# Multi-particle quantum systems below the four-particle threshold

# Multi-particle Quantum Systems below the Four-Particle Threshold [1])

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*Abstract.* In the time independent approach to the scattering theory of a quantum mechanical system of $N$ nonrelativistic particles, the wave operators are expressed in terms of amplitudes satisfying the Faddeev-Yakubovsky equations. We study these equations for energies below the four particle threshold $s_4$. A graphical method is introduced to classify and analyze the singular integrals occuring in the iterated Yakubovsky kernels. Under a certain spectral assumption, we establish and control the Fredholm alternative. Partial asymptotic completeness of the scattering states below $s_4$ follows.

## 1. Wave Operators and the Resolvent Operator

We consider a quantum mechanical $N$-body system with a Hamiltonian

$$H \equiv \sum_{i=1}^{N} \frac{p_i^2}{2\,m_i} + \sum_{1 \le i < j \le N} V_{ij} \equiv H^0 + V \tag{1.1}$$

acting on the C-M Hilbert space $\mathcal{H} = L^2(R^{3N-3})$. The $p_i$ and the $m_i$ are respectively the C-M momentum and the mass of particle $i$. Throughout this article we shall assume that the Fourier transforms $v_{ij}$ of the potentials satisfy Faddeev's conditions [2]:

$$|v(k)| \leqslant c\,(1 + |k|)^{-\theta}, \qquad \theta > 3/2\,, \tag{1.2}$$

$$|v\,(k+h) - v(k)| \leqslant c\,(1 + |k|^{-\theta}\,|h|^{\mu}\,, \qquad |h| \leqslant 1\,,\ \mu > 1/2\,, \tag{1.3}$$

$$v(-k) = \overline{v(k)}\,. \tag{1.4}$$

In particular, the potentials are real valued and $L^2$ in $x$-space.

Let $a_k$ be a partion of $\{1, \dots N\}$ into $k$ disjoint clusters $(A_1, \dots A_k)$. The Hamiltonian of the non interacting clusters is:

$$H_{a_k} \equiv H^0 + \sum_{A_i \in a_k} V_{a_i} \equiv H^0 + V_{a_k} \equiv \overline{H}^0_{a_k} + \hat{H}^0_{a_k} + V_{a_k} \equiv \overline{H}^0_{a_k} + \hat{H}_{a_k} \tag{1.5}$$

where

$$V_A \equiv \sum_{i<j \in A} V_{ij}\,, \tag{1.6}$$

$$\overline{H}^0_{a_k} \equiv \sum_{j=1}^{k} \frac{p^2(A_j)}{2\,m(A_j)} \equiv E_{a_k}(\overline{p}_{a_k}) \tag{1.7}$$

and

[1]) Extracted from the author's thesis [1].

$$\hat{H}^0_{a_k} \equiv \sum_{A_j \in a_k} \hat{E}_{A_j}(\hat{p}_{A_j}), \quad \hat{E}_{A_j}(\hat{p}_{A_j}) \equiv \sum_{i \in A_j} \frac{\hat{p}^2(A_j)}{2\, m_i} \tag{1.8}$$

The introduction of the total momenta $\overline{p}_{a_k} \equiv (p(A_1), \dots p(A_k))$ of the clusters $A_j$ and of the momenta $\hat{p}_i(A_j)$ of the particles in $A_j$ relative to the C-M of $A_j$ corresponds to a factorization of the Hilbert space:

$$\mathcal{H} = \mathcal{H}_{a_k} \otimes \hat{\mathcal{H}}_{a_k} = \overline{\mathcal{H}}_{a_k} \bigotimes_{j=1}^{k} \hat{\mathcal{H}}_{a_j} \tag{1.9}$$

$H_{a_k}$ acts in the obvious manner on the product (1.9).

The operators $H$, $H_{a_k}$ and $\hat{H}_{a_k}$ are self adjoint on $D(H^0)$ and $D(\hat{H}^0_{a_k})$ respectively and bounded from below [3].

The operators $H_{a_k}$ describe possible asymptotic motions of our system. In order to formulate this we define a channel as a pair $(a_k, \hat{\varphi}_{a_k})$ where $a_k$ is a partition of $\{1, \dots N\}$ and $\hat{\varphi}_{a_k} \in \hat{\mathcal{H}}_{a_k}$ is a bound state of $\hat{H}_{a_k}$: $\hat{H}_{a_k} \hat{\varphi}_{a_k} = -\varkappa^2_{a_k} \hat{\varphi}_{a_k}$. On the channel states $\Phi_{a_k} \in D_{a_k} \equiv \overline{\mathcal{H}}_{a_k} \otimes \hat{\varphi}_{a_k}$, $H_{a_k}$ reduces to the channel Hamiltonian:

$$H^c_{a_k} = \overline{H}^0_{a_k} - \varkappa^2_{a_k}\,. \tag{1.10}$$

It is well known [4] that the wave operators defined as the strong limits

$$\Omega^{\pm}_{a_k} \equiv s\text{-}\lim_{t \to \pm\infty} e^{iHt}\, e^{-iH^c_{a_k} t} \tag{1.11}$$

exist on all of $\mathcal{H}$. Thus $\exp\{-i\, H_{a_k} t\} \Phi_{a_k}$ tends strongly to $\exp\{-i\ H\ t\}\ \Omega^{\pm}_{a_k} \Phi_{a_k}$ as $t \to \pm\infty$.

In formal scattering theory the wave operators are expressed in terms of boundary values of the resolvent operator. On a rigorous level we have the following result by Hunziker [5]:

*Theorem:* (1.12)

Let $H$ and $H^c_a = \int \lambda\, dE^a(\lambda)$ be selfadjoint operators on $\mathcal{H}$ and let $\psi \in \mathcal{H}$ be such that

$$\psi^{\pm} \equiv s\text{-}\lim_{t \to \pm\infty} e^{iHt}\, e^{-iH^c_a t} \psi$$

exists. Then $\psi^{\pm}$ can be represented as

$$\psi^{\pm} = s\text{-}\lim_{\varepsilon \downarrow \uparrow 0} -i\, \varepsilon \int R\,(\lambda - i\, \varepsilon)\, dE^a(\lambda)\, \psi\,, \tag{1.13}$$

where $R(z) = (z - H)^{-1}$ and the integral exists as strong limit of Stieltjes sums.

The time independent approach to scattering theory consists in the use of (1.13) for the computation of the wave operators. Therefore it includes the investigation of the resolvent operator $R(z)$.

In order to describe the method of Faddeev and Yakubovsky [2, 6, 7] we need a few notations. Let $a_k$ and $b_l$ be two partitions. $a_k \subset b_l$ means that every cluster of $a_k$ is contained in some cluster of $b_l$. By definition $a_k \sqcup b_l$ is the partition arising from $a_k$ by the additional connections of $b_l$; i.e. two particles $i$ and $j$ belong to the same cluster of $a_k \llcorner b_l$ if and only if there is a particle $k$, such that the pairs $(i, k)$ and $(j, k)$ are each contained in some cluster of $a_k$ or $b_l$. A sequential connectivity $\alpha_k$ is a sequence of partitions of the form:

$$\alpha_k = (a_{N-1}, a_{N-2}, \dots a_k), \quad a_{i+1} \subset a_i, \quad N-1 \geqslant i \geqslant k. \tag{1.14}$$

The operation $\llcorner$ for sequential connectivities is defined by

$$\alpha_k \llcorner \beta_m \equiv (a_{N-1}, \dots a_k, c_{k-1}, \dots c_l), \tag{1.15}$$

where the subsequence $(a_k, c_{k-1}, \dots c_l)$ is obtained from $(a_k, a_k \llcorner b_{N-1}, \dots a_k \llcorner b_m)$ by retaining only one partition from any maximal subsequence of identical partitions. We shall use the notations:

$$R(a_k, z) \equiv (z - H_{a_k})^{-1}, \quad R_0(z) \equiv R(a_N, z), \quad R(z) \equiv R(a_i, z), \tag{1.16}$$

$$T_{a_k}(z) \equiv V_{a_k} + V_{a_k} R(a_k, z) V_{a_k}, \quad T(z) \equiv T_{a_1}(z),$$

$$T_{a_i}^{\alpha_k} \equiv V_{a_{N-1}} R(a_{N-1}, z) V(a_{N-2}/a_{N-1}) R(a_{N-2}, z), \dots V(a_k/a_{k+1}) \left(1 + R(a_i, z) V_{a_i}\right), \tag{1.17}$$

with

$$a_k \subset a_i \text{ and } V(a_l/a_{l+1}) \equiv V_{a_l} - V_{a_{l+1}}. \tag{1.18}$$

These quantities are related by [7, 8]:

$$T_{a_k}(z) = \sum_{\alpha_{k+1} \subset \alpha_k} T_{a_k}^{\alpha_{k+1}}(z) + \sum_{\alpha_{k+2} \subset \alpha_k} T_{a_{k+2}}^{\alpha_{k+2}}(z) + \dots + \sum_{\alpha_{N-1} \subset \alpha_k} T_{a_{N-1}}^{\alpha_{N-1}}(z), \tag{1.19}$$

$$R(a_k, z) = R_0(z) + R_0(z) T_{a_k}(z) R_0(z). \tag{1.20}$$

By $\alpha_l \subset \alpha_k$ we mean that $\alpha_l$ can be continued to $\alpha_k$. A graphical interpretation of the above definitions is given in Ref. [7]. The $T_{a_k}^{\alpha_{k+1}}(z)$ obey the Faddeev-Yakubovsky (F-Y) operator equations (cf. [2, 6, 7]):

$$T_{a_k}^{\alpha_{k+1}}(z) = T_{a_{k+1}}^{\alpha_{k+1}}(z) + \sum Q_{a_k}^{\alpha_{k+1} \beta_{k+1}}(z) R_0(z) T_{a_k}^{\beta_{k+1}}(z), \tag{1.21}$$

or in matrix notation $\mathcal{T}_{a_k} = \mathcal{T}_{a_k}^0 + Q_{a_k} R_0 \mathcal{T}_{a_k}$, where

$$Q_{a_k}^{\alpha_{k+1} \beta+1}(z) \equiv \Sigma' T_{a_l}^{\alpha_l}(z) R_0(z) V_{d_{N-1}} + \Sigma'' T_{a_{l+1}}^{\alpha_{l+1}}(z) R_0(z) V_{d_{N+1}}. \tag{1.22}$$

$\Sigma'$ denotes the sum over all $\alpha_l$ and $d_{N-1}$ such that $\alpha_l \subset \alpha_{k+1}, \alpha_l \llcorner \beta_{k+1} = \alpha_k, \alpha_l \llcorner \beta_{k+2} = \alpha_{k+1}$ and $d_{N-1} \subset a_l, d_{N-1} \llcorner b_{k+2} = a_{k+1}$. The summation $\Sigma''$ differs from $\Sigma'$ only by the additional restriction $d_{N-1} \llcorner a_{l+1} = a_l$ [7]. The advantage of the F-Y equations rests upon the following two properties:

a) Uniqueness: In the family of operators with a domain $D \supset D(H^0)$, the $T_{a_k}^{\alpha_{k+1}}(z)$, defined by (1.17), are the only solutions of the F-Y equations for $z$ in the resolvent set of $H$ [2, 6]. (1.23)

b) Connectedness: For any $n \geqslant N - k$ the $n$th power of the Yakubovsky kernel is connected, i.e. there is no factorization of the form $\{(Q_{a_k} R_0)^n\}^{\alpha\beta} = Q \otimes 1_{\overline{\mathcal{H}}_a}$ in $\mathcal{H} = \hat{\mathcal{H}}_a \otimes \overline{\mathcal{H}}_a$ [1, 8]. (1.24)

According to Ref. [8],

$$T_{a_k}^{\alpha_k}(z) = \sum_{\beta_{k+1}} Q_{a_k}^{\alpha_{k+1} \beta_{k+1}}(z) R_0(z) T_{a_k}^{\beta_{k+1}}(z). \tag{1.25}$$

Hence we are lead to the following procedure for the computation of the $T$-operators. After solving the Lippmann-Schwinger equations

$$T_{a_{N-1}}^{\alpha_{N-1}}(z) = V_{a_{N-1}} + V_{a_{N-1}} R_0(z) T_{a_{N-1}}^{\alpha_{N-1}}(z) \tag{1.26}$$

we can construct $Q_{a_{N-2}}$ according to (1.22) and thus formulate the F-Y equations for $k = N - 2$. After computing $\mathfrak{J}^0_{a_{N-3}}$ and $Q_{a_{N-3}}$ we pass to the equations for $k = N - 3$, etc.

In Ref. [1] a generalization of the F-Y equation is proposed for systems interacting via a sum of 2-, 3-, ... and $N$-body forces. These equations have the connectedness property and for $N = 4$ the uniqueness property is also verified. Here we shall only consider Hamiltonians of the form (1.1).

## 2. Faddeev's Program for the F–Y-Equations

We expect that the $T_{a_k}^{\alpha_k+1}(z)$ act as integral operators in momentum representation. This Ansatz leads to the F-Y integral equations for the corresponding kernels. In the passage to the F-Y integral equations, it is desirable to preserve the existence of a unique solution (1.23) and even in view of (1.13), to extend this property for $Im\, z \downarrow 0$. Faddeev's program serves to accomplish this by establishing the Fredholm alternative for the integral equations without loosing control over the solutions of the homogeneous system, even in the limit $Im\, z \downarrow 0$. For this aim, the Ansatz for the kernels has to be more differentiated.

The definitions (1.17) indicate the appearance of multiplicative singularities in ther kernels of the $T_{a_k}^{\alpha_k}(z)$ of the form [7]:

$$\begin{aligned} K(k, l, z) = {} & (z - \overline{E}_{a_{i_1}}(k))^{-1} (z - \overline{E}_{a_{i_2}}(k))^{-1} \ldots (z - \overline{E}_{a_{i_n}}(k))^{-1} F(k, l, z) \\ & \times (z - \overline{E}_{b_{j_m}}(l))^{-1} \ldots (z - \overline{E}_{b_{j_1}}(l))^{-1} \end{aligned} \tag{2.1}$$

with $a_{i_1} \subset a_{i_2} .. \subset \ldots \subset a_{i_n}$ and $b_{j_1} \subset \ldots \subset b_{j_m}$, where

$$\overline{E}_a(k) \equiv E_a(\overline{k}_a) - \varkappa_a^2 \,. \tag{2.2}$$

The handling of these singularities is greatly facilitated by the following two properties of the spectra $\sigma(\hat{H}_a)$:

a) The part of $\sigma(\hat{H}_a)$ in the complement of $\sigma_c(\hat{H}_a)$, the continuous spectrum of $\hat{H}_a$, consists of eigenvalues only. These are of finite multiplicity and can accumulate only at the lower end $\varepsilon_a$ of $\sigma_c(\hat{H}_a)$. A proof of this may be found in Refs. [9] and [10]. (2.3)

b) There is a $\delta_a > 0$ such that $[\varepsilon_a - \delta_a, +\infty)$ is free of eigenvalues. This is a standard assumption, cf. [1, 2, 7]. No general proof is known. For attempts of a partial answer we refer to [11]. (2.4)

In accordance with (2.3) and (2.4) we assume, for notational convenience, that $\hat{H}_a$ has exactly one non degenerate eigenvalue $-\varkappa_a^2$ strictly below the continuum and that $d \equiv \min_{a \supset b} (\varkappa_a^2 - \varkappa_b^2) > 0$. This implies for any $a \subset b$ that $Re[\{z - \overline{E}_a(\overline{k}_a)\} - \{z - \overline{E}_b(\overline{k}_b)\}] \geqslant d > 0$. Therefore the multiplicative singularities (2.1) can be separated by a Hölder continuous partition of the identity (cf. [1, 7]), which yields:

$$T_{a_k}^{\alpha_{k+1}}(k, l, z) = \delta_{a_k}(\bar{k}_{a_k} - \bar{l}_{a_k}) \Bigg\{ T_{a_k}^{\alpha_{k+1}}(R, R \mid k, l, z)$$

$$+ \Sigma' \left[ \frac{T_{a_k}^{\alpha_{k+1}}(a_i, R \mid k, l, z)}{z - \bar{E}_{a_i}(k)} + \frac{T_{a_k}^{\alpha_{k+1}}(R, d \mid k, l, z)}{z - \bar{E}_d(l)} + \frac{T_{a_k}^{\alpha_{k+1}}(a_i, d \mid k, l, z)}{(z - \bar{E}_{a_i}(k))(z - \bar{E}_d(l))} \right] \Bigg\} \tag{2.5}$$

$$Q_{a_k}^{\alpha_{k+1}\beta_{k+1}} R_0(k, l, z)$$

$$= \delta_{a_k}(\bar{k}_{a_h} - \bar{l}_{a_k}) \Sigma'' \left\{ \frac{Q_{k}^{\alpha_{k+1}\beta_{k+1}}(R, d \mid k, l, z)}{z - \bar{E}_d(l)} + \frac{Q_{a_k}^{\alpha_{k+1}\beta_{k+1}}(a_i, d \mid k, l, z)}{(z - \bar{E}_{a_i}(k))(z - \bar{E}_d(l))} \right\} \tag{2.6}$$

$$\delta_{a_k}(\bar{k}_{a_k} - \bar{l}_{a_k}) \equiv \prod_{A_j \varepsilon a_k} \delta^3(k(A_j) - l(A_j)) . \tag{2.7}$$

By (1.17) and (1.22) $\Sigma'$ extends over all $a_i \in \alpha_{k+1}$ and $d \subset a_k$, $d \neq a_k, a_N$, whereas $\Sigma''$ is the sum over all $a_i \in \alpha_{k+1}$ and $d \subset a_{k+1}$ such that either $d = d_N$ or $d \supset d_{N-1}$ with $d_{N-1} \llcorner b_{k+1} = a_k$. The index $R$ denotes a regular part. It is clear that (2.6) contains additional $\delta$-functions arising from contributions of a lower connectivity than $a_k$. By construction $T_{a_k}^{\alpha_{k+1}}(R, d \mid k, l, z)$ is different from 0 only in the region $| Re\, z - \bar{E}_d(l) | < 2/3\, d$; in this region all the other denominators are bounded by $3/d$. A similar statement holds for the other contributions.

This leads to the following representation of the F-Y equations:

$$T_{a_k}^{\alpha_{k+1}}(a_i, d \mid k, l, z) = T_{a_{k+1}}^{\alpha_{k+1}}(a_i, d \mid k, l, z)$$

$$+ \sum_{\beta_{k+1}, C} \int dp \frac{Q_{a_k}^{\alpha_{k+1}\beta_{k+1}}(a_i, c \mid k, p, z)\, T_{a_k}^{\beta_{k+1}}(b, d \mid p, l, z)}{z - \bar{E}_b(p)}$$

$$+ \sum_{\beta_{k+1}, C, b \in \beta_{k+1}} \int dp \frac{Q_{a_k}^{\alpha_{k+1}\beta_{k+1}}(a_i, c \mid k, p, z)\, T_{a_k}^{\beta_{k+1}}(b, d \mid p, l, z)}{(z - \bar{E}_c(p))(z - \bar{E}_d(p))} \tag{2.8}$$

It is understood that the variables $k$, $l$ in (2.8) are adapted to the $\delta$-functions in (2.5) and (2.6). In matrix notation (2.8) reads:

$$\mathfrak{T}_{a_k}(d \mid l, z) = \mathfrak{T}_{a_k}^0(d \mid l, z) + \mathcal{A}_{a_k}(l, z)\, \mathfrak{T}_{a_k}(d \mid l, z) . \tag{2.9}$$

Aequivalently we may discuss:

$${}^n\mathfrak{T}_{a_k}(d \mid l, z) = {}^n\mathfrak{T}_{a_k}^0(d \mid l, z) + \mathcal{A}_{a_k}(l, z)\, {}^n\mathfrak{T}_{a_k}(d \mid l, z) , \tag{2.10}$$

where

$${}^n\mathfrak{T}_{a_k}^0 \equiv (\mathcal{A}_{a_k})^n\, \mathfrak{T}_{a_k}^0 \quad \text{and} \quad {}^n\mathfrak{T}_{a_k} \equiv \mathfrak{T}_{a_k} - \sum_{r=0}^{n-1} {}^r\mathfrak{T}_{a_k}^0 .$$

In the discussion of (2.10) we shall frequently use the following terminology: The term $a$-connected amplitude' denotes a product of the form $\delta_a(\bar{k}_a - \bar{l}_a)\,(z - \bar{E}_b(k))^{-1} \times K(b, c \mid k, l, z)\,(z - \bar{E}_c(l))^{-1}$ where $K$ does not contain $\delta$-functions and $b \subset a$, $c \subset a$. The denominators will be referred to as '$b$-connected left propagators' and '$c$-connected right propagators', whereas the numerator $K$ will be called '$a$-connected component'. An $a$-connected component will be termed H-C if $K$ depends Hölder continuously on $k$, $l$ and $z \in \Pi_{\varepsilon_a}$ Here $\Pi_{\varepsilon_a}$ denotes the complex plane slit along $[\varepsilon_a, +\infty)$ and completed by its limit points from above and from below (which are not identified).

It follows from Privalov's lemma (Appendix I) that $\mathcal{A}_{a_k}(l, z)$ defines a linear operator in the Banach space $B_{a_k}(\theta, \mu)$. This suggests to discuss (2.10) in $B_{a_k}(\theta, \mu)$. In order to show that ${}^n\mathfrak{J}^0_{a_k}(d \mid l, z)$ is a $B_{a_k}(\theta, \mu)$-valued function which depends Hölder continuously on $l$ and $z \in \Pi_{\varepsilon_{ak}}$ and to establish the Fredholm alternative for the equation (2.10) in a region $\{z \in \Pi_{\varepsilon_{a_k}}, Re\, z \leqslant E_0\}$ it suffices, according to Refs. [12] and [7] (cf. also [1]), to prove the following two properties for the elements of the kernel matrix $(Q_{a_k} R_0)^n (k, l, z)$:

a) Any amplitude contributing to $(Q_{a_k} R_0)^n (k, l, z)$ has $a_k$-connected H-C components of an index $\alpha > \mu > 0$. (2.11)

b) For some $\beta > \theta > 0$ the components of $(Q_{a_k} R_0)^n (k, l, z)$ are bounded by $C\, N_{a_k}(k, \beta)\, N_{a_k}(l, \beta)$ uniformly in $l$ and $z \in \Pi_{\varepsilon_{a_e}} \cap \{Re\, z \leqslant E_0\}$. (2.12)

These properties also show that $(\mathcal{A}_{a_k}(l, z))^n$ depends, in the sense of the operator norm in $\mathfrak{L}(B_{a_k}(\theta, \mu), B_{a_k}(\theta, \mu))$, Hölder continuously on $l$ and $z$. Thus a solution of (2.10) is H-C at any regular point $(l, z)$.

In order to control the Fredholm alternative for $z = E + i\,0$, $E \to \sigma_c(\hat{H}_{a_k})$, it is necessary to require:

$\overline{S}$: For some $\delta > 0$ the interval $[\varepsilon_{a_k} - \delta, \infty)$ does not contain singular points of the equation (2.10). (2.13)

This is of course closely related to the spectral assumption (2.4).: Any $z \notin \sigma_c(H_{a_k})$ is a singular point of (2.10) if and only if it is an eigenvalue of $\hat{H}_{a_k}$ [2, 6]. The same holds for $E \pm i\,0 \in \sigma_c(\hat{H}_{a_k})$ if $\mu > 1/2$ [2].

The critical terms for the decrease at infinity (2.12) of $(Q_{a_k} R_0)^n (k, l, z)$ arise from the regular parts in $Q_{a_k} R_0$ since the remaining contributions have compact support with respect to the variables appearing in the propagators. From the way one proceeds, solving successively the F-Y equations for the $T^{\alpha_l + 1}_{a_l}$ with decreasing $l$, it becomes clear that the purely regular contributions to $(Q_{a_k} R_0)^n$ may be estimated by purely regular contributions to Born terms of sufficiently high order for $(Q_{a_k} R_0)^n$. The latter ones were estimated by Hepp ([7], Lemma (3.3)):

*Lemma:* (2.14)

Any purely regular Born term $G$ contributing to $(Q_{a_k} R_0)^n (k, l, z)$, with $n \geqslant 4 (N - k,)$ is bounded by $C\,(G, E_0)\, N_{a_k}(k, \theta)\, N_{a_k}(l, \theta)$ for some $\theta > 3/2$, uniformly in $\bar{l}_{a_k}$ and $Re\, z \leqslant E_0$. Furthermore it is H-C of an index $\mu > 0$ with respect to $k$, $l$, $z$.

This settles (2.12). In the sequel we concentrate on the proof of the regularity property (2.11), which amounts to a discussion of the singular integrals arising in the iterated Yakubovsky kernels. In § 3 we shall see that the iterad Yakububovsky kernels may be represented as a sum of generalized Feynman integrals with H-C numerators. The separation of the singularities confines the momenta occuring in the propagators to bounded regions. With arguments similar to those invoked for the verification of (2.12), it is always possible to assure the convergence of the integrations over the remaining variables, absolutely and uniformly in $Re\, z \leqslant E_0$ and in the propagator momenta. We shall refer to this fact as'sufficient decrease' of the integrand.

## 3. Graphical Representation of Singular Integrals

In this paragraph we develop a graphical tool for the classification and the analysis of the singular integrals arising in the computation of the iterated F-Y kernels. Our graphs will only specify the singular energy denominators and the Hölder continuity and connectivity structure of the numerator in the integrand.

In the integrand for the iterated F-Y kernel $[(Q_{a_k} R_0)^n]^{\alpha_{k+1}\,\beta_{k+1}}$ we insert H-C partitions of the identity in order to separate the multiplicative singularities as in (2.1). This yields a finite sum of products of amplitudes. To fix the notation, we recall that the $i^{\text{th}}$ factor consists of a $c^i_{k(i)}$-connected component $\delta_{c^i}(\bar{p}^{\,i-1}_{c^i} - \bar{p}^{\,i}_{c^i})\, f(p^{i-1}, p^i, z)$ and possibly an $l^i_{m(i)}$-connected left-propagator $\left(z - \overline{E}_l(\bar{p}^{\,i-1}_i)\right)^{-1}$ and an $r^i_{n(i)}$-connected right-propagator $\left(z - \overline{E}_r(\bar{p}^{\,i}_r)\right)^{-1}$. Components which are not H-C arise from the low order iterations of the F-Y equations. If in these iterations we still encounter non H-C components, we reiterate until we obtain H-C components. Thus the integrand for $[(Q_{a_k} R_0)^n]^{\alpha_{k+1}\,\beta_{k+1}}$ can always be written as a finite sum of products of amplitudes with H-C components. With each product we associate a sequence of connectivities

$$\{(l'_{m(1)})\; c'_{k(1)}(r'_{n(1)})\;, \ldots\; (l^s_{m(s)})\; c^s_{k(s)}(r^s_{n(s)})\}\;, \tag{3.1}$$

where ( ) denotes a possible omission. To such a sequence we relate in a one to one manner a graph as follows: We draw $N$ horizontal lines representing the $N$ particles of the system under consideration. For a $c_k = -\,(C', \ldots C^k)$-connected component we draw a '$c_k$-connected' straight vertical line ('$c$-line'), i.e. an assembly of straight vertical lines connecting the horizontal lines of the particles in the clusters $C' \ldots C^k$ respectively. For clarity we dot the intersections of the vertical lines with the linked particle lines. For any $l_m$-connected ($m < N$) left propagator we draw an $l_m$-connected wavy vertical line ('$p$-line') to the left of the corresponding component line and similarly for the right-propagators. The free propagators will be represented by undotted vertical wavy lines.

After integration over the $\delta$-functions, momentum conservation at every $c$-line has to be taken into account for the choice of a set of independent integration variables ('loop momenta').

Example ($N = 4$, $m_i = 1/2$):

The graph

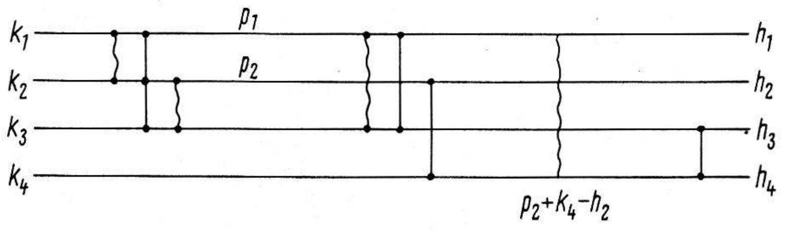



Fig. 1

represents the following type of integrand:

$$(z+\varkappa_{12}^2-1/2\,(k_3+k_4)^2-k_3^2-k_4^2)^{-1}\,f_{123}(k,p,z)\,(z+\varkappa_{23}^2-p_1^2-1/2\,(p_1+k_4)^2-k_4^2)^{-1}$$
$$(z+\varkappa_{13}^2-1/2\,(p_2+k_4)^2-p_2^2-k_4^2)^{-1}\,f_{13/24}(p,k,h,z)$$
$$(z-h_1^2-h_2^2-(p_2+k_4+h_1)^2-(p_2+k_4-h_2)^2)^{-1}\,f_{34}(p,k,h,z)\,.$$

A $c$-subgraph is a portion of a graph between two $c$-lines, including these. It is said to be contractible to a $c$-line, if the integration over the intermediate momenta yields an H-C component. A trivial contraction is possible whenever the subgraph contains no $p$-lines: it involves only integrations over $\delta$-functions and over H-C functions of sufficient decrease (cf. Sect. 2). A graph is called contractible if its maximal $c$-subgraph is contractible. It is called partially contracted, if only non maximal $c$-subgraphs have been contracted. After contraction a $c$-subgraph becomes a $c$-line, and we relabel the momenta and lines of the graph accordingly.

Let us state in a theorem those properties of the graphs which reflect the combinatorial structure of the F-Y kernel (2.6) and the fact that we have separated the multiplicative singularities.

*Theorem:* (3.2)

A finite number of graphs is associated with the iterated F-Y kernels. Graphs which are obtained from each other by an exchange of neighbouring $p$- and $c$-lines of equal connectivity are aequivalent. After the trivial contractions have been performed, the following relations hold for any $i \geqslant 1$:

$$r^i \subset c^i\,,\; l^i \subset c^i\,,$$
$$r^i \not\subset l^{i+1},\; r^i \not\supset l^{i+1}\,,$$
$$r^i \not\supset c^{i+1}\,,\; r_{n(i)} \not\subset c^{i+1} \text{ for } n(i) \neq N\,,$$
$$c^i \not\subset l^{i+1}\,,\; c^i \not\subset c^{i+1}\,.$$

Proof: The relations $r^i \not\subset l^{i+1}$ and $r^i \not\supset l^{i+1}$ exress the fact that we have separated the multiplicative singularities. All the remaining properties follow from the combinatorial structure (2.6) of $Q_{a_k} R_0$, except $r^i \not\supset c^{i+1}$. The latter is a consequence of the trivial contractions.

For the sequel the following lemma on the choice of loop momenta is useful:

*Lemma:*

Let $a = (A_1, \ldots A_r)$ and $b^i_{r(i)} = (B^i_1, \ldots B^i_{r(i)})$ be partitions of $\{1, \ldots N\}$. If for any $i$, $A_1 \cup A_2$ is contained in a cluster of $c^i = a \sqcup b^i = (C^i_1, \ldots C^i_{s(i)})$, then the momenta $k(B^i)$ do not determine $k(A_1)$ and $k(A_2)$ separately.

Proof: We label the clusters of $c^i$ such that $A_1 \cup A_2 \subset C^1_i$. Hence

$$k(A_1) + k(A_2) = \sum_{B^i_l \subset C^i_1} k(B^i_l) - \sum_{\substack{A_l \subset C^i_1 \\ l \neq 1,2}} k(A_l)\,, \quad i = 1, 2, \ldots$$

Obviously, these equations determine at most the sum $k(A_1) + k(A_2)$.

We conclude this paragraph with a notational remark. We shall write $A \sim \sum_{i=1}^{n} G_i$, whenever the graphs associated with the quantity $A$ are $G_1, G_2, \ldots G_n$.

## 4. Contractibility and Maximal Regularity below $s^4$

In terms of contractibility, the regularity statement (2.11) reads: For $n \geqslant n_0(a_k, N)$ and $Re\, z \leqslant E_0$ the graphs arising from $(Q_{a_k} R_0)^n\, (k, l, z)$ are $a_k$-connected and contractible. It is sufficient to prove this assertion for $n = n_0$ because of the

*Lemma:* (4.1)

A graph $G$ to $(Q_{a_k} R_0)^n$ containing an $a_k$-connected contractible $c$-subgraph is contractible.

Proof: Apply Privalov's lemma (Appendix I).

In the subsequent discussion we confine ourselves to $E_0 < s_4$ where $s_4$ is the 4-particle threshold $s_4 \equiv \inf_{k \geq 4} \{x \mid x \in \sigma(\hat{H}_{a_k})$. In this region only $a_l$-connected ($l \geqslant 3$ $p$-lines have to be considered, since the propagators of lower connectivity become non-singular and will be incorporated systematically in the components. It will turn out that this implies that the contractions based on Privalov's lemma ('Privalov contractions') and on Faddeev's lemma ('Faddeev contractions'), summarized in Appendix I, allow to show the contractibility of any graph. We only treat systems with more than 3 particles.

*Lemma:* (4.2)

$$(Q^{\alpha_k \beta_k}_{a_k} R_0)\, (k, h, z) \sim \sum_{\substack{c_l \in a_k \\ c_l \,\llcorner\, b_k = a_{k-1} \\ c_l \,\llcorner\, b_{k+1} = a_k}} \{c_l\}\,, \qquad k \geqslant 4$$

Proof: The lemma holds for $k = N$. We make the induction assumption that it is true for $k' = k + 1$. Therefore the graphs to $(Q_{a_k} R_0)^{4(N-k)}$ can be contracted trivially. This, together with the estimate (2.12) shows that the Fredholm alternative applies to the F-Y equations for $T^{\alpha_k+1}_{a_k}$. Since we consider the region $Re\, z < s_4$ only, there is a unique solution in $B_{a_k}(\theta, \mu)$. Hence $T^{\alpha_k+1}_{a_k} \sim \{c_k = a_k\}$. By (1.25) and (1.22) the lemma results.

The F-Y equations for $T^{\alpha_4}_{a_3}$ can be treated the same way. The only difference is the appearance of a simple pole at $z = \bar{E}_{a_3}(\bar{k}_{a_3})$, generated by the bound state of $\hat{H}_{a_3}$ Using (1.25) we obtain:

*Lemma:* (4.3)

$$T_{a_l}^{\alpha_l} \sim \{c_l = a_l\}\,,\ l \geqslant 4 \text{ and } T_{a_3}^{\alpha_3} \sim \{c_3 = a_3\} + \{l_3 = a_3,\, c_3 = a_3\}\,.$$

By (1.22) we may conclude (for $N \geqslant 4$):

*Lemma:* (4.4)

$$Q_{a_2}^{\alpha_3'\alpha_3^2}\, R_0 \sim \sum_{\substack{c_l \in \alpha_3' \\ c_l \,\llcorner\!\!\_\_\, a_3^2 = a_2 \\ c_l \,\llcorner\!\!\_\_\, a_4^2 = a_3'}} \{c_l\} + \{l_3 = a_3',\, c_3 = a_3'\}$$

*Lemma:* (4.5)

Any graph $G$ to $(Q_{a_2} R_0)^n$, $n \geqslant N - 2$, is connected and contractible.

Proof: $G$ is trivially contractible if there is no $p$-line in the interior. The case where $G$ contains inner $p$-lines is only seemingly more difficult since these arise in $c$-subgraphs of the form

$$\{c_l^{i-1},\, l_3^i = a_3^i,\, c_3^i = a_3^i\} \text{ or } \{c_3^{i-1} = a_3^{i-1},\, r_3^i = a_3^{i-1},\, l_3^i = a_3^i,\, c_3^i = a_3^i\}\,. \tag{4.6}$$

Both are $a_2$-connected by (4.4) and trivially contractible since the $p$-lines can be shifted to the exterior of these subgraphs by theorem (3.2). Lemma (4.1) implies that $G$ is contractible.

By arguments we are already familiar with, we conclude that the Fredholm alternative applies, in some $B_{a_2}(\theta, \mu)$, to the equation:

$${}^n\mathfrak{J}_{a_2}(d \mid l, z) = {}^n\mathfrak{J}_{a_2}^0(d \mid l, z) + \mathcal{A}_{a_2}(l, z)\, \mathfrak{J}_{a_2}(d \mid l, z) \tag{4.7}$$

for $n \geqslant 4$ $(N - 2)$. Together with our spectral assumption (2.13) this allows to conclude that (4.7) has a unique solution in $B_{a_2}(\theta, \mu)$ for $z \in \Pi_{\epsilon a_3} \cap \{Re\, z < s_4\}$, except for a bound state pole at $z = \overline{E}_{a_2}(\overline{k}_{a_2})$. Now it is helpful to remember the definition of $B_{a_2}(\theta, \mu)$ as direct sum [7]:

$$B_{a_2}(\theta, \mu) = \bigoplus_{a_3 \subset a_2} B_{a_2}(\theta, \mu, \alpha_3)\,, \tag{4.8}$$

where, according to (2.5)

$$B_{a_2}(\theta, \mu, \alpha_3) = B_{a_2}(\theta, \mu, R) \bigoplus_{a_i \in \alpha_3} B_{a_2}(\theta, \mu \mid a_i)\,. \tag{4.9}$$

By construction ${}^nT_{a_3}^{\alpha_3}(a_i, d \mid k, h, z)$ is equal to the $B_{a_2}(\theta, \mu, a_i)$-component of the solution ${}^n\mathfrak{J}_{a_2}(d \mid h, z)$ of (4.7). Summarizing our discussion of (4.7) in graphical terms we have:

*Lemma:* (4.10)

$$\begin{aligned} T_{a_2}^{\alpha_3} \sim{} & \{c_3 = a_3\} + \{l_3 = a_3,\, c_3 = a_3\} + \{c_2 = a_2\} + \{l_3 = a_3,\, c_2 = a_2\} \\ & + \sum_{r_3 \subset a_2} \{l_3 = a_3,\, c_2 = a_2,\, r_3\} + \{l_2 = a_2,\, c_2 = a_2\}\,. \end{aligned}$$

From (4.4) and (4.10) we deduce that the graphs to $T_{a_2}^{\alpha_2} = \Sigma\, Q_{a_2}^{\alpha_3\,\beta_3}\, R_0\, T_{\beta_3}^{\alpha_2}$ are $a_2$-connected and contain a $c$-subgraph which is trivially contractible. This leads to the

*Lemma:* (4.11)

$$T^{\alpha_2}_{a_2} \sim \{c_2 = a_2\} + \{l_3 = a_3, c_2 = a_2\} + \sum_{r_3 \subset a_2} \{l_3 = a_3, c_2 = a_2, r_3\}$$
$$+ \{l_2 = a_2, c_2 = a_2\}, \tag{4.12}$$

$$Q^{\alpha_2' \alpha_2^2}_{a_1} R_0 \sim \Sigma''' \{c_l\} + \{l_3 = a_3', c_3 = a_3'\} + \{l_3 = a_3', c_2 = a_2'\}$$
$$+ \{l_3 = a_3', c_2 = a_2', r_3\} + \{l_2 = a_2', c_2 = a_2'\}.$$

In $\Sigma'''$ $\alpha_2'$ is restricted by the conditions

$a_3' \llcorner a_2^2 = a_1$, $a_3' \llcorner a_3^2 = a_2'$, $a_2' \llcorner a_2^2 = a_1$, $a_2' \llcorner a_3^2 = a_2'$ and $c_l$ ranges over all $c_3 \in \alpha_2'$ such that $c_l \llcorner a_2^2 = a_1$, and $c_l \llcorner a_3^2 = a_2'$, whereas $r_3$ is subject to $r_3 \llcorner a_3^2 = a_2'$. (4.13)

The formula (4.13) is the basis for our study of the graphs $G$ arising from $(Q_a R_0)^n$. By performing all possible trivial and Privalov contractions we are lead to a graph $G'$. $G'$ (and therefore $G$) is contractible if it contains a connected $c$-line. If there is no connected $c$-line in $G'$, we cut $G'$ at the 2-connected $p$-lines into sectors. We shall essentially show (Lemma 4.75) that the $t$th sector contains $r(t)$; $1 \leqslant r(t) \leqslant 3$, propagators of the form:

$$[z + \varkappa^2 - n_l^i p_i^2 - n_i'' (p_i + p_{i+1})^2 + n_i''' p_{i+1}^2]^{-1},$$
$$\text{for} \quad 1 + \sum_{t' < t} r(t') \leqslant i \leqslant \sum_{t' \leq t} r(t'). \tag{4.14}$$

Hence if $G'$ contains at least 4 sectors, Faddeev's lemma (cf. Appendix I), applied to the integration over $p_2, p_3, p_4$, provides the analytical tool for the contraction of $G'$ (and $G$). Example:

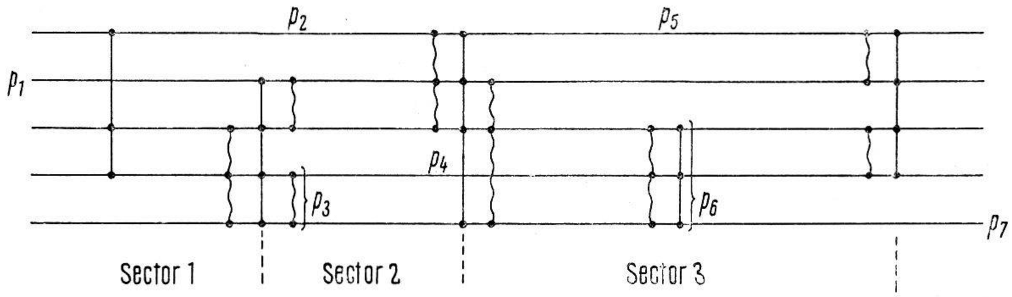



Fig. 2

It will turn out that $G'$ contains at least 4 sectors provided $n \geqslant 5\,(N-2)$. This shows that $n_0 = 5\,(N-2)$ is a possible choice for the contractibility statement (2.11). It remains to verify (4.14). The following lemmas on the structure of the sectors are devoted to this task.

*Lemma:* (4.15)

Let $G$ be a graph obtained from $(Q_{a_1} R_0)^n$, after all possible trivialcontractions have been carried out. Then, every $c$-subgraph of $G$, containing more than one $c$-line, contains at least one 2-connected $c$-line.

Proof: By assumption, the $c$-subgraph between $c^i_{k(i)}$ and $c^{i+1}_{k(i+1)}$ contains at least one $p$-line. If $k(i)$, $k(i+1) \geqslant 3$, the $p$-lines $r^i$ and $l^{i+1}$ must be 3-connected. Therefore, they can be shifted to the exterior of the subgraph, which contradicts the assumption that no $c$-subgraph is trivially contractible.

*Lemma:* (4.16)

Let $c^i$, $i > 1$, be the $i$th $c$-line. If $c^i$ is 2-connected, $c^{i-1} \llcorner c^i = a_1$.

Proof: $c^{i-1}$ arises from the contraction of a $c$-subgraph to some $Q_{a_1}^{\alpha_2^{i-k-1}, \alpha_2^{i-h}} R_0, \ldots$ $Q_{a_1}^{\alpha_2^{i-1}, \alpha_2^{i}} R_0$. Formula (4.13) implies: $c^{i-1} \supset a_l^{i-1}$ with $a_l^{i-1} \llcorner a_2^i = a_1$. But $c^i$ was obtained from the contraction of a $c$-subgraph to $Q_{a_1}^{\alpha_2^{i}, \alpha^{i+1}} R_0, \ldots Q_{a_1}^{\alpha_2^{i+l}, \alpha_2^{i+l+1}} R_0$. Therefore $c^i = a_2^i$. Hence $c^{i-1} \llcorner c^i \supset a_l^{i-l} \llcorner a_2^i = a_1$.

*Lemma:* (4.17)

Let $c_{k(i)}^i$, $c_{h(i+1)}^{i+1}$ be two $c$-lines in a graph $G$ where trivial contractions and Privalov contractions are not possible. If $c^i$ is 2-connected, $c_2^i \llcorner c_{k(i+1)}^{i+1} = a_1$.

Proof: In view of (4.16) we can assume $k\,(i+1) \geqslant 3$. The $c$-subgraph between $c^i$ and $c^{i+1}$ takes the form $\{c_2^i, r_3^i, c_{k(i+1)}^{i+1}\}$ after shifting $l^{i+1}$ to the exterior, if necessary. $r^i$ must be 3-connected; otherwise it could be shifted to the exterior. Let $r_3^i = (R_1^i, R_2^i, R_3^i)$ and $c_2^i = (R_1^i, R_2^i, R_3^i)$. Theorem (3.8) implies $c_{k(i+1)}^{i+1} \not\subset r_3^i$. Therefore $c^{i+1}$ connects either $R_1^i$ and $R_2^i$ or $R_1^i \cup R_2^i$ and $R_3^i$. In the latter case $c^i \llcorner c^{i+1} = a_1$, as the lemma states. In the first case lemma (3.10); on the choice of loop momenta, tells us that $p \equiv k(R_1^i)$ is independent of the external momenta of the subgraph and may therefore be used as loop momentum. Denoting the remaining loop momenta with $q$, the $c$-subgraph in question represents an integral of the form:

$$\int dq\, d^3p\, f(k, h, p, q, z)$$
$$\times \{z + \varkappa^2 - n(R_1^i)\, p^2 - n(R_2^i)\, (p + k(R_3^i))^2 - n(R_3^i)\, k^2(R_3^i)\}^{-1},$$

where

$$n(A) \equiv (2\, m(A))^{-1}. \tag{4.18}$$

Therefore the contractibility of the $c$-subgraph follows from Privalov's lemma. This contradicts the assumption of the lemma.

*Lemma:* (4.19

Let $G$ be a graph to $(Q_{a_1} R_0)^n$, which does not allow trivial or Privalov contractions. Then any subgraph of $G$, containing more than one $c$-line, is connected.

Proof: (4.15), (4.16), (4.17).

*Lemma:* (4.20)

Under the assumption of lemma (4.19), $p$-lines which cannot be commuted to the exterior of the graph are 3-connected.

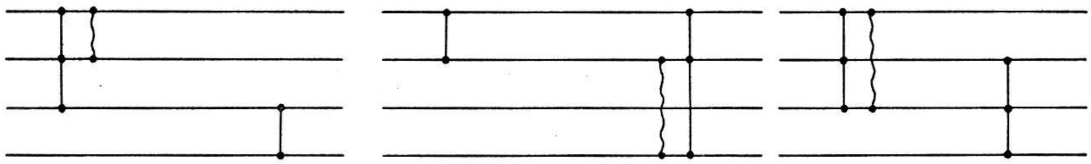
Proof: Suppose there were a 2-connected $p$-line $l_2^i$ in the interior of the graph. The $c$-subgraph between $c^{i-1}$ and $c^{i+1}$ must be of the form $\{c_2^{i-1}, r_3^{i-1}, l_2^i = a_2, c_1^i = a_2, l_2^{i+1}, c_2^{i+1}\}$. Theorem (3.8) states $r_3^{i-1} \not\subset l_2^i = a_2$. Therefore $r_3^{i-1} \llcorner c_2^i = a_1$. But, repeating the arguments in the proof of (4.17), we see that the conditions $r_3^{i-1} \subset c_2^{i-1}$ and $r_3^{i-1} \llcorner c_2^i = a_1$ are sufficient for the $c$-subgraph $\{c_2^{i-1}, r_3^{i-1}, c_2^i\}$ to admit a Privalov contraction, contrary to the assumption.

*Lemma:* (4.21)

Let $G$ be as in (4.19). If $\{c_2^i, r_3^i, c_{k(i+1)}^{i+1}\}$ is a $c$-subgraph of $G$, then $r_3^i \llcorner\, c_{k(i+1)}^{i+1} = \tilde{a}_2$ with $c_2^i \llcorner\, \tilde{a}_2 = a_1$. If $\{c_{k(i)}^i, l_3^{i+1}, c_2^{i+1}\}$ is a $c$-subgraph of $G$, then $c_{k(i)}^i \llcorner\, l_3^{i+1} = \hat{a}_2$ with $\hat{a}_2 \llcorner\, c_2^{i+1} = a_1$.

Proof: We consider $\{c_2^i, r_3^i, c_{k(i+1)}^{i+1}\}$. By theorem (3.10) $r_3^i \not\supset c_{k(i+1)}^{i+1}$. Therefore $r_3^i \llcorner\, c_{k(i+1)}^{i+1}$ is at least 2-connected. If it were connected, the subgraph would admit a Privalov contraction, according to the arguments in the proof of (4.17). The same applies to the case where $r_3^i \llcorner\, c_{k(i+1)}^{i+1} \equiv \tilde{a}_2 = c_2^i$. Therefore, to avoid a contradiction with the assumption, we must necessarily have $r_3^i \llcorner\, c_{k(i+1)}^{i+1} = \tilde{a}_2$, with $c_2^i \llcorner\, \tilde{a}_2 = a_1$. The discussion for the subgraph $\{c_{k(i)}^i, l_3^{i+1}, c_2^{i+1}\}$ is similar.

Examples:



$c_2^i \neq \tilde{a}_2$

$\hat{a}_2 = c_2^{i+1}$
$\rightarrow G$ contractible

$c_2^i \subset r_3^i \llcorner\, c_{k(i+1)}^{i+1}$
$\rightarrow G$ contractible

Fig. 3

From now on we shall refrain from graphical illustrations. For the readers ease we remark that the 5 particle system provides examples for all the statements following below.

*Lemma:* (4.22)

Let $G$ be as in (4.19), and let $\{c_2^i, r_3^i, l_3^{i+1}, c_2^{i+1}\}$ be a $c$-subgraph of $G$. Then either the relations a), (4.23), or the relations b), (4.24), are satisfied.

a) $r_3^i \llcorner\, c_2^{i+1} = a_1$, $c_2^i \llcorner\, l_3^{i+1} = a_1$, $r_3^i \llcorner\, l_3^{i+1} \neq a_1$, (4.23)

b) $r_3^i \llcorner\, c_2^{i+1} \neq a_1$, or $c_2^i \llcorner\, l_3^{i+1} \neq a_1$. (4.24)

Proof: The relations $r_3^i \llcorner\, c_2^{i+1} \neq a_1$ and $c_2^i \llcorner\, l_3^{i+1} \neq a_1$ cannot be satisfied simultaneously because this would imply $r_3^i \subset c_2^{i+1}$ and $l_3^{i+1} \subset c_2^i$. However, two partitions $a_k \neq b_k$ have at most one common refinement $c_{k+1}$ [6]. Hence $l_3^i = l_3^{i+3}$. in contradiction with theorem (3.2). It remains to show that the relation

$$r_3^i \llcorner\, c_2^{i+1} = c_2^i \llcorner\, l_3^{i+1} = r_3^i \llcorner\, l_3^{i+1} = a_1 \tag{4.25}$$

is contradictory to the assumption of the lemma. Suppose (4.25) holds. Let

$$r_3^i = (R_1^i, R_2^i, R_3^i)\ ,\ c_2^i = (R_1^i \cup R_2^i, R_3)\ , \tag{4.26}$$

$$l_3^{i+1} = (L_1^{i+1}, L_2^{i+1}, L_3^{i+1})\ ,\ c_2^{i+1} = (L_1^{i+1} \cup L_2^{i+1}, L_3^{i+1})\ . \tag{4.27}$$

$k_1 \equiv k(R_3^i)$ and $h_1 = k(L_3^{i+1})$ are external momenta of the subgraph. The relations $r_3^i \subset c_2^i$ and $r_3^i \llcorner\, c_2^{i+1} = a_1$ imply that both, $c^i$ and $c^{i+1}$, connect the clusters $R_i^1$ and $R_i^2$. By lemma (3.3) $p_1 \equiv k(R_i^1)$ may be chosen as a loop momentum. Similarly, it follows from (4.25) that each of the connectivities $c^i$, $c^{i+1}$ and $r^i$ connect $L_1^{i+1}$ and $L_2^{i+1}$. Therefore $p_2 \equiv k(L_2^{i+1})$ may be chosen independently as a second loop momentum.

Thus a $c$-subgraph $\{c_2^i, r_3^i, l_3^{i+1}, c_2^{i+1}\}$, satisfying (4.25), represents an integral of the form

$$\int dp\, f(k, h, p, z)\, [z + \varkappa_r^2 - n(R_3^i)\, k_1^2 - n(R_2^i)\, (p_1 + k_1)^2 - n(R_1^i)\, p_1^2]^{-1}$$
$$\times\, [z + \varkappa_l^2 - n(L_1^{i+1})\, p_2^2 - n(L_2^{i+1})\, (p_2 + h_1)^2 - n(L_3^{i+1})\, h_1^2]^{-1} . \tag{4.28}$$

By applying Privalov's lemma twice, we see that the graph is contractible, which contradicts the assumption of the lemma.

*Lemma:* (4.29)

Let $\{c_2^i, r_3^i, c_k^{i+1}, l_3^{i+2}, c_2^{i+2}\}$, $k \geqslant 3$, be a $c$-subgraph of a graph $G$ satisfying the hypothesis of lemma (4.19). Then the relation $r_3^i \llcorner c_k^{i+3} \llcorner l_3^{i+2} \neq a_1$ holds.

Proof: Suppose $r_3^i \llcorner c_k^{i+1} \llcorner l_3^{i+1} = a_1$. The partitions $c_2^i, r_3^i, l_3^{i+2}, c_2^{i+2}$ are given by

$$r_3^i = (R_1^i, R_2^i, R_3^i)\ ,\quad c_2^i = (R_1^i \cup R_2^i, R_3^i)\ , \tag{4.30}$$

$$l_3^{i+2} = (L_1^{i+2}, L_2^{i+2}, L_3^{i+2})\ ,\quad c_2^{i+2} = (L_1^{i+2} \cup L_2^{i+2}, L_3^{i+2})\ . \tag{4.31}$$

By lemma (4.21), $r_3^i \llcorner c_k^{i+1}$ and $c_k^{i+1} \llcorner l_3^{i+2}$ are 2-connected partitions. On the other hand it follows from lemma (4.19) that $c_2^i \llcorner c_k^{i+1} = c_k^{i+1} \llcorner c_2^{i+2} = a_1$. Thus $c_k^{i+1}$ connects $R_1^i \cup R_2^i$ with $R_3^i$ and $L_1^{i+1} \cup L_2^{i+2}$ with $L_3^{i+2}$. If the clusters are suitably labelled, we can write therefore:

$$r_3^i \llcorner c_k^{i+1} = (R_1^i, R_2^i \cup R_3^i)\ , \tag{4.32}$$

$$c_k^{i+1} \llcorner l_3^{i+2} = (L_1^{i+2}, L_2^{i+2} \cup L_3^{i+2})\ . \tag{4.33}$$

$k_1 \equiv k(R_3^i)$ and $h_1 \equiv k(L_3^{i+2})$ are external momenta of the subgraph. $c_2^i \llcorner c_k^{i+1} = c_k^{i+1} \llcorner c_2^{i+1} = a_1$ shows that $p_1 \equiv k(R_1^i)$ and $p_2 \equiv k(L_1^{i+2})$ cannot be determined by external momenta. If furthermore $r_3^i \llcorner (c_k^{i+1} \llcorner l_3^{i+2}) = a_1, p_1$ cannot be determined from the external momenta and $p_2$. Therefore $p_1$ and $p_2$ are independent loop momenta. Privalov's lemma, applied to the $p_1$ and to the $p_2$-integration shows that the subgraph is contractible. Therefore $r_3^i \llcorner c_k^{i+1} = l_3^{i+2} = a_1$ must be false.

We are now prepared to verify the structure (4.14) of the propagators in a sector of a maximal $c$-subgraph satisfying the assumption of lemma (4.19). There are two kinds of sectors; exterior sectors, to the left of the leftmost cut and to the right of the rightmost cut, and interior sectors. Let us turn to the interior sectors first. These are in a one to one correspondence with the minimal $c$-subgraphs between two 2-connected $c$-lines.

We shall consistently use the following notation for the clusters in the connectivities of the $p$- and $c$-lines:

$$l_3^i = (L_1^i, L_2^i, L_3^i);\quad r_3^i = (R_1^i, R_2^i, R_3^i)\ . \tag{4.35}$$

$$c_2^i = (L_1^i \cup L_2^i, L_3^i) = (R_1^i \cup R_2^i, R_3^i)\ . \tag{4.36}$$

i.e. $L_3^i$ and $R_3^i$ are the clusters of $l_3^i$ and $r_3^i$ respectively, which also appear in $c_2^i$.

$$c_3^i = (L_1^i, L_2^i, L_3^i)\ . \tag{4.37}$$

If no $p$-line is attached to $c_k^i$, we write: (4.38)

$$c_k^i = (C_1^i, \dots C_k^i)\ . \tag{4.39}$$

We now start with the classification of minimal $c$-subgraphs $g$ between two 2-connected $c$-lines. The main classes are given by:

I: $g$ contains two $c$-lines: $c_2^1$ and $c_2^2$. (4.40)

II: $g$ contains three $c$-lines: $c_2^1, c_k^2, c_2^3, k \geqslant 3$. (4.41)

Class I is subdivided further by:

I.1: $g$ contains one $p$-line: $r_3^1$ or $l_3^2$, (4.42)

I.2: $g$ contains two $p$-lines: $r_3^1$ and $l_3^2$. (4.43)

I.1: The structure of $g$ is discussed in lemma (4.21). It implies $r_3^1 \subset c_2^1, r_3^1 \subset c_2^2$ and $c_2^1 \llcorner c_2^2 = a_1$. Similarly $l_3^2 \subset c_2^2, l_3^2 \subset c_2^1$ and $c_2^1 \llcorner c_2^2 = a_1$. Therefore $r_3^1 = l_3^2$ is the unique common refinement of $c_2^1$ and $c_2^2$. Hence, if $r_1^3$ is the $p$-line in $g$;

$$c_2^1 = (R_1^1 \cup R_2^1, R_3^1), \quad c_2^2 = (R_1^1, R_2^1 \cup R_3^1). \tag{4.44}$$

In (4.44) and in all following similar discussions, it is understood that the clusters are suitably labelled.

$k_1 \equiv k(R_3^1)$ and $h_1 \equiv k(R_1^1)$ are external momenta of $g$. By (4.44), $k(R_2^1) = -(k_1 + h_1)$. Therefore, the quadratic form $Q_{r^{3^1}}$ in the propagator $r_3^1$ depends only on external momenta:

$$Q_{r_3^1}(k, h) = n(R_3^1)\, k_1^2 + n(R_2^1)\, (k_1 + h_1)^2 + n(R_1^1)\, h_1^2 \tag{4.45}$$

I.2: $g = \{c_2^1, r_3^1, l_3^2, c_2^2\}$. According to lemma (4.22), there are two subcases:

a) $r_3^1 \llcorner c_2^2 = a_1, \quad c_2^1 \llcorner l_3^2 = a_1, \quad r_3^1 \llcorner l_3^2 \neq a_1$. (4.46)

b) $r_3^1 \llcorner c_2^2 \neq a_1$ or $c_2^1 \llcorner l_3^2 \neq a_1$. (4.47)

I.2.a: It is easily verified [1] that there are essentially two possibilities: $\alpha$, $\beta$:

$\alpha$) $R_1^1 \cup R_3^1 = L_1^2 \cup L_3^2, \; R_2^1 = L_2^2$. (4.48)

We set $k_1 \equiv k(R_3^1)$ and $h_1 \equiv k(L_3^2)$.

According to lemma (3.10) $p \equiv k(R_2^1)$ is a loop momentum. By (4.48) we obtain:

$$Q_{r_3^1}(k, h, p) = n(R_3^1)\, k_1^2 + n(R_2^1)\, p^2 + n(R_1^1)\, (k_1 + p)^2, \tag{4.49}$$

$$Q_{l_3^2}(k, h, p) = n(L_3^2)\, h_1^2 + n(L_2^2)\, p^2 + n(L_1^2)\, (h_1 + p)^2. \tag{4.50}$$

$\beta$) $R_1^1 \cup R_3^1 = L_2^2, \; R_2^1 = L_1^2 \cup L_3^2$. (4.51)

Here we set $k_1 \equiv k(R_3^1)$, $h_1 \equiv k(L_3^2)$.

$p \equiv k(R_2^1) = -k(L_2^2)$ is a loop momentum (lemma 3.10). (4.51) yields

$$Q_{r_3^1}(k, p) = n(R_3^1)\, k_1^2 + n(R_1^1)\, (k_1 + p)^2 + n(R_2^1)\, p^2, \tag{4.52}$$

$$Q_{l_3^2}(h, p) = n(L_2^2)\, p^2 + n(L_1^2)\, (p - h_1)^2 + n(L_3^2)\, h_1^2. \tag{4.53}$$

I.2.b: We only treat the case $r_3^1 \llcorner c_2^2 \neq a_1$, $c_2^1 \llcorner l_3^2 = a_1$, the discussion for $r_3^1 \llcorner c_2^2 = a_1$, $c_2^1 \llcorner l_3^2 \neq a_1$ is identical. $r_3^1$ is the common refinement of $c_2^1$ and $c_2^2$. Because of $c_2^1 \llcorner l_3^2 = a_1$, $p \equiv k(L_1^2)$ may be chosen as loop momentum. The external momenta in the quadratic forms are $k \equiv k(R_3^1)$ and $h \equiv k(L_3^2)$. The explicit discussion [6, 1] shows that the clusters of the common refinement $r_3^1$ are $(R_1^1, L_3^2, R_3^1)$. Thus, the quadratic forms associated with the $p$-lines $r_3^1$ and $l_3^2$ are:

$$Q_{r_3^1} = n(R_3^1)\, k^2 + n(L_3^2)\, h^2 + n(R_1^1)\, (k+h)^2\,, \tag{4.54}$$

$$Q_{l_3^2} = n(L_1^2)\, p^2 + n(L_2)\, (p+h)^2 + n(L_3^2)\, h^2\,. \tag{4.55}$$

Since Privalov's lemma, applied to the $p$-integration, furnishes a new H-C numerator of sufficient decrease, only the surviving denominator $(z + \varkappa_{r_3^1}^2 - Q_{r_3^1})^{-1}$ is of interest.

The graphs of class II may be further characterized by the number of $p$-lines:

$$\text{II.1:}\ g = \{c_2^1, r_3^1, c_k^2, l_3^3, c_2^3\}\,,\ k \geqslant 3\,,\ \text{(two } p\text{-lines)}\,. \tag{4.56}$$

$$\text{II.2:}\ g = \{c_2^1, r_3^1, l_3^2 = c_3^2, c_3^2, l_3^3, c_2^3\}\ \text{(three } p\text{-lines)}\,. \tag{4.57}$$

We discuss these cases separately:

II.1: By lemma (4.29) $r_3^1 \llcorner c_k^2 \llcorner l_3^3 \neq a_1$. (3.9) therefore implies $r_3^1 \llcorner c_k^2 = c_k^2 \llcorner l_3^3 = a_2$. Using $c_2^1 \llcorner c_k^2 = c_k^2 \llcorner c_2^3 = a_1$ we conclude that

$$a_2 = (R_1^1, R_2^1 \cup R_3^1) = \begin{cases} \text{either} & (L_1^3, L_2^3 \cup L_3^3) \\ \text{or} & (L_2^3 \cup L_3^3, L_1^3)\,. \end{cases} \tag{4.58}$$

$k \equiv k(R_3^1)$ and $h \equiv k(L_3^3)$ are external momenta. $p \equiv k(R_1^1)$ is a loop momentum. With this choice we obtain $k(R_2^1) = -(k+p)$, $k(L_1^3) = \pm p\; k(L_2^3) = -(h \pm p)$, according to (4.58).

$$Q_{r_3^1} = n(R_3^1)\, k^2 + n(R_2^1)\, (k+p)^2 + n(R_1^1)\, p^2\,, \tag{4.59}$$

$$Q_{l_3^3} = n(L_1^3)\, p^2 + n(L_2^3)\, (h \pm p)^2 + n(L_3^3)\, h^2\,. \tag{4.60}$$

II.2: We distinguish between two cases:

$$\text{a)}\quad r_3^1 \llcorner c_3^2 \llcorner l_3^3 = a_1\,, \tag{4.61}$$

$$\text{b)}\quad r_3^1 \llcorner c_3^2 \llcorner l_3^3 \neq a_1\,. \tag{4.62}$$

II.2.a: As in the proof of (4.29), we apply lemma (3.3) twice in order to verify that $p_1 \equiv k(R_1^1)$ and $p_2 \equiv k(L_1^3)$ can be chosen as independent loop momenta. By lemma (4.21) $c_3^2 \llcorner l_3^3 = \hat{a}_2$. We label the clusters of $c_3^2$ and $\hat{a}_2$ such that $\hat{A}_1 = C_1^2$. Since $c_3^2 \llcorner c_2^3 = a_1$, by (4.19), $c_3^2$ connects $L_1^3 \cup L_2^3$ with $L_3^3$. Therefore $\hat{A}_i \neq L_3^3$, $i = 1, 2$. This yields essentially the alternatives:

$$\alpha)\quad C_1^2 = L_3^3 \cup L_2^3 \ \text{and}\ \beta)\ C_1^2 = L_1^3\,. \tag{4.63}$$

Similarly we have $r_3^1 \llcorner c_3^2 = \tilde{a}_2 = (\tilde{A}_1, \tilde{A}_2)$. $\tilde{a}_2 \neq \hat{a}_2$ since $\tilde{a}_2 \llcorner \hat{a}_2 = (r_3^1 \llcorner c_3^2) \llcorner (c_3^2 \llcorner l_3^3) = r_3^1 \llcorner c_3^2 \llcorner l_3^3 = a_1$. Therefore we may label the clusters of $c_3^2$ such that $\tilde{A}_1 = C_2^2$. $c_2^1 \llcorner c_3^2 = a_1$ implies $R_3^1 \neq \tilde{A}_i$, $i = 1, 2$. Again two cases are possible:

$$\gamma)\quad C_2^2 = R_3^1 \cup R_2^1\,,\quad \delta)\ C_2^2 = R_1^1\,. \tag{4.64}$$

Putting $k \equiv k(R_3^1)$ and $h = k(L_3^3)$, we have $k(C_1^2) = \mp p_2$ in the cases $\alpha$) and $\beta$) respectively, and $k(C_2^2) = \mp p_1$ in the cases $\gamma$) and $\delta$).

Thus:

$$Q_{r_3^1} = n(R_3^1)\, k^2 + n(R_2^1)\, (k+p_1)^2 + n(R_1^1)\, p_1^2\,, \tag{4.65}$$

$$Q_{l_3^2} = n(C_2^2)\, p_1^2 + n(C_3^2)\, (p_1 \pm p_2)^2 + n(C_1^2)\, p_2^2\,, \tag{4.65}$$

$$Q_{l_3^3} = n(L_1^3)\, p_2^2 + n(L_2^3)\, (p_2 + h)^2 + n(L_3^3)\, h^2\,. \tag{4.67}$$

In (4.66) the + sign holds for the combinations $(\alpha, \gamma)$, $(\beta, \delta)$, the $-$ sign for $(\alpha, \delta)$ and $(\beta, \gamma)$.

II.2.b: As in II.1, we have $r_3^1 \llcorner c_3^2 \llcorner l_3^3 \mp a_1$. Repeating the discussion of II.1, we obtain essentially two alternative identification schemes:

$$(R_1^1, R_2^1 \cup R_3^1) = \begin{cases} \text{either } (L_1^3, L_2^3 \cup L_3^3) \\ \text{or } (L_2^3 \cup L_3^3, L_1^3)\,, \end{cases} \tag{4.68}$$

and

$$(R_1^1, R_2^1 \cup R_3^1) = \begin{cases} \text{either } (L_1^2, L_2^2 \cup L_2^2) \\ \text{or } (L_2^2 \cap L_3^2, L_1^2)\,. \end{cases} \tag{4.69}$$

As in II.1, we may choose $p_1 \equiv k(R_1^1)$. Since the partitions $c_2^1$, $r_3^1$ and $c_2^3$ all connect the clusters $L_2^2$ and $L_3^2$ of $l_3^3$, $p_2 \equiv k(L_2^2)$ is a second independent loop momentum. Setting $k \equiv k(R_3^1)$ and $h \equiv (L_3^3)$, we obtain:

$$Q_{r_3^1} = n(R_3^1)\, k^2 + n(R_2^1)\, (k + p_1)^2 + n(R_1^1)\, p_1\,, \tag{4.70}$$

$$Q_{l_3^3} = n(L_1^2)\, p_2^1 + n(L_2^3)\, (p_1 \pm p_2)^2 + n(L_2^2)\, p_2^2\,, \tag{4.71}$$

$$Q_{l_3^3} = n(L_1^3)\, p_1^2 + n(L_2^3)\, (p_1 \pm h)^2 + n(L_3^3)\, h^2\,. \tag{4.72}$$

The alternatives $\pm$ in (4.71) and (4.72) arise from (4.68) and (4.69). After applying Privalov's lemma to the $p_2$ integration, only $Q_{r_3^1}$ and $Q_{l_3^3}$ survive.

This concludes our discussion of the interior sectors of maximal $c$-subgraphs. An exterior sector to the left is confined by the $c$-lines $c_k^1$ $(k \geqslant 3)$ and $c_2^2$. We may assume that it contains one $p$-line $l_3^2$, since $r_3^1$ could be commuted with $c_3^1$. Lemma (4.21) implies that $c_k^1$ connects $L_1^2 \cup L_2^2$ with $L_3^2$ but does not connect $L_2^1$ with $L_2^2$. If $G$ contains at least 3 $c$-lines, $p \equiv k(L_3^2)$ is a loop momentum of $G$ and

$$Q_{l_3^2} = n(L_1^2)\, k^2 + n(L_2^2)\, (k + p)^2 + n(L_3^2)\, p^2\,, \tag{4.73}$$

with $k \equiv k(L_1^2)$. Similarly, an exterior sector to the right yields

$$Q_{r_3} = n(R_3)\, p^2 + n(R_2)\, (p + h)^2 + n(R_1)\, h^2\,, \tag{4.74}$$

with $h \equiv k(R_1)$ and $p \equiv k(R_3)$.

We summarize our discussion:

*Lemma:* (4.75)

Let $G$ be a graph which does not allow for trivial or Privalov contractions. Then in each sector of its maximal $c$-subgraph, the number of surviving propagators exceeds the number of (internal) $c$-lines at least by 1. The propagators are of the form

$$[z + \varkappa^2 - n(A_1)\, p_i^2 - n(A_2)\, (p_i + p_{i+1})^2 + n(A_3)\, p_{i+1}^2]^{-1}\,. \tag{4.76}$$

An immediate consequence is the theorem:

*Theorem:* (4.77)

In a quantum mechanical system satisfying (1.2–4) and the spectral condition (2.13) the graphs arising from $(Q_{a_1} R_0)^n\, (k, h, z)$, $n \geqslant 5\,(N-2)$, $Re\, z < s_4$, are connected and contractible.

Proof: Either the graph is contractible by trivial or Privalov contractions or lemma (4.75) applies. If $n \geqslant 5\,(N-2)$, it contains at least 5 $c$-lines, because the graphs to $(Q_{a_1} R_0)^k$ are at least $N-k$ connected. Hence, by lemma (4.75), $G$ contains at least 4 surviving propagators of the form (4.76). Thus Faddeev's lemma can be applied to the $p_2, p_3, p_4$ integration, proving that $G$ is contractible.

This, together with the fall-off estimate (2.12), proves the Fredholm alternative for the F-Y equation (2.10) in the region $\{z \mid z \in \Pi_{\varepsilon a_1}, Re\, z < s_4\}$. Thus the singularity structure of the full amplitude $T^{\alpha_2}_{a_1}(k, h, z)$ is determined, apart from the bound state poles, by the low order iterations to the F-Y equations ('maximal regularity'):

*Theorem:* (4.78)

In a quantum mechanical system, satisfying (1.2–4) and the spectral condition (2.13), the operators $T^{\alpha_3}_{a_1}(z)$ act as integral transformations in the momentum representation of the C-M Hilbert space for $z \in \Pi_{\varepsilon a_1} \cap \{Re\, z < s_4\}$. We have (using the matrix notation (1.21)):

$$\mathfrak{T}_{a_1}(k, h, z) = \sum_{n=0}^{5(N-2)} (Q_{a_1} R_0)^n\, \mathfrak{T}_{a_1}(k, h, z) + \tilde{\mathfrak{T}}_{a_1}(k, h, z)\,. \tag{4.79}$$

For $z$ in the above region, the elements of the kernel matrix have, apart from a boundstate pole $\psi_{a_1}(k)\, \psi_{a_1}(h)^*\, (z + \varkappa^1_{a_1})^{-1}$, H-C components $\tilde{T}^{\alpha_2}_{a_1}(R, R)$, $\tilde{T}^{\alpha_2}_{a_1}(a_i, R)$, $\tilde{T}^{\alpha_2}_{a_1}(R, d)$ and $\tilde{T}^{\alpha_2}_{a_1}(a_i, d)$, where $a_i \in (a_2, a_3)$ and $d$ ranges over all partitions of $\{1, \dots N\}$ into 2 or 3 clusters. $\psi_{a_1}$ is H-C and satisfies an estimate

$$|\psi_{a_1}(k)| \leqslant c\, N(\theta, k)\,, \quad \theta > 3/2\,. \tag{4.80}$$

The components, as well as their Hölder derivatives, are bounded uniformly in $k$, $h$ and $z \in \Pi_{\varepsilon a_1} \cap \{Re\, z \leqslant E_0 < s_4\}$ by const $N(\theta, k)$, $N(\theta, h)$.

## 5. Remarks on Applications

In Ref. [1] the previous results are applied to the problem of asymptotic completeness (cf. e.g. [13]).

The channel states of a system are called asymptotocally complete if the time evolution of any state of the system tends for $t \to \pm\infty$ to a superposition of freely moving fragments; i.e. if

$$R^{\pm} \equiv \bigoplus_{a_k} \Omega^{\pm}_{a_k} D_{a_k} = \mathcal{H}\,. \tag{5.1}$$

A proof of (5.1), using as main ingredients the formula

$$[E(\beta) - E\,(\beta - 0)]\,\psi = \operatorname*{s-lim}_{\varepsilon \downarrow 0}\, 2\,\varepsilon/\pi \int_{-\infty}^{\beta} d\lambda\, R\,(\lambda - i\,\varepsilon)\, R\,(\lambda + i\,\varepsilon)\,\psi\,,$$

(cf. [14]) and the connection between time dependent and time independent scattering theory (1.12), is possible, provided the interchange of certain limits is allowed [7]. Theorem (4.78) (maximal regularity below $s_4$) provides a justification for $\beta < s_4$. Partial asymptotic completeness follows [1]:

$$R^{\pm}(\lambda) \equiv \bigoplus_{a_k} \Omega^{\pm}_{a_k} E^{a_k}(\lambda)\, D_{a_k} = E(\lambda)\, \mathcal{H}\,, \text{ for } \lambda < s_4\,. \tag{5.2}$$

A second application, given in Ref. [1], deals with a proposal by Hunziker [13] for the general definition of scattering cross sections $\sigma_{b\,a}(\Omega, \Phi_a)$ for the scattering of $m$ fragments in the initial channel state $\Phi_a$ into $n$ final fragments (channel $b$) with momenta in the region $\Omega < R^{3n}$. One expects that the definition is only meaningful if $\Omega$ is chosen such that certain rescattering processes are screened out. Using maximal regularity below $s_4$ we are able to enumerate these processes for $2 - n$ processes below $s_4$ and for the $3 - 3$ scattering in a 3 particle system.

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## Appendix I: Singular Integrals

For the reader's convenience we state here the lemmas on singular integrals used in this article; both due to Faddeev: A 3-dimensional version of Privalov's lemma and 'Faddeev's lemma' [2] (p. 44–45).

*Privalov's lemma:* (I.1)

Let $f(.,.)$: $R^m \times R^3 \to C$ be H-C of index $\mu > 0$ with an estimating function $N(k)$:

$$|f(k, p)| \leqslant N(k)\,,$$

$$|f(k + h, p + q) - f(k, p)| \leqslant N(k)\,(|h|^\mu + |p|^\mu) \text{ for } |h| \leqslant 1,\ |p| \leqslant 1\,,$$

and let $f$ vanish outside some ball $|p| < R$.

Then

$$F(k, z) \equiv \int d^3p\, \frac{f(k, p)}{z + \varkappa^2 - E(k, p)} \tag{I.2}$$

is H-C in $R^m \times \Pi_{-\varkappa^2}$ with Hölder indices $\mu' < \mu$ and $\nu < \min(1/2, \mu/2)$. In (I.2) $E(k, p)$ denotes a positive definite quadratic form and $\Pi_{-\varkappa^2}$ is the complex plane slit along $[-\varkappa^2, \infty)$ and completed by the limit points from above and from below.

*Corollary:* (I.3)

Let $f(k, p)$: $R^{3n} \times R^{3n} \to C$ be H-C of index $\mu > 0$ with an estimating function $N(k)$ and let $f$ be absolutely integrable in $p$, uniformly in $k$. Let furthermore $\varkappa_a^2 < \varkappa_b^2$ for $a \subset b$. Then

$$F(k, z) \equiv \int d^{3n}p\, \frac{f(k, p)}{\big(z + \varkappa_a^2 - E_a(p)\big)\big(z + \varkappa_b^2 - E_b(p)\big)} \tag{I.4}$$

is H-C in regions $R^{3n} \times [\Pi_{-\varkappa^2} \cap \{Re\, z < E_0\}]$, $\varkappa^2 \equiv \max(\varkappa_a^2, \varkappa_b^2)$, with indices $\mu' < \mu$ and $\nu < \min(1/2, \mu/2)$ and estimating function $N(k)$.

Proof: Outside a sufficiently large ball $|p| < R(E_0)$, the denominator is non singular for $Re\, z < E_0$. The contribution to $F(k, z)$ from the integration over

$|p| \geqslant R(E_0)$ has the desired regularity properties. If $a \subset b$ or $b \subset a$, the singularities can be separated and the corollary results from applying Privalov's lemma in both subregions of $|p| < R(E_0)$. If $a \not\subset b$ and $b \not\subset a$ then, by lemma (3.3), the loop momenta can be chosen such that $E_2(p) = p_1^2 + E_1(p)$, $E_b(p) = p_2^2 + E_2(p)$ and $E_{1,2}(p)$ do not depend on $p_1$ and $p_2$. The corollary follows from applying Privalov's lemma twice.

*Faddeev's lemma:* (I.5)

Let $f(k_1, \dots k_n, p_1, p_2, p_3)$ be H-C with index $\mu > 0$ and estimating function $N(k)$ such that $f(k_1, \dots p_3) = 0$ for $|p_i| > R$, $1 \leqslant i \leqslant 3$. Then

$$F(l_1, \dots k_n, z) \equiv \int \prod_{i=1}^{3} d^3p_i\, f(k, p) \prod_{l=1}^{4} [z + x_l^2 - Q_l(k, p)]^{-1}, \tag{I.6}$$

where

$$Q_l(k, p) = n_l' p_{l-1}^2 + n_2'' (p_{l-1} \pm p_l)^2 + n_l''' p_l^2 , \quad 1 \leqslant l \leqslant 4 ,$$

with

$$p_0 = p_0(k) \text{ and } p_4 = p_4(k) ,$$

is H-C with some index $\mu' > 0$ and estimating function $N(k)$ in $R^{3n} \times \Pi_{-\varkappa^2}$, where $\varkappa^2 = \max_{1 \leq l \leq 4} \varkappa_l^2$.

## Appendix II: On the Definition of $B_{a_k}(\theta, \mu)$

In analogy with the definition of $B(\theta, \mu)$ in Ref. [7] we define $B_{a_k}(\theta, \mu)$ as direct sum of function spaces $R_{a_k}^{3N} \to C$ normed by

$$\|f\|_{a_k, \Theta, \mu} \equiv \sup_{p,q \in R_{a_k}^{3N}} N_{a_k}(p, \theta)^{-1} \left\{ |f(p)| + \frac{|f(p+q) - f(p)|}{|q|^{\mu}} \right\}, \tag{II.1}$$

$$N_{a_k}(p, \theta) \equiv \sum_i{}' \prod_{i=1}^{N-k} (1 + |p_{ij}|)^{-\Theta} ,$$

$\Sigma'$ extends over all $(N-k)$-tupels $p_{ij}$ of partial sums

$$p_{ij} = \sum_{r=1}^{N} \sigma_{ijr} p_r , \quad \sigma_{ijr} = 0, \pm 1 ,$$

which span $R_{a_k}^{3N}$ and such that $\sigma_{ijr} \sigma_{ijr.} = 0$ if $r$ and $r'$ belong to different clusters of $a_k$.

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