B} \to [0, 1]$ and the following equivalence relation in O_1: $A_1 \sim A_2$ if $p_1(\alpha, A_1, .) = p_2(\alpha, A_2, .)$ for all α in K. Let $O_2 = O_1/\!\sim$ and let $[A]$ in O_2 denote the $\sim$-equivalence class which contains O_1-observable A. Then the map p_2: $K \times O_2 \times \mathcal{B} \to [0, 1]$ defined by $p_2(\alpha, [A], .) = p_1(\alpha, A, .)$ is a general information system satisfying 2.1 and 2.2 (possibly 2.3, too). For any convexity basis R of S_1, $K \cap R$ is convexly independent in K. Moreover, $\overline{K \cap R} = \bar{K} \cap \bar{R} = K \cap S_1 = K$ which means that $K \cap R$ is a convexity basis for the system defined by p_2. By application of the theorem to p_2, $|K \cap R| = \mu(K)$ is independent of choice of R. The set $\mathcal{K}(S)$ of convex subsets of the states for some p: $S \times O \times \mathcal{B} \to [0, 1]$ is partially ordered by the inclusion relation which will be denoted $<$. In fact, $\mathcal{K}(S)$ is a complete lattice with meets $\bigwedge_{i\varepsilon I} K_i = \bigcap_{i\varepsilon I} K$: for all I-indexed families of convex sets and corresponding joins $V_{i\varepsilon I} K_i = \overline{\bigcup_{i\varepsilon I} K_i}$. Given a convexity basis $R \subset S$, these lattice operations are conveniently expressed in the form

3.9. (a) $\bigwedge_{i\varepsilon I} K_i = \overline{\bigcap_{i\varepsilon I} k_i}$,

(b) $V_{i\varepsilon I} K_i = \overline{\bigcup_{i\varepsilon I} k_i}$

where $k_i = R \cap K_i$ is the generator of K_i in the sense $\overline{k}_i = K_i$. The unit of $\mathcal{K}(S)$ is S and the zero is ϕ, the empty set. Most of the properties assumed for suitable extensions (such as Q_3) of the Q_0 associated with p are easily proved for $\mathcal{K}(S)$, as is shown in the following

3.10. *Theorem*

The complete lattice $\mathcal{K}(S)$ is orthocomplemented, weakly modular [12, 14] but not modular (a fortiori not distributive) and atomic. The covering law [7] is also true in $\mathcal{K}(S)$.

Proof: (i) Atomicity. The singleton sets $\{\alpha\} \in \mathcal{K}(S)$ are atoms, for $K < \{\alpha\}$ and $K \neq \{\alpha\} \Rightarrow K = \phi$.

(ii) Orthocomplementation. The axioms of orthocomplementation are

(a) $K'' = K$, (b) $K_1 < K_2 \Longleftrightarrow K_2' < K_1'$, (c) $K \vee K' = \phi, K \vee K' = S$,

given a convexity basis R, define $k = (R \cap K) \subset K$ as above and let $K' = \overline{R - k}$. To show that this is independent of basis let R_0 be a convexity basis different from R and let $K^0 = R_0 - k_0$ where $k_0 = R_0 \cap K$. If $K' < K^0$, then by interchanging R and R^0 $K^0 < K'$, proving the identity of $'$ and ^{0}. Let $\alpha = \sum_{i=1}^{n \leqslant \infty} t_i \alpha_i$ be an element of K' so that each of the α_i's lies outside K. But each α_i outside K is a mixture of β_j's in k_0: $\alpha_i = \sum_{j=i}^{m \leqslant \infty} S_{ij} \beta_j$ where the S_{ij}'s are suitable weights. Hence α is a mixture of β_j's in k_0, or $\alpha \in K^0$. To prove (a) we note that $k' = R - k$, $k'' = R - (R - k) = k$, therefore $\overline{k''} = K'' = \overline{k} = K$. To prove (b) we note that $K_1 < K_2$ implies that $k_1 \subset k_2$ or, taking complements in R, that $k_2' \subset k_1'$. Hence $\overline{k_2'} < \overline{k_1'}$. To prove (c), we note that $k \cap k' = \phi$ and $\overline{\phi} = \phi$; also that $k \cup k' = R$ and $\overline{R} = S$.

(iii) Non-modularity. Denote $\{\alpha = t\,\alpha_i + (1-t)\,\alpha_j;\ 0 < t < 1\}$ by K_{ij} and let α_s be some point in the interior of the interval K_{ij}. Then $K_{is} \wedge ((\{\alpha_i\} \vee \{\alpha_j\}) = K_{ij}) = K_{is}$, $(K_{is} \wedge \{\alpha_i\}) \vee \{\alpha_j\} = K_{ij}$ and these must be equal if $\mathcal{K}(S)$ is modular.

(iv) Weak modularity. This property can be defined in three equivalent ways:

(a) $K_1 < K_2 \Rightarrow K_2 = (K_2 \Lambda K_1') V K_1$;

(b) $K_1 < K_2$ and $K_1' \Lambda K_2 = \phi \Rightarrow K_1 = K_2$;

(c) $K_1 < K_3$ and $K_2 < K_3' \Rightarrow (K_1 \vee K_2) \wedge K_3 = K_1$.

The equivalence is proved in Zierler's [14] lemma 1.3. Form (b) is easily proved: $K_1' \wedge K_2 = \phi \Rightarrow k_1' \cap k_2 = \phi$ and $K_1 < K_2 \Rightarrow k_1 \subset k_2$; by Boolean algebra in the power set of R, $k_1 = k_2$ and hence $K_1 = K_2$.

(v) Covering axiom. This takes the form $K_1 < K_1 \vee \{\alpha\}$ for some atom $\{\alpha\} \Leftarrow$ either $K = K_1$ or $K = K_1 \vee \{\alpha\}$. The case $\{\alpha\}$ belongs to K_1 is trivial since in this case $K_1 \vee \{\alpha\} = K_1$. In any other case $\alpha = t\alpha_1 (1-t)\,\alpha_2$, $0 < t < 1$ and $\alpha_1 \in K_1$, $\alpha_2 \in K_1'$; thus $R_2 = k_1 \cup \alpha_2$ is a basis for $K_1 \vee \{\alpha\}$. Now $K < K_2$ implies that $K \cap R_2 \subseteq R_2$. Let $K \cap R_2 = k \cap k_0$ so that $k_0 \subseteq \{\alpha_2\}$ and since $\{\alpha_2\}$ is an atom in the Boolean algebra of the power set of R_2, this implies that either $k_0 = \phi$ $(K = K$) or $k_0 = \{\alpha_2\}$ $(K = K_1 \vee \{\alpha\})$.

Not only does $\mathcal{K}(S)$ have the properties desired of the usual lattices of questions, but in some weak sense elaborated below, it contains the questions. Moreover, the containment is such that when the hull Q_1 of Q_0 is formed, a fairly serious contradiction can be produced. The set Q_0 is first embedded in a set $\mathcal{K}(S)^\infty$ constructed as follows.

Let $\{^1/_2\}'$ be the set $[0, ^1/_2 [\cup] ^1/_2, 1] \subset [0, 1]$ and let $\mathcal{K}(S)^\infty$ be the Kronecker product $\{f\colon \{^1/_2\}' \to \mathcal{K}(S) \mid f \text{ a map}\}$. In $\mathcal{K}(S)^\infty$ let

3.11. (a) $\mathfrak{f}_1 \leqslant \mathfrak{f}_2$ if $\mathfrak{f}_1(t) \leqslant \mathfrak{f}_2(t) \quad \forall t \in \{^1/_2\}'$,
(b) $\mathfrak{f} = V_{i\in I}\,\mathfrak{f}_i$ if $\mathfrak{f}(t) = V_{i\in I}\,\mathfrak{f}_i(t) \quad \forall t \in \{^1/_2\}'$,
(c) $\mathfrak{f} = \Lambda_{i\in I}\,\mathfrak{f}_i$ if $\mathfrak{f}(t) = \Lambda_{i\in I}\,\mathfrak{f}_i(t) \quad \forall t \in \{^1/_2\}'$,
(d) $\mathfrak{f}_1 = \mathfrak{f}_2$ if $\mathfrak{f}_1(t) = \mathfrak{f}_2(t)' \quad \forall t \in \{^1/_2\}'$.

That $\mathcal{K}(S)_\infty$ possesses essentially the same properties as $\mathcal{K}(S)$ is shown in the following.

3.12. *Proposition*

$\mathcal{K}(S)^\infty$ with the pointwise operations 3.11 is a (i) complete, (ii) atomic, (iii) orthocomplemented, (iv) weakly modular but (v) non-modular (a fortiori non-distributive) lattice in which (vi) the covering axiom is true.

Proof: (i) The R.H.S.'s of 3.11 (b) and (c) are defined since $\mathcal{K}(S)$ is complete. (ii) The atoms of $\mathcal{K}(S)^\infty$ are obviously the functions $\mathfrak{f}\colon \{^1/_2\}' \to S$ whose values are singleton sets (atoms) of $\mathcal{K}(S)$. (iii)–(vi) are simple consequences of the corresponding properties of $\mathcal{K}(S)$ which hold for all points of $\{^1/_2\}'$.

The embedding of Q_0 in $\mathcal{K}(S)^\infty$ is described by the following

3.13. *Theorem*

The map $j\colon Q_0 \to \mathcal{K}(S)^\infty$ defined by $j(Q)(t) = S_t(Q) = \{\alpha \in S \mid M_\alpha(Q) = t\}$ $\forall t \in \{^1/_2\}'$ is (i) one-to-one, (ii) monotone, and (iii) preserves complements only in jhe weak sense $j(Q') \leqslant j(Q)'$, (iv) is not generally lattice-continuous in the sense $t(Q_1 \, V \, Q_2) = j(Q_1)\, V\, j(Q_2)$ when the join on the left is defined in Q_0.

Proof: (i) Let $j(Q_1) = j(Q_2)$ or $S_t(Q_1) = S_t(Q_2)$ if $t \neq {}^1/_2$. Since $S = U_t\, S_t(Q_1) = U_t\, S_t(Q_2)$, $S_{1/2}(Q_1) = S_{1/2}(Q_2)$ also; and by separation $Q_1 = Q_2$. (ii) Let $Q_1 \leqslant Q_2$ or $M_\alpha(Q_1) \leqslant M_\alpha(Q_2)\ \forall\ \alpha \in S$. Then $\forall t \in [0, 1]$ $U_{s<t}\, S_s(Q_1) \subseteq U_{s<t}\, S_s(Q_2)$ and also $U_{s\leq t}\ S_s(Q_1) \subseteq U_{s\leq t} S_s(Q_2)$. Taking complements of the smaller unions in larger, $S_t(Q_1) \subseteq S_t(Q_2)$, hence $j(Q_1) \leqslant j(Q_2)$. (iii) For $t \neq {}^1/_2$, $j(Q')(t) = \{\alpha \in \mid M_\alpha(Q) = 1 - t\} \subset \{\alpha \mid M_\alpha(Q) \neq t\} = j(Q)'(t)$. (iv) Let $Q_1\, V\, Q_2$ be defined in Q_0, and let $Q_1 \neq 0$. Then $\{j(Q_1)\ V\ j(Q_2)\}(t) = S_t(Q_1)\ V\ S_t(Q_2)$ and $j(Q_1\ V\ Q_2)(t) = \{\alpha \in S \mid \sup[M_\alpha(Q_1), M_\alpha(Q_2)] = t\} = S_1(t)\ U\ S_2(t)$ where $S_1(t) = \{\alpha \mid M_\alpha(Q_1) = t$ and $M_\alpha(Q_2) < t\}$ and $S_2(t) = \{\alpha \mid M_\alpha(Q_2) = t$ and $M_\alpha(Q_1) \leqslant t\}$. Both these sets are convex, and so is their union, hence $j(Q_1\ V\ Q_2)(t) = S_1(t)\ V\ S_2(t)$; and since $S_1(t) \subseteq S_t(Q_1)$ and $S_2(t) \subseteq S_t(Q_2)$, it follows that $j(Q_1\ V\ Q_2) \leqslant j(Q_1)\ V\ j(Q_2)$. The equality leads, however, to a contradiction: $j(Q_1\ V\ Q_2)(t) = \{j(Q_1)\ V\ j(Q_2)\}(t) \Rightarrow S_1(t) = S_t(Q_1)$ and $S_2(t) = S_t(Q_2)$ or, more explicitly, $S_t(Q_1) \subseteq U_{s<t}\ S_s(Q_2)$ and $S_s(Q_2) \subseteq U_{r\leq s}\, S_r(Q_1)$, which together imply that $S_t(Q_1) \subseteq U_{r<t}\, S_r(Q_1)$ or that $S_t(Q_1)$ is empty for all t, or that $Q_1 = 0$, the null question.

Although this embedding theorem is not quite as tidy in the mathematical sense as that, for example, quoted by Jauch [12] (p. 127), it has the merit of involving states, rather than some mathematical extension of Q_0 which could always be further extended by a monomorphism without an obvious stop on the process of extension.

The extension of Q_0 stops here at $\mathcal{K}(S)^\infty$, a set which is built out of the physical data contained in the states.

4. Construction of a Hilbert Space

The lattice $\mathcal{K}(S)$ already resembles the lattice of subspaces of a normed space: convex subsets of states are highly reminiscent of unit spheres of subspaces. Not unexpectedly there is a 'dual' of $\mathcal{K}(S)$ which is indeed a normed space – in fact it is a Hilbert space. The ensuing definitions are a first step towards constructing a 'dual' of $\mathcal{K}(S)$.

Let $\mathsf{R} \subseteq C$ and C be a sfield with a valuation $|\ |: C \to \mathsf{R}^+$ and an involution $*$: $C \to C$ for which $c^* c = |c|^2 \ \forall c \in C$. A function $f: S \to C$ is *convex* if $f(\sum_i t_i \alpha_i) = \sum_i t_i f(\alpha_i)$ for all mixtures $\sum_i t_i \alpha_i$ of states. Such functions exist in abundance, for any map $\varphi: R \to C$ from a convexity basis R of S extends to a convex $f_\varphi(\sum_i t_i \alpha_i) = \sum_i t_i \varphi(\alpha_i)$. Let $R = (\alpha_i)_{i\varepsilon I}$ be a convexity base for S. A convex function $f: S \to C$ is R-summable if $\|f\|_R = \sum_i |f(\alpha_i)|^2$ is finite. Such f's exist: a function $\varphi: R \to C$ which is nonzero on only a finite number of states in the basis generates an f_φ with $\|f_\varphi\|_R = \sum_i |\varphi(\alpha_i)|^2$, the sum of a finite number of real positive terms. The map $\|.\|_R$ clearly satisfies the triangle inequality and $\|cf\|_R = |c|^2 \|f\|_R$, which makes $\|.\|_R$ a semi-norm on the linear vector space of R-summable functions. In fact $\|.\|_R$ is a norm, for $\|f\|_R = 0$ implies that $|f(\alpha_i)|^2 = 0 \ \forall i \in I$ and hence that f is the zero function. A sequence of functions which is Cauchy with respect to the $\|.\|_R$ norm clearly has a limit which is R-summable, so the space $\mathcal{M}(S, \mathsf{C})$ of R-summable, convex functions is a Banach space. In fact it is a Hilbert space whose inner product is given in the following.

4.1. *Proposition*

The Banach space $\mathcal{M}(S, \mathsf{C})$ of C-valued R-summable convex functions on S is a Hilbert space with respect to the sesquilinear form $(f_1, f_2)_R = \sum_{i\varepsilon I} f_1^*(\alpha_i)\, f_2(\alpha_i)$. Moreover, if $R' = (\alpha_j)_{j\varepsilon J}$ is some other convexity basis for S the sesquilinear form $(f_1, f_2)_{R'}$ is a multiple of the former $(f_1, f_2)_R$. The Hilbert space will be denoted $\mathcal{H}(S, \mathsf{C})$.

Proof: By the Schwartz inequality $(f_1, f_2)_R$ converges. Sesquilinearity, completeness and definiteness follow from above. The second part, $(f_1, f_2)_R = (f_1, f_2)_{R'}$ is proved by showing that the two inner products generate the same orthocomplementation, hence the result by straightforward application of a theorem of Birkoff von Neumann (Ann. Math. ,*37* 823 (1936). Now $(f_1, f_2)_R = 0$ implies $|(f_1, f_2)_R^{**}|^2 = 0$ and $|(f_1, f_2)_R|^2 = |\sum_i f_i(\alpha_i)\, f_1(\alpha_i)|^2 = |\sum_i \sum_j t^2_{ij}\, f_1^*(\alpha_j)\, f_2(\alpha_j)|^2$ where $\alpha_i = \sum_j t^2_{ij}\, \alpha_j$. Let $T(j) = \sum_i t_{ij}$ and let the infimum of these (positive, finite) numbers be T_1. Then

$$\{|(f_1, f_2)_R|^2 > T_1^2\, |\textstyle\sum_j f_1^*(\alpha_j)\, f_2(\alpha_j)|^2 = T_1^2\, |(f_1, f_2)_{R'}|^2\}$$

thus $(f_1, f_2)_R = 0$ implies $(f_1, f_2)_{R'} = 0$. Similarly if T_2 denotes the supremum of the $T(j)$, $|(f_1, f_2)_R|^2 < T_2\, |(f_1, f_2)_{R'}|^2$ and $(f_1, f_2)_{R'} = 0$, thus allowing the Birkhoff von Neumann theorem to be invoked.

The proposition 4.1 establishes that the Hilbert space $H(S, \mathsf{C})$ does not depend on choice of basis for the states. It remains to show that questions can be made into projection operators on $H(S, \mathsf{C})$ for any reasonable C.

For any question Q, let $p(Q)$ be the characteristic function of the set $S_1(Q)$ of states for which Q is true. Then define $\hat{Q}: H(S, C) \to H(S, C)$ by $(\hat{Q} f)(\alpha) = p(Q)(\alpha) f(\alpha)$. The operator is obviously linear. Moreover $(\hat{Q} f, \hat{Q} f)_R = \sum_i |p(Q)(\alpha_i) f(\alpha_i)|^2$ and the effect of the $p(Q)(\alpha_i)$ is to cut out some of the summands which occur in $\| f \|^k$, ensuring the convergence of the sum $\| \hat{Q} f \|_R$. The operator is a projector because $p(Q)(\alpha)^2 = p(Q)(\alpha)$, a property of characteristic functions.

5. Conclusions

The developments reported here, though not absolutely conclusive, seem to suggest that the quantum logic approach to the foundations of microphysics is unlikely to tell physicists that they must not use complex-valued wave-functions. The whole question of the sfield is avoided by fixing attention only on the lattice $\mathcal{K}(S)$ which has a 'dual' $H(S, C)$ for any suitable C. In any case only the probabilities $p(\alpha, A, E)$ enter the formation of statistical hypotheses to be tested by experiment.

Acknowledgements

This work was completed while the author was visiting the University of Sussex as a U.K. Science Research Council Fellow. The author is extremely grateful to the S.R.C. for financial support, to the University of Sussex for its hospitality and to a colleague, Dr. M. J. Wilford, for critical comments on an early draft. The kind response of Professor J. M. Jauch of Université de Genève to an early version of these ideas encouraged the author to develop them further. Only the author is responsible for any errors that may remain.

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