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Some Potential Perturbations of the Laplacian

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1. Introduction

Let $-\Delta$ denote the negative Laplacian acting in free space of dimension d. For the case of dimensions $d = 3$, Povzner [A.1] and Ikebe [A.2] formulated conditions on the potential p which ensured that the continuous part of $-\Delta + p$ is unitarily equivalent to $-\Delta$. Their growth condition at infinity was generalized concurrently and independently by Kato [A.25] and elsewhere [A.23].

In this paper, we further generalize this growth condition which is characterized by a decay exponent at infinity. This is described in more specific terms in Theorem 2.2, which is our main theorem.

In Section 2 we introduce two conditions on the potential and formulate two theorems. Our Condition I is a limiting case of a condition of Kato and Kuroda. In Theorem 2.1 we show that if in addition the potential p is small then for a given interval $\mathfrak{J}$ the continuous part of $(-\Delta + p)_F$ over $\mathfrak{J}$ is unitarily equivalent to the corresponding part of $-\Delta$. Here $(-\Delta + p)_F$ denotes the Friedrichs extension of $-\Delta + p$. In particular, this part has no singularly continuous spectrum. Next we introduce Condition II, which is more stringent than Condition I. It holds for bounded potentials whose decay exponent exceeds $(1 + 1/6)$. In particular, it holds for potentials whose decay exponent exceeds $(1 + 1/4)$, which was introduced by Kato [A.25.b]. In Theorem 2.3, which is our main theorem, we show that Condition II together with a mild continuity assumption on p implies that the entire continuous part of $(-\Delta + p)_F$ is unitarily equivalent to $-\Delta$.

In Section 3 we state a previously formulated abstract set of criteria for the unitary equivalence of two operators. These criteria are stated with reference to a given Banach space $\mathfrak{G}$ which is not unique. In fact, our choice of this norm has been motivated by the Kato-Kuroda theory [A.10], [A.13] of smooth perturbations. For, our norm $\mathfrak{G}$ is defined with the aid of a factorization of the potential which is appropriate in their sense.

In Section 4 we establish Theorem 2.1. The main difficulty of this proof is to show that Condition I on the potential implies Conditions $G_{1,2,3}(\mathfrak{J})$ of Theorem 3.1. The proof of this implication, in turn, is based on an estimate formulated elsewhere [B.19].

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In Section 5 we establish Theorem 2.2. The main difficulty of this proof to show that the more stringent Condition II implies Conditions $A_{1,2}(\mathfrak{J})$ of Theorem 3.1. The proof of this implication, in turn, is based on two facts. The first one is formulated in Theorem 5.1. The second one is a deep result of Kato [B.3] describing the eigenfunctionals of the operator $-\Delta + p$.

After this work was completed the author learned about the deep results of Agmon[2]) [42] announced in Nice at the International Congress of Mathematics.

Acknowledgment. It is a pleasure to thank Professor Jauch for his interest and for valuable conversations.

2. Formulation of the Results

Let $\mathcal{E}_d$ denote the real Euclidean space of dimension d, and let $\ddot{\mathfrak{C}}_\infty(\mathcal{E}_d)$ denote the class of infinitely differentiable complex valued functions on $\mathcal{E}_d$ whose support is bounded and does not contain the origin. As is well-known [A.34] the negative Laplacian is essentially self-adjoint on $\ddot{\mathfrak{C}}_\infty(\mathcal{E}_d)$ and we denote its closure by $-\Delta$. For a given potential-function p we denote by $M(p)$ the corresponding multiplication operator, i.e. the closure of the operator

$$M(p)\, f(x) = p(x)\, f(x)\,, \quad x \in \mathcal{E}_d\,, \quad f \in \ddot{\mathfrak{C}}_\infty({}_d\mathcal{E})\,. \tag{2.1}$$

For the case of dimensions $d = 3$ Povzner [A.1] formulated a condition on the potential p, which ensures that $-\Delta + M(p)$ is essentially self-adjoint on $\ddot{\mathfrak{C}}_\infty(\mathcal{E}_d)$ and that the continuous part of its closure is unitarily equivalent to $-\Delta$. His result was extended by Ikebe [A.3] who showed that it suffices to assume the condition that follows. To describe it we need a notation. Specifically for a given potential p we set

$$\tilde{p}(\xi) = \sup_{|x|=\xi} |p(x)|\,, \quad x \in \mathcal{E}_d\,. \tag{2.2}$$

Condition P-I. *The potential-function $p(x)$ of the variable x in $\mathcal{E}_3$ is real and square integrable over all of $\mathcal{E}_3$. There is a positive number ε such that for the function $\tilde{p}$ of definition* (2.2) *we have*

$$\tilde{p}(\xi) = 0(1)\left(\frac{1}{\xi}\right)^{2+\varepsilon} \quad \textit{at} \quad \xi = \infty\,. \tag{2.3}$$

Furthermore p is Holder-continuous with the exception of finitely many points.

The Povzner-Ikebe growth condition at infinity has been generalized concurrently and independently by Kato [A.25] and elsewhere [23]. Specifically Kato replaced the exponent in (2.3) by $1 + 1/4 + \varepsilon$.

Our first condition is implied by the absolute integrability of $\tilde{p}$ and by

$$\lim_{\xi=\infty} \xi\, \tilde{p}(\xi) = 0\,. \tag{2.4}$$

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[2]) *Added in proof.* An informal conversation with Agmon at the recent Scattering Theory Conference at Oberwolfach led to a sharper result formulated in the Appendix. At the same conference Kuroda and Lavine announced similar results.

To describe it in more specific terms for each positive ν define an integral-mean for the function $\tilde{p}$ by setting

$$I(\nu, \tilde{p}) = \Sigma\, \nu^{1/3} \left(\frac{1}{1+|k|}\right)^{1/2} \int_{\nu + k\nu^{1/3}}^{\nu + (k+1)\nu^{1/3}} \tilde{p}(\xi)\, d\xi\,, \tag{2.5}$$

where the summation is extended over those integers k for which

$$k \in \left(-\frac{1}{2}\nu^{2/3} - 1\,,\quad +\frac{1}{2}\nu^{2/3} + 1\right),$$

and over the endpoints. The condition that follows imposes a more stringent local requirement on the potential p than the one of Povzner-Ikebe. At the same time it is stated for arbitrary dimensions d.

Condition I. *The potential-function $p(x)$ of the variable x in $\mathcal{E}_d$ is such that the function $\tilde{p}(\xi)$ of definition* (2.2) *is locally square-integrable away from $\xi = 0$. At the same time we have*

$$\int_0^1 \begin{Bmatrix} 1, & d = 1 \\ \eta, & d \geqslant 2 \end{Bmatrix} \tilde{p}(\xi)\, d\xi + \int_1^\infty \tilde{p}(\xi)\, d\xi < \infty\,, \tag{2.6}$$

and for $d \geqslant 2$

$$\sup_{\nu > 0} I(\nu, \tilde{p}) < \infty \quad \textit{and} \quad \lim_{\nu = \infty} I(\nu, \tilde{p}) = 0\,. \tag{2.7}$$

With the aid of this condition we formulate our first theorem.

It implies that for small potentials the singularly continuous spectrum is absent, at least over a given interval.

Theorem 2.1. *Let $\mathfrak{J}$ be a compact interval of the positive axis which does not contain zero. Suppose that the potential p satisfies Condition I. Then to the interval $\mathfrak{J}$ there is a strictly positive constant δ_0 such that for every δ in $[-\delta_0, +\delta]$ the part of the Friedrichs extension of $-\Delta + \delta M(p)$ over the interval $\mathfrak{J}$ is unitarily equivalent to the part of $-\Delta$ over $\mathfrak{J}$.*

We have not been able to remove completely the smallness condition of this theorem. However, we can replace it by a more stringent growth condition at infinity, which follows.

Condition II. *To the potential p there is a positive number ε such that for the potential*

$$q(x) = (1 + |x|)^{1/6+\varepsilon}\, p(x)\,, \tag{2.8}$$

Condition I holds.

Incidentally note that this condition holds for bounded potentials whose decay exponent at infinity is greater that $(1 + 1/6)$. We use this condition to replace the smallness condition of Theorem 2.1.

Theorem 2.2. *Suppose that the potential p satisfies Condition II. Suppose further that it is Holder continuous with the possible exception of finitely many points. Then the continuous part of the Friedrichs extension of $-\Delta + M(p)$ is unitarily equivalent to $-\Delta$.*

3. The Previous Theorem on Partly Gentle Perturbations

Let the operators A_0 and A_1 act in an abstract Hilbert space $\mathfrak{H}$ and assume that they are self-adjoint on the given domains $\mathfrak{D}(A_0)$ and $\mathfrak{D}(A_1)$. These domains need not be equal but we assume that their intersection is dense and we set

$$V = A_1 - A_0 \quad \text{on} \quad \mathfrak{D}(A_0) \cap \mathfrak{D}(A_1) .$$

Let $\mathfrak{J}$ be a given bounded interval and let $E_0(\mathfrak{J})$ and $E_1(\mathfrak{J})$ denote the spectral projector of these operators over $\mathfrak{J}$. We denote by $A_0(\mathfrak{J})$ and $A_1(\mathfrak{J})$ the part of these operators over $\mathfrak{J}$, that is their restriction to $E_0(\mathfrak{J})\,\mathfrak{H}$ and $E_1(\mathfrak{J})\,\mathfrak{H}$.

Next let $\mathfrak{G}$ be a Banach space[3]) such that both $\mathfrak{G}$ and $\mathfrak{H}$ can be embedded in a metric space $\mathfrak{M}$. We assume that this embedding is such that $\mathfrak{G} \cap \mathfrak{H}$ is dense in $V(\mathfrak{D}(A_0) \cap \mathfrak{D}(A_1))$ with reference to the $\mathfrak{H}$-norm and in $\mathfrak{G}$ with reference to the $\mathfrak{G}$-norm. We assume further that V considered as a mapping of $\mathfrak{D}(A_0) \cap \mathfrak{D}(A_1)$ into $\mathfrak{M}$ is continuous with reference to the $\mathfrak{M}$-metric, and hence it can be extended to all of $\mathfrak{H}$. In applications the abstract Hilbert space $\mathfrak{H}$ is an $\mathfrak{L}_2$ space and for $\mathfrak{M}$ we choose the space of measurable functions. Then these requirements are practically no restrictions.

As is well-known [B.7.d], the unperturbed resolvent set, $\varrho(A_0)$ contains the points of the open upper or lower half planes that we denote by μ. For a given interval $\mathfrak{J}$ and angle α between 0 and $\pi/2$ we define the regions $\mathcal{R}_\pm(\mathfrak{J})$ by the relations

$$\mathcal{R}_\pm(\mathfrak{J}) = \{\mu\colon Re\,\mu_\pm \in \mathfrak{J}\,, \quad 0 < \arg \mu < \alpha\}\,. \tag{3.1$_\pm$}$$

Note that if $\mathfrak{J}$ is in the spectrum of A_0 then the resolvent $R_0(\mu) = (\mu - A_0)^{-1}$ can not be continued onto $\mathfrak{J}$ as a bounded operator on $\mathfrak{H}$.

Now we formulate the previously mentioned criteria which allow us to continue the perturbed resolvent onto $\mathfrak{J}$ as a form on $\mathfrak{G} \times \mathfrak{G}$. To describe this in more specific terms we introduce a convention. We say that the operators $R_0(\mu)$ on $\mathfrak{H}$ determine bounded forms on $\mathfrak{G} \times \mathfrak{G}$, if the forms[4])

$$[R_0(\mu)]_{\mathfrak{G}}\,(f, g) = \big(R_0(\mu)\, f, g\big) \quad \text{on} \quad (\mathfrak{G} \cap \mathfrak{H}) \times (\mathfrak{G} \cap \mathfrak{H})$$

are bounded with reference the $\mathfrak{G}$-norm. We denote the closure of these forms which are defined on all of $\mathfrak{G} \times \mathfrak{G}$, by the same symbols $[R_0(\mu)]_{\mathfrak{G}}$. We say that the operator $VR_0(\mu)$ in $\mathfrak{H}$ determines a bounded operator on $\mathfrak{G}$ if

$$V\, R_0(\mu)\, (\mathfrak{G} \cap \mathfrak{H}) \subset \mathfrak{G}$$

and this mapping is bounded with reference the $\mathfrak{G}$-norm. Note that in general $V\, R_0(\mu)$ maps $\mathfrak{H}$ into $\mathfrak{M}$.

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[3]) Previously this Banach space was denoted by $\mathfrak{B}$. To emphasize its key role in defining the partial gentleness criteria we denote it by $\mathfrak{G}$.

[4]) Previously following Dirac the inner product was linear in the second argument. At present we follow Kato and the inner product is linear in the first argument.

Condition $G_1(\mathfrak{J})$. *For each μ in the open regions $\mathfrak{R}_\pm(\mathfrak{J})$ the operators $R_0(\mu)$ on $\mathfrak{H}$ determine bounded forms on $\mathfrak{G}\times\mathfrak{G}$ and the forms $[R_0(\mu)]_{\mathfrak{G}}$ admit weakly continuous extension onto the closures $\mathfrak{R}_\pm(\mathfrak{J})$. Furthermore the norms of these forms remain bounded independently of μ.*

Condition $G_2(\mathfrak{J})$. *For each μ in the open regions $\mathfrak{R}_\pm(\mathfrak{J})$ the operators $VR_0(\mu)$ in $\mathfrak{H}$ determine bounded operators on $\mathfrak{G}$. These operators, $(VR_0(\mu))_{\mathfrak{G}}$, depend continuously in norm on μ and admit continuous extension onto the closures $\mathfrak{R}_\pm(\mathfrak{J})$.*

Actually in these two conditions it would be sufficient to assume the existence of the radial limit only, but we shall not be concerned with this fact. Next we assume that the operator V can be approximated in the following manner.

Condition $G_3(\mathfrak{J})$. *There is a sequence of operators, $\{V_k\}$, such that for each k and μ in the open region $\mathfrak{R}_\pm(\mathfrak{J})$, the operators $V_k\, R_0(\mu)$ are defined on all of $\mathfrak{H}$ and are bounded. The pair (A_0, A_0+V_k) satisfies Conditions $G_{1,2}(\mathfrak{J})$. Furthermore*

$$\lim_{k=\infty} \|(VR_0(\mu))_{\mathfrak{G}} - (V_k\, R_0(\mu))_{\mathfrak{G}}\| = 0\,.$$

Note that if V is A_0 bounded with reference the $\mathfrak{H}$-norm then we can set $V_k = V$. That is, in this case, Conditions $G_1(\mathfrak{J})$ and $G_2(\mathfrak{J})$ imply Condition $G_2(\mathfrak{J})$. We refer to these three conditions by saying that the pair of operators (A_0, A_1) is gentle over the interval $\mathfrak{J}$, in short partly gentle. Next we state the two additional conditions.

Condition $A_1(\mathfrak{J})$. *For every ω in the closed and bounded interval $\mathfrak{J}$, the operators $(1-VR_0^\pm(\omega))_{\mathfrak{G}}$ are invertible. That is, they admit bounded inverses defined on all of $\mathfrak{G}$.*

Condition $A_2(\mathfrak{J})$. *For each μ in $R_\pm(\mathfrak{J})$ on $\mathfrak{H}$ determine bounded forms on $\mathfrak{G}\times\mathfrak{G}$. These forms are related to the unperturbed resolvent via the second resolvent equation,*

$$[R_1(\mu)]_{\mathfrak{G}} - [R_0(\mu)]_{\mathfrak{G}} = [R_1(\mu)]_{\mathfrak{G}}\,(VR_0(\mu))_{\mathfrak{G}}\,.$$

An elementary argument [A.13] that if the operator V is A_0-bounded with reference the $\mathfrak{H}$-norm then the gentleness conditions and Condition $A_1(\mathfrak{J})$ imply Condition $A_2(\mathfrak{J})$.

After these preparations we formulate a theorem on such perturbations which was established elsewhere [A.13].

Theorem 3.1. *Suppose that the pair of operators (A_0, A_1) is satisfies Conditions $G_{1,2,3}(\mathfrak{J})$ over the closed and bounded interval $\mathfrak{J}$. Suppose further that Conditions $A_1(\mathfrak{J})$ and $A_2(\mathfrak{J})$ hold. Then $A_0(\mathfrak{J})$ and $A_1(\mathfrak{J})$, the part of these operators $\mathfrak{J}$, are unitarily equivalent.*

4. The Proof of Theorem 2.1

To derive Theorems 2.1 and 2.2 from the abstract Theorem 3.1 it is convenient to introduce a new perturbed and unperturbed operator. Let $\mathfrak{S}_{d-1}$ be the $(d-1)$-dimensional unit sphere and define the unitary transformation T mapping $\mathfrak{L}_2(\mathcal{E}_d)$

onto $\mathfrak{L}_2((0, \infty), \mathfrak{L}_2(\mathcal{S}_{d-1}))$ by

$$T f(\xi)(u) = \xi^{(d-1)/2} f(\xi u), \quad \xi \in (0, \infty), \quad u \in \mathcal{S}_{d-1}. \tag{4.1}$$

Clearly the adjoint is given by

$$T^* f(x) = \left(\frac{1}{|x|}\right)^{(d-1)/2} f(|x|) \left(\frac{x}{|x|}\right), \quad x \in \mathcal{E}_d. \tag{4.1*}$$

We define the new unperturbed operator, acting in $\mathfrak{L}_2((0, \infty), \mathfrak{L}_2(\mathcal{S}_{d-1}))$, to be the Friedrichs extension of

$$A_0 = T(-\Delta)\, T^* \quad \text{on} \quad T\, \mathfrak{C}_\infty(\mathcal{E}_d). \tag{4.2}$$

To define the new perturbation first each ξ define the operator $p_0(\xi)$ on $\mathfrak{L}_2(\mathcal{S}_{d-1})$ by

$$p_0(\xi)\, \varphi(u) = p(\xi u)\, \varphi(u), \quad u \in \mathcal{S}_{d-1}, \quad \varphi \in \mathfrak{L}_2(\mathcal{S}_{d-1}). \tag{4.3}$$

Then we see from definitions (4.1), (4.1)* and (2.1) that T carries V into multiplication by this $\mathfrak{L}_2(\mathcal{S}_{d-1})$ operator valued function. Specifically,

$$M(p_0) = T\, M(p)\, T^* \tag{4.4}$$

and we take this to be the perturbation.

Next we introduce a gentleness norm with reference to which the conditions of the abstract Theorem 3.1 hold for the pair of operators $((A_0 + M(p_0))_F, A_0)$. We define this norm with the aid of the positive function $\tilde{p}$ of (2.2) by setting

$$\mathfrak{H} = \mathfrak{L}_2((0, \infty), \mathfrak{L}_2(\mathcal{S}_{d-1})) \tag{4.5}$$

where

$$\|f\|_{\mathfrak{G}} = \left\| M\left(\frac{1}{\tilde{p}}\right)^{1/2} f \right\|_{\mathfrak{H}}. \tag{4.6}$$

The choice of this norm has been motivated by the considerations of Kato and Kuroda [A.24], [A.27]. For it defines a factorization of the perturbation which is appropriate in their sense.

a) Condition $G_1(\mathfrak{J})$. To verify this condition we first observe that the norm of the sequilinear form of $R_0(\mu)$ on $\mathfrak{G} \times \mathfrak{G}$ is majorized by the $\mathfrak{H}$ norm of a corresponding operator. Specifically

$$\|[R_0(\mu)]\|_{\mathfrak{G}} < \|M(\tilde{p})^{1/2}\, R_0(\mu)\, M(\tilde{p})^{1/2}\|_{\mathfrak{H}}. \tag{4.7}$$

For, remembering definition (4.5) we see that

$$\mathfrak{H} \cap \mathfrak{G} \subset \mathfrak{R}(M(\tilde{p})^{1/2}).$$

Hence

$$(R_0(\mu) f, g)_{\mathfrak{H}} = \left(R_0(\mu)\, M(\tilde{p})^{1/2}\, M\left(\frac{1}{\tilde{p}}\right)^{1/2} f, \quad M(\tilde{p})^{1/2}\, M\left(\frac{1}{\tilde{p}}\right)^{1/2} g\right).$$

Appling the Schwarz inequality and using definition (4.5) again, we obtain

$$|[R_0(\mu)]_{\mathfrak{G}}(f, g)| \leqslant \|M(\tilde{p})^{1/2} R_0(\mu) M(\tilde{p})^{1/2}\|_{\mathfrak{H}} \cdot \|f\|_{\mathfrak{G}} \cdot \|g\|_{\mathfrak{G}} .$$

This completes the proof of relation (4.7). Hence the uniform boundedness part of Condition $G_1(\mathfrak{J})$ is implied by the uniform of the family of operators on $\mathfrak{H}$,

$$F(\mu) = M(\tilde{p})^{1/2} R_0(\mu) M(\tilde{p})^{1/2} . \tag{4.8}$$

This uniform boundedness was established elsewhere [A.43.a]. A repetition of the arguments leading to relation (4.7) shows that for each pair of points μ and ω in $R_{\pm}(\mathfrak{J})$,

$$\|[R_0(\mu)] - [R_0(\omega)]\|_{\mathfrak{G}} \leqslant \|M(\tilde{p})^{1/2} \big(R_0(\mu) - R_0(\omega)\big) M(\tilde{p})^{1/2}\|_{\mathfrak{H}} .$$

Hence the boundary value requirement of Condition $G_1(\mathfrak{J})$ is implied by the existence of the boundary values of the family of operators in (4.8). This was shown elsewhere [A.43.a], for every compact interval of the open positive axis. These two facts together establish the validity of Condition $G_1(\mathfrak{J})$ for every such interval $\mathfrak{J}$.

b) Condition $G_2(\mathfrak{J})$. To verify this condition first recall definition (4.5). This shows that

$$\|M(p_0) R_0(\mu) f\|_{\mathfrak{G}} = \left\| M\left(\frac{1}{\tilde{p}}\right)^{1/2} M(p_0) R_0(\mu) f \right\|_{\mathfrak{H}} .$$

It is clear from definition (2.2) that

$$\left\| M\left(\frac{1}{\tilde{p}}\right)^{1/2} M(p_0) M\left(\frac{1}{\tilde{p}}\right)^{1/2} \right\| < 1 .$$

These two relations together with definition (4.5) show that for each f in $\mathfrak{H} \cap \mathfrak{G}$,

$$\|M(p_0) R_0(\mu) f\|_{\mathfrak{G}} < \|M(\tilde{p})^{1/2} R_0(\mu) M(\tilde{p})^{1/2}\|_{\mathfrak{H}} \cdot \|f\|_{\mathfrak{G}} . \tag{4.9}$$

At the same time it follows that

$$\|M(p_0) R_0(\mu) - M(p_0) R_0(\omega)\|_{\mathfrak{G}} \leqslant \|M(\tilde{p})^{1/2} \big(R_0(\mu) - R_0(\omega)\big) M(\tilde{p})^{1/2}\|_{\mathfrak{H}} .$$

Hence the validity of Condition $G_2(\mathfrak{J})$ is implied by the existence of the boundary values, with reference the $\mathfrak{H}$-operator norm, of the family of operators in (4.8). This existence was established elsewhere [A.43.b], for every compact interval $\mathfrak{J}$ of the open positive axis. This, in turn, establishes the validity of Condition $G_2(\mathfrak{J})$ for such intervals.

c) Condition $G_3(\mathfrak{J})$. To verify this condition for each positive k define the truncated potential by

$$p_k(x) = \begin{cases} p(x) & \dfrac{1}{k} < |x| < k \\ 0 & \text{otherwise} \end{cases} .$$

According to an argument used elsewhere [B.8] the operator $M(p_k) R_0(\mu)$ is defined on all of $\mathfrak{H}$ and it is compact, in particular bounded. Clearly the truncated potentials

satisfy conditions $G_{1,2}(\mathfrak{J})$. Definitions (2.2) and (4.5) show that

$$\|M(p_0)\, R_0(\mu) - M(p_{k,0})\, R_0(\mu)\|_{\mathfrak{G}} \leqslant \|M\big((\tilde{p} - \tilde{p}_k)\,(1/p)^{1/2}\big)\, R_0(\mu)\, M(\tilde{p})^{1/2}\|_{\mathfrak{H}}\,.$$

According to an estimate formulated elsewhere [A.43.c] the right-member tends to zero as k tends to infinity. This establishes the validity of Condition $G_3(\mathfrak{J})$. At the same time it follows that the operator $\big(M(p_0)\, R_0(\mu)\big)_{\mathfrak{G}}$ is compact. We shall make essential use of this fact in Section 5.

d) Condition $A_1(\mathfrak{J})$. To verify this condition we need an additional smallness assumption. Specifically set

$$\delta_0 = \frac{1}{\sup\limits_{\mu \in \mathcal{R}_\pm(\mathfrak{J})} \|M(\tilde{p})^{1/2}\, R_0(\mu)\, M(\tilde{p})^{1/2}\|_{\mathfrak{H}}}\,.$$

According to the already established Condition $G_2(\mathfrak{J})$ the supremum in the denominator is bounded from above and hence δ_0 is bounded from below. Clearly, for each real number δ and complex number μ in $\mathcal{R}_\pm(\mathfrak{J})$, we have

$$|\delta| < \delta_0 \quad \text{implies} \quad \|M(\delta\tilde{p})^{1/2}\, R_0(\mu)\, M(\delta\tilde{p})^{1/2}\|_{\mathfrak{H}} \leqslant \frac{1}{2}\,.$$

Hence, remembering relation (4.9), we obtain

$$\|M(\delta p_0)\, R_0(\mu)\|_{\mathfrak{G}} \leqslant \frac{1}{2}\,.$$

According to the already-established Condition $G_2(\mathfrak{J})$, this family of operators can be continuously extended in the μ-variable onto the closures $\overline{\mathcal{R}_\pm(\mathfrak{J})}$. Hence, for each ω in these closures, the operators $\big(I - M(\delta p_0)\, R_0(\omega)\big)_{\mathfrak{G}}$ are invertible. That is to say Condition $A_1(\mathfrak{J})$ holds.

e) Condition $A_2(\mathfrak{J})$. To verify this condition all that we do is to refer to an argument used elsewhere [A.23.c]. This shows that under general circumstances Condition $G_3(\mathfrak{J})$ and Condition $A_1(\mathfrak{J})$ together with the compactness of the operator $M(p_0)\, R_0(\mu)$ imply Condition $A_2(\mathfrak{J})$.

Having established these conditions we can easily establish Theorem 2.1. In fact, inserting the validity of these conditions in the abstract Theorem 3.1 we arrive at the validity of Theorem 2.1.

5. The Proof of Theorem 2.2

We derive this theorem from the abstract Theorem 3.1. According to Section 4 for each compact interval $\mathfrak{J}$ of the open positive axis Condition I alone implies Conditions $G_{1,2,3}(\mathfrak{J})$. Since according to Section 2 Condition II is more stringent than Condition I, it also implies Conditions $G_{1,2,3}(\mathfrak{J})$.

a) Condition $A_1(\mathfrak{J})$. To verify this condition we show that for each ω in $\mathfrak{J}$ the operators $\big(1 - M(p_0)\, R_0^{\pm}(\omega)\big)_{\mathfrak{G}}$ are invertible. According to Subsection 4.c the second terms are compact and hence according to the Fredholm alternative [B.15.a] it

suffices to show that each of these two operators is one to one. This is the statement of the theorem that follows.

Theorem 5.1. *Suppose that the potential p satisfies Condition II. Suppose further that ω is an exceptional point of the open positive axis and h is an exceptional vector in $\mathfrak{G}$, that is to say*

$$\big(1 + M(p_0)\, R_0^+(\omega)\big)_{\mathfrak{G}}\, h = 0 \quad \textit{or} \quad \big(1 + M(p_0)\, R_0^-(\omega)\big)_{\mathfrak{G}}\, h = 0\,. \tag{5.1}$$

Then

$$h = 0\,. \tag{5.2}$$

To establish this theorem we first note that the exceptional sesquilinear form $[R_0^{\pm}(\omega)]_{\mathfrak{G}}$ and the exceptional vector h define a linear functional on $\mathfrak{G}$. Namely, the functional which assigns to the given vector g in $\mathfrak{G}$ the complex number $[R_0^{\pm}(\omega)]_{\mathfrak{G}}(g, h)$. In view of our choice of $\mathfrak{G}$ this functional corresponds to an $\mathfrak{L}_2(\mathfrak{S}_{d-1})$ valued function and symbolically we set

$$R_0^{\pm}(\omega)\, h(\xi) = \int_0^{\infty} R_0^{\pm}(\omega)\, (\xi, \eta)\, h(\eta)\, d\eta\,. \tag{5.3}$$

One of the ingredients of the proof of Theorem 5.1 is an asymptotic description of the norm of this $\mathfrak{L}_2(\mathfrak{S}_{d-1})$-valued function. For brevity set

$$\mathfrak{A} = \mathfrak{L}_2(\mathfrak{S}_{d-1})\,.$$

This asymptotic description makes essential use of the fact that the operator A_0 admits a family of reducing subspaces and on each of them it acts like an ordinary differential operator. To describe this in more specific terms we need the notion of the Laplace-Beltrami operator acting in $\mathfrak{L}_2(\mathfrak{S}_{d-1})$. It was emphasized by Kato [B.3], that this is the operator B_0 determined by the requirement that for every smooth function f in $\mathfrak{H}$ for which $f(0) = 0$

$$A_0\, f(\xi) = f''(\xi) - \frac{1}{\xi^2}\left(B_0 + \frac{(d-1)\,(d-3)}{4}\right) f(\xi)\,. \tag{5.4}$$

As is well known [B.12], B_0 is self-adjoint, its spectrum is discrete and it is given by

$$\lambda(l) = \left\{\begin{array}{ll} 0 & d = 1\,, \\ l\,(l + d - 2) & d \geqslant 2\,, \quad l = 0, 1, 2, \ldots\,. \end{array}\right\} \tag{5.5}$$

Let $\mathfrak{E}(l)$ denote the eigen-space of B_0 with eigen-value $\lambda(l)$, and let $0(l)$ denote the ortho-projector on $\mathfrak{E}(l)$. That is set

$$0(l)\, \mathfrak{L}_2(\mathfrak{S}_{d-1}) = \mathfrak{E}(l)\,, \quad 0(l) = 0^*(l)\,, \quad 0(l)^2 = I\,.$$

Following Dixmier [B.14] we set

$$\mathfrak{L}_2\big((0, \infty), \mathfrak{E}(l)\big) = \overline{\mathfrak{L}_2(0, \infty) \otimes \mathfrak{E}(l)}\,, \tag{5.6}$$

and denote by $I \otimes 0(l)$ the ortho-projector onto this subspace. An elementary argument shows that this subspace reduces the operator A_0. Let $L(l, d)$ be the $\mathfrak{L}_2(0, \infty)$-closure of the operator defined by

$$L(l, d)\, \varphi(\xi) = \varphi''(\xi) - \frac{1}{\xi^2}\left(\lambda(l) + \frac{(d-1)\,(d-3)}{4}\right)\varphi(\xi)\,, \tag{5.7}$$

for those complex valued smooth functions which satisfy the boundary condition $\varphi(0) = 0$, we have bounded support and for which the right-member of (5.7) is in $\mathfrak{L}_2(0, \infty)$. Then the resolvent of this ordinary differential operator is related to the restriction of the resolvent of A_0 to the reducing subspace (5.6) by

$$I \otimes 0(l) \cdot R_0(\mu) = (\mu - L(l, d))^{-1} \otimes 0(l)\,. \tag{5.8}$$

After these preparations we formulate a lemma, which is the first step in the asymptotic description of the norm of the $\mathfrak{A}$-valued function (5.3).

Lemma 5.1. *For the exceptional value ω and exceptional vector h of Theorem* 5.1 *we have*

$$|\, R_0^{\pm}(\omega)\, h(\xi)\,|_{\mathfrak{A}} = 0(\xi^{1/6}) \quad \textit{at} \quad \xi = \infty\,. \tag{5.9}$$

Let $y(l, m)$ denote the spherical harmonics in the usual notation [B.12] and set

$$h = \sum_{l,m} h(l, m) \otimes y(l, m)\,. \tag{5.10}$$

As is well known [B.12] for each l the spherical harmonics $y(l, -l), \ldots, y(l, +l)$ span the eigenspace of the Laplace-Beltrami operator with eigenvalue $\lambda(l)$. In other words

$$\{y(l, -l), \ldots, y(l, -l)\} = 0(l)\ \mathfrak{L}_2(\mathbf{S}_{d-1})\,.$$

This fact together with relation (5.8) and (5.10) shows that

$$R_0^{\pm}(\omega)\, h(\xi) = \sum_{l,m} (\omega - L(l, d))^{-1}\, h(l, m)\,(\xi) \otimes y(l, m)\,. \tag{5.11}$$

Since the $\{y(l, m)\}$ form a complete ortho-normal set in $\mathfrak{A}$ this yields

$$|\, R_0^{\pm}(\omega)\, h(\xi)\,|_{\mathfrak{A}}^2 = \sum_{l,m} |\,(\omega - L(l, d))^{-1}\, h(l, m)\,(\xi)\,|^2\,. \tag{5.12}$$

To estimate this sum for each value of ξ we break it up into two parts, as follows. First, for each integer d, define a sequence by setting

$$\nu(l, d) = \left(l\,(l + d - 2) + \frac{(d-2)^2}{4}\right)^{1/2}, \qquad l = 0, 1, 2, \ldots\,. \tag{5.13}$$

Second, for each positive ν and for the fixed ω, we define three intervals by setting

$$\mathfrak{J}_1(\nu) = \left[0, \frac{1}{2}\,\frac{\nu}{\sqrt{\omega}}\right], \tag{5.14$_1$}$$

$$\mathfrak{J}_2(\nu) = \left[\frac{1}{2}\frac{\nu}{\sqrt{\omega}}, \frac{3}{2}\frac{\nu}{\sqrt{\omega}}\right], \tag{5.14$_2$}$$

$$\mathfrak{J}_3(\nu) = \left[\frac{3}{2}\frac{\nu}{\sqrt{\omega}}, \infty\right]. \tag{5.14$_3$}$$

Third for each positive ξ we define $\mathfrak{J}^1(\xi)$ to be the set of those integers l for which the function $\nu(l, d)$ of definition (5.13) has the property that

$$\xi \in \mathfrak{J}_1(\nu(l, d)) .$$

The set of integers $\mathfrak{J}^{2,3}(\xi)$ is defined similarly. Finally we define

$$s_1^2(\xi) = \sum_{l \in \mathfrak{J}^1(\xi)} |(\omega - L(l, d))^{-1} h(l, m)(\xi)|^2 . \tag{5.15$_1$}$$

and

$$s_2^2(\xi) = \sum_{l \in \mathfrak{J}^2(\xi) \cup \mathfrak{J}^3(\xi)} |(\omega - L(l, d))^{-1} h(l, m)(\xi)|^2 . \tag{5.15$_2$}$$

Here and in the future the summation over m runs between $-l$ and $+l$ and for brevity we do not indicate these limits explicitely.

It is clear from definition (5.12) and $(5.15)_{1,2}$ that

$$|R_0^{\pm}(\omega) h(\xi)|_{\mathfrak{A}}^2 = s_1^2(\xi) + s_2^2(\xi) . \tag{5.16}$$

To estimate the first sum, $s_1^2(\xi)$, recall that by definition of the kernel

$$|(\omega - L(l, d))^{-1} h(l, m)(\xi)|^2 = |\int (\omega - L(l, d))^{-1}(\xi, \eta) h(l, m)(\eta) d\eta|^2 . \tag{5.17}$$

According to definition (4.5) to the function h in $\mathfrak{G}$ there is a function $\tilde{h}$ such that

$$h(\eta) = (\tilde{p}(\eta))^{1/2} \tilde{h}(\eta) , \quad \tilde{h} \in \overline{\mathfrak{L}_2(0, \infty) \otimes \mathfrak{A}} . \tag{5.18}$$

Since $\tilde{p}$ is spherically symmetric this yields

$$h(l, m)(\eta) = (\tilde{p}(\eta))^{1/2} \cdot \tilde{h}(l, m)(\eta) , \quad \tilde{h}(l, m) \in \mathfrak{L}_2(0, \infty) . \tag{5.18$_{lm}$}$$

Inserting this relation in equation (5.17) and applying the Schwarz inequality we obtain

$$|(\omega - L(l, m))^{-1} h(l, m)(\xi)|^2 \leqslant \int |(\omega - L(l, d))^{-1}(\xi, \eta)|^2 \tilde{p}(\eta) d\eta \cdot \|\tilde{h}(l, m)\|^2 . \tag{5.19}$$

To estimate the right member we need an estimate formulated elsewhere [B.19]. To describe this estimate let the function φ_ν be a solution of the differential equation

$$(\varphi_\nu'(z))^2 \varphi_\nu(z) = \nu^2 \left(1 - \frac{1}{z^2}\right), \quad \varphi_\nu(1) = 0 . \tag{5.20$_{\nu^2}$}$$

The fact that we incorporated ν in the equation is a technicality. Aside from this technicality, it is the basis of the asymptotic theories of Erdelyi [B.5] and Olver [B.7] and it goes back to Langer [B.1]. Let $C(\varphi_\nu)$ be the adjusted composition operator corresponding to this function. Specifically, for a given complex valued function $f(z)$ of the complex variable z set

$$C(\varphi_\nu)\, f(z) = \left(\frac{1}{\varphi_\nu'(z)}\right)^{1/2} f(\varphi_\nu(z))\,. \tag{5.21}$$

We shall apply this operator to the function n defined by

$$n(z) = \min\left(1,\ \left|\frac{1}{z}\right|^{1/4}\right). \tag{5.22}$$

With the aid of these notations the previously mentioned estimate [B.19.a] can be formulated as follows; there is a constant 0(1) such that for every μ and l and (ξ, η) in $(0, \infty) \times (0, \infty)$ we have

$$(\mu - L(l, d))^{-1}(\xi, \eta) = 0(1)\left|\frac{1}{\sqrt{-\mu}}\right|(\nu + 1)$$
$$\times \left|C(\varphi_\nu)\, n\left(\frac{\sqrt{-\mu}}{-i\nu}\,\eta\right) C(\varphi_\nu)\, n\left(\frac{\sqrt{-\mu}}{-i\nu}\,\eta\right)\right|.$$

Since the supremum of the first factor for μ in $\mathcal{R}_\pm(\mathfrak{J})$ is finite this yields

$$(\mu - L(l, d))^{-1}(\xi, \eta) = 0(1)\,(\nu + 1)\left|C(\varphi_\nu)\, n\left(\frac{\sqrt{-\mu}}{-i\nu}\,\xi\right) C(\varphi_\nu)\, n\left(\frac{\sqrt{-\mu}}{-i\nu}\,\eta\right)\right|. \tag{5.23}$$

This, in turn, inserted in (5.19) yields

$$\left.\begin{aligned}\int |(\omega - L(l, d))^{-1}(\xi, \eta)|^2\, \tilde{p}(\eta)\, d\eta = 0(1)\cdot(\nu + 1)^2 \cdot \left|C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2 \\ \times \int \left|C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\eta\right)\right|^2 \tilde{p}(\eta)\, d\eta\,.\end{aligned}\right\} \tag{5.24}$$

By definition the sum $s_1^2(\xi)$ is extended over those values of l for which ξ is in the interval $\mathfrak{J}_1(\nu)$, where $\nu = \nu(l, d)$ is defined (5.13). It is clear from definitions $(5.20)_{\nu^2}$ and (5.22) that

$$|C(\varphi_\nu)\, n(z)| = \left|\frac{1}{\nu^2}\,\frac{z^2}{z^2 - 1}\right|^{1/4} \min(|\varphi_\nu(z)|^{1/4}, 1)\,. \tag{5.25}$$

Remembering definition $(5.14)_1$ this shows that for ξ in the interval $\mathfrak{J}_1(\nu)$,

$$\left|C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2 = 0\left(\frac{1}{\nu + 1}\right). \tag{5.26}$$

According to an estimate formulated elsewhere [A.43.c]

$$\int_0^\infty \left| C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\eta\right)\right|^2 \tilde{p}(\eta)\, d\eta = 0\left(\frac{1}{\nu+1}\right). \tag{5.27}$$

Inserting these two estimates in estimate (5.24), for such ξ we obtain

$$\int_0^\infty |(\omega - L(l, d))^{-1}(\xi, \eta)|^2\, \tilde{p}(\eta)\, d\eta = 0(1)\,.$$

Inserting this estimate, in turn, in estimate (5.19) we obtain

$$s_1^2(\xi) = 0(1) \sum_{l \in \mathfrak{J}^1(\zeta)} \|\tilde{h}(l, m)\|^2\,, \tag{5.28}$$

if we remember definition $(5.15)_1$. According to definitions (5.18) and $(5.18)_{lm}$ the sum extended over all (l, m) is finite. In fact,

$$\sum_{l,m} \|\tilde{h}(l, m)\|^2 = \|\tilde{h}\|^2 < \infty\,.$$

Inserting this fact in estimate (5.28) we arrive at

$$\sup_\xi s_1^2(\xi) < \infty\,. \tag{5.29}$$

Incidentally note that in the proof of this estimate all that we needed was that the potential p satisfy Condition I.

To estimate the second sum, $s_2^2(\xi)$, we shall make essential use of the fact that p satisfies the more restrictive Condition II. Let U_0 by a spectral transformation of the unperturbed operator A_0. More specifically it is a unitary transformation which carries A_0 into the multiplication operator on $\mathfrak{L}_2((0, \infty), \mathfrak{A})$. Then according to the basic Lemma 3.1 of [A.16], assumption (5.1) implies that

$$U_0\, h(\omega) = 0\,. \tag{5.30}$$

Let $U_0(l, d)$ denote a spectral-transformation of the operator $L(l, d)$. Then relations (5.8) and (5.30) together with definition (5.4) show that

$$U_0(l, d)\, h(l, m)\, (\omega) = 0\,. \tag{$(5.30)_{lm}$}$$

Since $L(l, d)$ is an ordinary differential operator we can construct a spectral transformation for it according to the considerations of Titchmarsh [A.31]. Specifically, we can obtain its kernel from the boundary values of the kernel of the resolvent. For brevity we omit the details and just give the result. Namely,

$$U_0(l, d)\, (\xi, \eta) = \beta(\nu)\, f_{\nu,0}\left(\frac{\sqrt{-\xi}}{-i\nu}\,\eta\right), \tag{5.31}$$

where $\beta(\nu)$ is some constant depending on $\nu = \nu(l, d)$ and the function $f_{\nu,0}$ is related to the boundary value of the resolvent by the formula

$$(\omega - L(l, d))^{-1}(\xi, \eta) = \alpha(\nu, \omega) \left\{ \begin{array}{ll} f_{\nu,\infty}\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right) f_{\nu,0}\left(\frac{\sqrt{-\omega}}{-i\nu}\eta\right) & \eta < \xi, \\ f_{\nu,0}\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right) f_{\nu,\infty}\left(\frac{\sqrt{-\omega}}{-i\nu}\eta\right) & \eta < \xi, \end{array} \right\} \tag{5.32}$$

where $\alpha(\nu, \omega)$ is some constant. Insertion of formula (5.31) in relation $(5.30)_{lm}$ yields

$$\int_0^\infty f_{\nu,0}\left(\frac{\sqrt{-\omega}}{-i\nu}\eta\right) h(l, m)(\eta)\, d\eta = 0 .$$

This, in turn, inserted in formula (5.32) yields

$$\left. \begin{array}{l} (\omega - L(l, d))^{-1} h(l, m)(\xi) = \alpha(\nu, \omega) \\ \times \int_\xi^\infty \left[f_{\nu,\infty}\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right) f_{\nu,0}\left(\frac{\sqrt{-\omega}}{-i\nu}\eta\right) - f_{\nu,0}\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right) \right] h(l, m)(\eta)\, d\eta . \end{array} \right\} \tag{5.33}$$

Remembering definition (5.18) and applying the Schwarz inequality, we arrive at

$$\left. \begin{array}{l} |(\omega - L(l, d))^{-1} h(l, m)(\xi)|^2 \leqslant |\alpha(\nu, \omega)|^2 \int_\xi^\infty |\tilde{h}(l, m)(\eta)|^2\, d\eta \\ \times \int_\xi^\infty \left| f_{\nu,0}\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right) f_{\nu,\infty}\left(\frac{\sqrt{-\omega}}{-i\nu}\eta\right) - f_{\nu,0}\left(\frac{\sqrt{-\omega}}{-i\nu}\eta\right) f_{\nu,\infty}\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right) \right|^2 \tilde{p}(\eta)\, d\eta . \end{array} \right\} \tag{5.34}$$

According to definition $(5.17)_2$ the sum $s_2^2(\xi)$ is extended over those values of l for which ξ is the union of the intervals $\mathfrak{J}_2(\nu)$ and $\mathfrak{J}_3(\nu)$. In view of definitions $(5.14)_{2,3}$ this yields in

$$\xi \geqslant \frac{1}{2}\frac{\nu}{\sqrt{\omega}} \quad \text{and} \quad \eta > \frac{1}{2}\frac{\nu}{\sqrt{\omega}},$$

if we remember the limits of integration in relation (5.34). According to an estimate formulated elsewhere [B.19.b], for this range of the variables ξ and η we have

$$f_{\nu,0}\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right) = 0(1) \left| C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right) \right|, \tag{5.35$_0$}$$

and

$$f_{\nu,\infty}\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right) = 0(1) \left| C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right) \right|. \tag{5.35$_\infty$}$$

At the same time it follows that for fixed ω the constant $\alpha(\nu, \omega)$ is of the order

$$|\alpha(\nu, \omega)| = 0\,(\nu + 1) . \tag{5.36}$$

Insertion of these estimates in relation (5.34) yields

$$\left.\begin{aligned} |(\omega - L(l,d))^{-1} h(l,m)(\xi)|^2 &= 0(1)\,(\nu+1)^2 \int_\xi^\infty |h(l,m)(\eta)|^2\, d\eta \\ &\times \int_\xi^\infty \left| C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2 \left| C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\eta\right)\right|^2 \tilde{p}(\eta)\, d\eta\,. \end{aligned}\right\} \tag{5.37}$$

According to assumption (2.8) the potential $\tilde{p}$ can be written in the form,

$$\tilde{p}(\eta) = \left(\frac{1}{1+\eta}\right)^{1/6+\varepsilon} \tilde{q}(\eta)\,,$$

where $\tilde{q}$ satisfies Condition I. Hence

$$\int_\xi^\infty \left| C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\eta\right)\right|^2 \tilde{p}(\eta)\, d\eta \leqslant \left(\frac{1}{1+\xi}\right)^{1/6+\varepsilon} \int_\xi^\infty \left| C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\eta\right)\right|^2 \tilde{q}(\eta)\, d\eta\,. \tag{5.38}$$

Since $\tilde{q}$ satisfies Condition I we can replace $\tilde{p}$ by $\tilde{q}$ in estimate (5.27). This yields

$$\int_\xi^\infty \left| C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\eta\right)\right|^2 \tilde{q}(\eta)\, d\eta = 0\left(\frac{1}{\nu+1}\right). \tag{5.39}$$

Inserting estimates (5.38) and (5.39) in estimate (5.37) we obtain

$$\left.\begin{aligned} |(\omega - L(l,d))^{-1} h(l,m)(\xi)|^2 &= 0(1)\,(\nu+1)\left(\frac{1}{1+\xi}\right)^{1/6+\varepsilon} \\ &\times \left| C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2 \int_\xi^\infty |\tilde{h}(l,m)(\eta)|^2\, d\eta\,. \end{aligned}\right\} \tag{5.40}$$

It was observed elsewhere [B.19.d] that definition $(5.20)_{\nu^2}$, (5.21) and (5.22) imply for each positive ξ

$$\left| C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right| = 0(1)\left(\frac{1}{\nu+1}\right)^{1/3} \min\left(1, \left|\frac{\sqrt{-\omega}}{-i\nu}\right| \xi\right).$$

This estimate together with definitions $(5.14)_{2,3}$ yields for each $\xi\, \mathfrak{J}_2(\nu) \cup \mathfrak{J}_3(\nu)$ in

$$\nu \left| C(\varphi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2 \left(\frac{1}{1+\xi}\right)^{1/6+\varepsilon} = 0(1)\,(1+\xi)^{1/6-\varepsilon}\,. \tag{5.41}$$

Inserting relation (5.41) in estimate (5.40) we arrive at the existence of a constant 0(1) such that for every such ξ and for $l = 0, 1, \ldots,$

$$|(\omega - L(l,d))^{-1} h(l,m)(\xi)|^2 = 0(1)\,(1+\xi)^{1/6-\varepsilon} \int_\xi^\infty |\tilde{h}(l,m)(\eta)|^2\, d\eta\,. \tag{5.42}$$

The summing these estimates over those values of l for which the given fixed ξ is in the union of the intervals $\mathfrak{J}_2(\nu)$ or $\mathfrak{J}_3(\nu)$ and remembering definition $(5.15)_2$ we arrive at

$$s_2^2(\xi) = 0(1)\ (1+\xi)^{1/6-\varepsilon} \sum_{l,m} \int_\xi^\infty |\tilde{h}(l, m)\ (\eta)|^2\, d\eta\,.$$

From this, in turn, remembering definitions (5.18) and $(5.18)_{lm}$ we arrive at

$$s_2^2(\xi) = 0(1)\ (1+\xi)^{1/6-\varepsilon} \int_\xi^\infty |\tilde{h}(\eta)|_{\mathfrak{A}}^2\, d\eta\,. \tag{5.43}$$

At the same time it follows from definition (5.18) that

$$\int_0^\infty |\tilde{h}(\eta)|_{\mathfrak{A}}^2\, d\eta < \infty\,.$$

Finally inserting estimates (5.29) and (5.43) in relation (5.16) we arrive at the validity of conclusion (5.9) of Lemma 5.1.

In the proof of Theorem 5.1 instead of Lemma 5.1 we need a sharper result. This is formulated in the lemma that follows. The proof will make essential use of the estimates of Lemma 5.1 and of a induction argument that goes back to Ikebe [A.3].

Lemma 5.2. *The exceptional vector h of Theorem 5.1 is such that*

$$\lim_{\xi=\infty} |R_0^{\pm}(\omega)\ h(\xi)|_{\mathfrak{A}} = 0\,. \tag{5.44}$$

To establish this lemma first we maintain that

$$\lim_{\xi=\infty} s_1^2(\xi) = 0\,. \tag{5.45}$$

For, it is clear from inequality (5.34) that for each fixed l

$$|(\omega - L(l, d)|^{-1}\, h(l, m)\ (\xi)|^2 = 0(1) \int_\xi^\infty |\tilde{h}(l, m)\ (\eta)|^2\, d\eta\,.$$

This estimate together with estimate (5.28) yields the validity of relation (5.45) if we repeat the arguments leading to relation (5.29).

Insertion of relation (5.45) and estimate (5.43) in definition (5.16) yields

$$|R_0^{\pm}(\omega)\ h(\xi)|_{\mathfrak{A}} = 0(1)\ (1+\xi)^{1/12-\varepsilon/2} \left(\int_\xi^\infty |\tilde{h}(\eta)|_{\mathfrak{A}}^2\, d\eta \right)^{1/2} + o(1)\,. \tag{5.46}$$

In case the exponent of the second factor is negative or zero conclusion (5.44) clearly holds. Accordingly assume that

$$\frac{1}{12} - \frac{\varepsilon}{2} > 0\,. \tag{$(5.47)_0$}$$

Recall assumption (5.1) which shows that

$$|h(\xi)|_{\mathfrak{A}} \leqslant \tilde{p}(\xi)\ |R_0^{\pm}(\omega)\ h(\xi)|_{\mathfrak{A}}\,.$$

Hence remembering definition (4.5) we obtain

$$|\tilde{h}(\xi)|_{\mathfrak{A}} \leqslant (\tilde{p}(\xi))^{1/2}\ |R_0^{\pm}(\omega)\ h(\xi)|_{\mathfrak{A}}\,.$$

From this, in turn, remembering assumption (2.8) we obtain

$$|\tilde{h}(\xi)|_{\mathfrak{A}} \leqslant \left(\frac{1}{1+\xi}\right)^{1/12+\varepsilon/2} (\tilde{q}(\xi))^{1/2}\ |R_0^{\pm}(\omega)\ h(\xi)|_{\mathfrak{A}}\,, \tag{5.48}$$

where q satisfies Condition I. Inserting estimate (5.46) in inequality (5.48) we arrive at

$$|\tilde{h}(\xi)|_{\mathfrak{A}} = o(1)\left(\frac{1}{1+\xi}\right)^{2\varepsilon/2} (\tilde{q}(\xi))^{1/2}\,, \tag{5.48$_1$}$$

if we use $(5.47)_0$ and that according to definition (5.18)

$$\int_{\xi}^{\infty} |\tilde{h}(\eta)|_{\mathfrak{A}}^2\, d\eta = O(1)$$

Inserting estimate $(5.48)_1$, in turn, in estimate (5.46) we arrive at

$$|R_0^{\pm}(\omega)\ h(\xi)|_{\mathfrak{A}} = o(1)\ (1+\xi)^{1/12-(1+1/2)\varepsilon} + o(1)\,, \tag{5.46$_1$}$$

if we use that since $\tilde{q}$ satisfies Condition I

$$\int_{1}^{\infty} \tilde{q}(\xi)\, d\xi < \infty\,.$$

In case the exponent of the second factor in $(5.46)_1$ is negative conclusion (5.44) clearly holds. Accordingly assume that

$$\frac{1}{12} - \left(1+\frac{1}{2}\right)\varepsilon > 0\,. \tag{5.47$_1$}$$

This allows us to repeat the previous argument and we obtain

$$|\tilde{h}(\xi)|_{\mathfrak{A}} = o(1)\left(\frac{1}{1+\xi}\right)^{4\varepsilon/2} (\tilde{q}(\xi))^{1/2} \tag{5.48$_2$}$$

Next let k be the smallest integer for which relation $(5.47)_k$ does not hold, that is for which

$$\frac{1}{12} - \left(k+\frac{1}{2}\right)\varepsilon \leqslant 0\,.$$

Then it is clear from the previous argument that

$$| R_0^{\pm}(\omega)\, h(\xi) |_{\mathfrak{A}} = o(1)\, (1 + \xi)^{1/12 - (k+1/2)\varepsilon} + o(1)\,. \tag{5.46$_k$}$$

This implies the validity of conclusion (5.44) and completes the proof of Lemma 5.2.

Having established Lemma 5.2 we return to the proof of Theorem 5.1. According to the other basic Lemma 3.2 of [A.16] assumption (5.1) implies that the function

$$\tilde{f}(\xi) = R_0^{\pm}(\omega)\, h(\xi)$$

is a weak solution of the equation

$$(A_0 + M(p_0)\, \tilde{f} = \omega\, \tilde{f}\,.$$

Remembering that A_0 is unitarily equivalent to the Laplacian, we can invoke a version of a theorem of Kato [B.3]. This implies that $\tilde{f}(\xi) = 0$ for large enough ξ which, in turn, together with the unique continuation principle [B.10] implies that $\tilde{f}(\xi) = 0$ for every ξ. Finally remembering assumption (5.1) we arrive at the validity of conclusion (5.2). This completes the proof of Theorem 5.1. At the same time this establishes the validity of Condition $A_1(\mathfrak{J})$.

b) Condition $A_2(\mathfrak{J})$. To verify this condition all that we do is to refer to an argument used elsewhere [A.23.c]. This shows that under general circumstances Condition $G_3(\mathfrak{J})$ and Condition $A_1(\mathfrak{J})$ together with the compactness of the operator $(M(p_0)\, R_0(\mu))_{\mathfrak{G}}$ imply Condition $A_2(\mathfrak{J})$.

Having established these conditions we can easily establish Theorem 2.2. For, inserting the validity of these conditions in the abstract Theorem 3.1 we arrive at the following: over each compact interval $\mathfrak{J}$ of the positive axis which does not contain zero, the parts of the perturbed and unperturbed operator are unitarily equivalent. This fact together with the countable additivity of the spectral projectors implies unitary equivalence of the parts over the entire positive axis. The arguments of Subsection 4.c show that the difference of the perturbed and unperturbed resolvents is compact. Hence the part of the perturbed operator over the positive axis equals the entire continuous part. This establishes the validity of Theorem 2.2.

APPENDIX

A Charper Version of Theorien 2.2

In this appendix we show that the conclusion of Theorem 2.2 holds under the following, more general condition.

Condition A. *To the potential p there is a positive number ε such that the potential*

$$(1 + |x|)^{\varepsilon}\, p(x)$$

satisfies Condition I.

More specifically we show that for such potentials the following theorem holds.

Theorem A. *Suppose that the potential* p *satisfies Condition A. Suppose further that it is Holder continuous with the possible exception of finitely many points. Then the continuous part of the Friedrichs extension of $-\Delta + M(p)$ is unitarily equivalent to $-\Delta$.*

We derive this theorem from the abstract Theorem 3.1. Clearly Condition A implies Condition I. Hence remembering the proof of Theorem 2.2 we see that Conditions $G_{1,2,3}(\mathfrak{J})$ hold for the present perturbation problem. At the same time we see that the validity of Conditions $A_{1,2}(\mathfrak{J})$ is implied by the validity of Theorem 5.1 for the class of potentials satisfying Condition A.

We establish Theorem 5.1 for this class of potentials with the aid of the two lemmas that follow. They are similar to Lemmas 5.1 and 5.2. The argument used to establish these lemmas was called a boot-strap argument by Agmon. In fact the norm for the present boot-strap argument was suggested by him during an informal conversation at Oberwolfach.

Lemma A-1. *Suppose that the potential* p *satisfies Condition A and define the positive number* ε *by this condition. Let the positive number* ω *and the vector* h *in* $\mathfrak{G}$ *be the exceptional value and exceptional vector of Theorem* 5.1. *Then for each positive integer* n *we have*

$$\int_0^\infty |R_0^{\pm}(\omega)\, h(\xi)\,|_{\mathfrak{A}}^2\, \tilde{p}(\xi)\,(1+\xi)^{n\varepsilon}\, d\xi < \infty\,. \tag{A-1$_n$}$$

First we establish this conclusion for the value $n = 1$. In other words we claim that

$$\int_0^\infty |\, R_0^{\pm}(\omega)\, h(\xi)\,|_{\mathfrak{A}}^2\, \tilde{p}(\xi)\,(1+\xi)\, d\xi < \infty\,. \tag{A-1$_1$}$$

To verify this recall estimate (5.24) which says that

$$\int_0^\infty |\,(\omega - L(l,d))^{-1}\,(\xi,\eta)\,|^2\, \tilde{p}(\eta)\, d\eta = 0(1)\cdot(\nu+1)^2\cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2$$
$$\cdot \int_0^\infty \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\eta\right)\right|^2 \tilde{p}(\eta)\, d\eta\,.$$

By assumption the potential $(1+|x|)^\varepsilon p(x)$ satisfies Condition I. This assumption allows us to apply to this potential the basic lemma of [A.43]. Its conclusion says that there is a constant 0(1) such that for every ν

$$\nu = \nu(l,d)\,, \quad l = 0, 1, \ldots,$$

we have

$$(\nu+1)\int_0^\infty \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2 \tilde{p}(\xi)\,(1+\xi)^\varepsilon\, d\xi = 0(1)\,. \tag{A-2}$$

It follows, in particular, that

$$(\nu+1)\int_0^\infty \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2 \tilde{p}(\xi)\, d\xi = 0(1)\,.$$

Inserting this estimate in (5.24), which was recalled before (A-2), we obtain

$$\int_0^\infty |(\omega - L(l, d))^{-1}(\xi, \eta)|^2 \tilde{p}(\eta)\, d\eta = 0(1) \cdot (\nu + 1) \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right)\right|^2. \quad \text{(A-3)}$$

Inserting estimate (A-3), in turn, in inequality (5.19) we obtain

$$\left.\begin{aligned} |(\omega - L(l, d))^{-1} h(l, m)(\xi)|^2 = 0(1) \cdot (\nu + 1) \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right)\right|^2 \\ \cdot \|\tilde{h}(l, m)\|^2 . \end{aligned}\right\} \quad \text{(A-4)}$$

Finally inserting this estimate, in definition (5.12) we arrive at

$$|R_0^{\pm}(\omega)\, h(\xi)|_{\mathfrak{A}}^2 = 0(1) \sum_{l, m} (\nu + 1) \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right)\right|^2 \cdot \|\tilde{h}(l, m)\|^2 . \quad \text{(A-5)}$$

Recall relation (5.25) which shows that for each ξ the family of functions

$$(\nu + 1) \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right)\right|^2$$

is bounded. Note that we do not need and it is not true that this bound is uniform in ν. All that we need is that for almost every ξ the family of positive functions

$$f_n(\xi) = \sum_{l=0, m}^{l=n} (\nu + 1) \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right)\right|^2 \cdot \tilde{p}(\xi) \cdot (1 + \xi)^\varepsilon$$

does converge. Hence application of Fatou's lemma [B.15] yields

$$\left.\begin{aligned} &\int_0^\infty \sum_{l=0, m}^{\infty} (\nu + 1) \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right)\right|^2 \cdot \tilde{p}(\xi)\,(1 + \xi)\, \|\tilde{h}(l, m)\|^2\, d\xi < \\ &< \sum_{l=0, m}^{\infty} \int_0^\infty (\nu + 1) \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right)\right|^2 \cdot \tilde{p}(\xi)\,(1 + \xi)^\varepsilon\, d\xi \cdot \|\tilde{h}(l, m)\|^2 . \end{aligned}\right\} \quad \text{(A-6)}$$

Inserting conclusion (A-2) in this estimate we obtain

$$\left.\begin{aligned} &\int_0^\infty \sum_{l=0, m}^{l=\infty} (\nu + 1) \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right)\right|^2 \cdot \tilde{p}(\xi)\,(1 + \xi)^\varepsilon \cdot \|\tilde{h}(l, m)\|^2\, d\xi \\ &\qquad = 0(1) \sum_{l=0, m}^{l=\infty} \|\tilde{h}(l, m)\|^2 . \end{aligned}\right\} \quad \text{(A-7)}$$

From this estimate, in turn, we obtain

$$\int_0^\infty |R_0^{\pm}(\omega)\, h(\xi)|_{\mathfrak{A}}^2\, \tilde{p}(\xi)\,(1 + \xi)^\varepsilon\, d\xi = 0(1)\, \|\tilde{h}\|^2 , \quad \text{(A-8)}$$

if we multiply estimate (A-5) by $\tilde{p}(\xi)\,(1+\xi)^{\varepsilon}$ and integrate over ξ and recall relations $(5.18)_{l,m}$. Since according to (5.18) the right-member is finite we arrive at the validity of conclusion $(A.1)_1$.

Next we establish conclusion $(A\text{-}1)_2$. First we maintain that the function s_1^2 of definition $(5.15)_1$ is such that

$$\int_0^\infty s_1^2(\xi)\cdot\tilde{p}(\xi)\,(1+\xi)^{2\varepsilon}\,d\xi<\infty\,. \tag{A-9}$$

To establish this estimate set

$$t_{1,\nu}^2(\xi)=\int_0^{\xi/2}|\,(\omega)-L(l,d))^{-1}\,(\xi,\eta)\,|^2\,\tilde{p}(\eta)\,d\eta \tag{A-10$_1$}$$

and

$$t_{2,\nu}^2(\xi)=\int_{\xi/2}^{\infty}|\,(\omega-L(l,d))^{-1}\,(\xi,\eta)\,|^2\,\tilde{p}(\eta)\,d\eta\,. \tag{A-10$_2$}$$

Then clearly

$$t_{1,\nu}^2(\xi)+t_{2,\nu}^2(\xi)\int_0^\infty|\,(\omega-L(l,d))^{-1}\,(\xi,\eta)\,|^2\,\tilde{p}(\eta)\,d\eta$$

and in analogy to estimate (A-3) we have

$$\left.\begin{aligned}|\,(\omega-L(l,d))^{-1}\,h(l,m)\,(\xi)\,|^2 &\leqslant 2\,t_{1,\nu}^2(\xi)\cdot\|\,\tilde{h}(l,m)\,\|^2\\ &\quad+2\,t_{2,\nu}^2(\xi)\int_{\xi/2}^{\infty}|\,\tilde{h}(l,m)\,(\eta)\,|^2\,d\eta.\end{aligned}\right\} \tag{A-11}$$

To estimate $t_{1,\nu}^2(\xi)$ we need an estimate which is implicit in [B.19]. It says that to each bounded subset of $\mathcal{E}_1\times\mathcal{E}_1$ there is a constant $0(1)$ such that for every (ξ/ν) and (η/ν) in this subset,

$$\left.\begin{aligned}|\,(\omega-L(l,d))^{-1}\,(\xi,\eta)\,| &= (0(1)\cdot(\nu+1)\cdot\min\left(\left(\frac{\xi}{\nu}\right)^{\nu},\left(\frac{\eta}{\xi}\right)^{\nu}\right)\\ &\quad\cdot\left|\,C(\phi_\nu)\,n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\,C(\phi_\nu)\,n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\eta\right)\right|.\end{aligned}\right\} \tag{A-12}$$

Next we choose the value l and the corresponding value ν in terms of ξ. To do this recall definition $(5.14)_1$ which shows that

$$l\,\varepsilon\,\mathcal{J}^1(\xi)\quad\textit{implies}\quad\left|\frac{\xi}{\nu}\right|<\frac{1}{2}\,\frac{1}{\sqrt{\omega}}\,. \tag{A-13}$$

Hence for such values of l the kernel in definition $(\text{A-10})_1$ can be estimated by the right-member of (A-12). This yields

$$t_{1,\nu}^2(\xi) = 0(1) \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2 \cdot (\nu+1)^2$$

$$\times \int_0^{\xi/2} \left(\frac{\eta}{\xi}\right)^{2\nu} \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\eta\right)\right|^2 \tilde{p}(\eta)\, d\eta\,.$$

This in turn, yields

$$t_{1,\nu}^2(\xi) = 0(1) \cdot (\nu+1) \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2 \cdot \left(\frac{1}{2}\right)^{2\nu}, \tag{A-14}$$

if we replace the first factor in the integral by its maximum and estimate the resulting integral by conclusion (A-2). At the same time remembering (A-13) we see that for such values of l the corresponding value of ν is such that

$$\sqrt{\omega}\,\xi < 2\nu\,.$$

Hence for each positive integer k we have

$$\left(\frac{1}{2}\right)^{\nu} = 0\left(\frac{1}{\xi+1}\right)^{k}.$$

Inserting this estimate in (A-14) we obtain

$$t_{1,\nu}^2(\xi) = 0(1) \cdot (\nu+1) \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2 \cdot \left(\frac{1}{1+\xi}\right)^{k}. \tag{A-15$_1$}$$

To estimate $t_{2,\nu}^2(\xi)$ recall the key estimate (5.23). Inserting it in definition $(\text{A-10})_2$ we obtain

$$t_{2,\nu}^2(\xi) = 0(1) \cdot (\nu+1)^2 \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2$$

$$\times \int_0^{\xi/2} \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\eta\right)\right|^2 \tilde{p}(\eta)\, d\eta\,.$$

Clearly

$$\int_{\xi/2}^{\infty} \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\eta\right)\right|^2 \tilde{p}(\eta)\, d\eta \leqslant \left(\frac{2}{2+\xi}\right)^{\varepsilon}$$

$$\cdot \int_{\xi/2}^{\infty} \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\eta\right)\right|^2 \cdot \tilde{p}(\eta)\,(1+\eta)^{\varepsilon}\, d\eta\,.$$

This estimate together with conclusion (A-2) yields

$$\int_{\xi/2}^{\infty} \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\eta\right)\right|^2 \tilde{p}(\eta)\, d\eta = 0(1)\left(\frac{1}{1+\xi}\right)^{\varepsilon}\cdot\left(\frac{1}{\nu+1}\right). \tag{A-16}$$

Inserting this estimate in the previous one we obtain

$$t_{2,\nu}^2(\xi) = 0(1)\cdot(\nu+1)\cdot\left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2\cdot\left(\frac{1}{1+\xi}\right)^{\varepsilon}. \tag{A-15$_2$}$$

Incidentally note that in contrast to estimate (A-15)$_1$ the present estimate (A-15)$_2$ holds for all values of l.

Inserting these two estimates in estimate (A-11) we obtain

$$|(\omega - L(l,d))^{-1} h(l,m)(\xi)|^2 = 0(1)\cdot(\nu+1)\cdot\left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2$$
$$\cdot\left[\left(\frac{1}{1+\xi}\right)^k \|\tilde{h}(l,m)\|^2 + \int_{\xi/2}^{\infty} |\tilde{h}(l,m)(\eta)|^2\, d\eta\right].$$

Note that this estimate is a variant of estimate (A-3). This variation is due, so to speak to the variation in the range of values of l. From this estimate, in turn, we obtain

$$\left.\begin{aligned} s_1^2(\xi) = 0(1)\sum_{l\varepsilon\mathfrak{J}^1(\xi),\, m}(\nu+1)\left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2\cdot\left(\frac{1}{1+\xi}\right)^k\cdot\|\tilde{h}(l,m)\|^2 \\ + 0(1)\sum_{l\varepsilon\mathfrak{J}^1(\xi),\, m}(\nu+1)\cdot\left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-j\nu}\,\xi\right)\right|^2\cdot\int_{\xi/2}^{\infty}|\tilde{h}(l,m)(\eta)|^2\, d\eta\,, \end{aligned}\right\} \tag{A-17}$$

if we remember definition (5.15)$_2$. Recall that estimate (A-15)$_1$ holds for each positive integer k and hence so does this estimate. This shows that

$$\left(\frac{1}{1+\xi}\right)^k (1+\xi)^{2\varepsilon} = 0(1)\,.$$

This estimate together with conclusion (A-2) yields

$$\int_0^{\infty}(\nu+1)\cdot\left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right|^2\cdot\left(\frac{1}{1+\xi}\right)^k(1+\xi)^{2\varepsilon}\,\tilde{p}(\xi)\, d\xi = 0(1)\,. \tag{A-18$_1$}$$

To estimate the integral corresponding to the second term in (A-17) recall assumption (5.1) which shows that

$$|h(\eta)|_{\mathfrak{A}} \leqslant \tilde{p}(\eta)\cdot|R_0^{\pm}(\omega)\, h(\eta)|_{\mathfrak{A}}\,.$$

Hence remembering definition (4.5) we obtain

$$|\tilde{h}(\eta)|_{\mathfrak{A}}^2 \leqslant \tilde{p}(\eta)\cdot|R_0^{\pm}(\omega)\, h(\eta)|_{\mathfrak{A}}^2. \tag{A-19}$$

This inequality together with the already established conclusion $(\text{A-1})_1$ implies that setting

$$\tilde{h}_1(\eta) = (1+\eta)^{\varepsilon/2}\tilde{h}(\eta) \quad \text{and} \quad \|\tilde{h}_1 = \|^2 = \int_0^\infty (1+\eta)^\varepsilon)|\tilde{h}(\eta)|^2_{\mathfrak{A}}\, d\eta \tag{A-20$_1$}$$

we have

$$\|\tilde{h}_1\|^2 < \infty\,. \tag{A-21}$$

Defining the function $\tilde{h}_1(l, m)$ similarly yields

$$\int_{\xi/2}^\infty |\tilde{h}(l, m)(\eta)|^2\, d\eta = 0(1)\left(\frac{1}{1+\xi}\right)^\varepsilon \|\tilde{h}_1(l, m)\|^2\,. \tag{A-22}$$

This estimate together with conclusion (A-2) yields

$$\left.\begin{aligned}\int_0^\infty (\nu+1)\cdot\left|C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right)\right|^2 \cdot \int_{\xi/2}^\infty |\tilde{h}(l, m)(\eta)|^2\, d\eta \cdot \tilde{p}(\xi)\cdot(1+\xi)^{2\varepsilon}\, d\xi \\ = 0(1)\,\|\tilde{h}_1(l, m)\|^2\,.\end{aligned}\right\} \tag{A-23$_1$}$$

Estimates $(\text{A-18})_1$, $(\text{A-23})_1$ and (A-17) together imply

$$\int_0^\infty s_1^2(\xi)\,\tilde{p}(\xi)\cdot(1+\xi)^{2\varepsilon}\, d\xi = 0(1)\,\|\tilde{h}_1\|^2\,, \tag{A-24$_1$}$$

similarly to the way estimates (A-4) and (A-7) did imply estimate (A-8). At the same time we see from estimate (A-21) that the right number is finite. This establishes the validity of estimate (A-9).

Second we maintain that

$$\int_0^\infty s_2^2(\xi)\,\tilde{p}(\xi)\,(1+\xi)^{3\varepsilon}\, d\xi < \infty\,. \tag{A-25}$$

To establish this recall estimate (5.37) which says that

$$\begin{aligned}|(\omega - L(l, d))^{-1}\, h(l, m)(\xi)|^2 = 0(1)\cdot(\nu+1)^2\cdot\int_\xi^\infty |\tilde{h}(l, m)(\eta)|^2\, d\eta \\ \cdot\left|C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right)\right|^2 \cdot \int_\xi^\infty \left|C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\eta\right)\right|^2 \tilde{p}(\eta)\, d\eta\,.\end{aligned}$$

At the same time it follows that this estimate is uniform for those values of l which are not in $\mathfrak{J}^1(\xi)$, that is to say over which the sum $s_2^2(\xi)$ is extended. Insertion of

estimates (A-16) and $(\text{A-}22)_1$ in this estimate yields

$$\left.\begin{aligned} |(\omega - L(l, d))^{-1} h(l, m)(\xi)|^2 (1+\xi)^{2\varepsilon} = 0(1) \cdot (\nu + 1) \\ \cdot \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right)\right|^2 \cdot \|\tilde{h}_1(l, m)\|^2 . \end{aligned}\right\} \quad (\text{A-}26)_1$$

Note that this estimate is another variant of estimate (A-3). This variation is due, again, to the variation of the range of values of l. Summing estimates $(\text{A-}26)_1$ over those values of l which are not in $\mathfrak{J}^1(\xi)$ and remembering definition $(5.15)_2$ we obtain

$$s_2^2(\xi)(1+\xi)^{2\varepsilon} = 0(1) \sum_{l \notin \mathfrak{J}^1(\xi),\, m} (\nu + 1) \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right)\right|^2 \cdot \|\tilde{h}_1(l, m)\|^2 . \quad (\text{A-}27)_1$$

Similarly to the way conclusion (A-2) did imply estimate (A-7) we see that it also implies

$$\left.\begin{aligned} \int_0^\infty \sum_{l=0,\, m}^{l=\infty} (\nu + 1) \left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\xi\right)\right|^2 \cdot \tilde{p}(\xi)(1+\xi)^\varepsilon \cdot \|\tilde{h}_1(l, m)\|^2 d\xi \\ = 0(1) \sum_{l=0,\, m}^{l=\infty} \|\tilde{h}_1(l, m)\|^2 . \end{aligned}\right\} \quad (\text{A-}7)_1$$

Similarly to the way estimates (A-4) and (A-7) did imply estimate (A-8) we see that the present estimates $(\text{A-}27)_1$ and $(\text{A-}7)_1$ do imply

$$\int_0^\infty s_2^2(\xi) \cdot \tilde{p}(\xi)(1+\xi)^{3\varepsilon}\, d\xi = 0(1) \sum_{l=0,\, m}^{l=\infty} \|\tilde{h}_1(l, m)\|^2 . \quad (\text{A-}28)_1$$

Since according to estimate (A-21) the right member is finite we arrive at the validity of estimate (A-25). Then combining estimates (A-9) and (A-25) we arrive at the validity of conclusion $(\text{A-}1)_2$.

To complete the proof of conclusion $(\text{A-}1)_n$ in the general case we proceed by induction. Accordingly assume that it holds for n and show that it also holds for $n + 1$. In analogy with definition $(\text{A-}20)_1$ set

$$\tilde{h}_n(\eta) = (1+\eta)^{n\varepsilon/2}\, \tilde{h}(\eta) \quad \text{and} \quad \|\tilde{h}_n\|^2 = \int_0^\infty |\tilde{h}_n(\eta)|_{\mathfrak{A}}^2\, d\eta . \quad (\text{A-}20)_n$$

Then conclusion $(\text{A-}1)_n$ implies

$$\|\tilde{h}_n\|^2 < \infty \quad (\text{A-}21)_n$$

similarly to the way conclusion $(\text{A-}1)_1$ did imply estimate $(\text{A-}21)_1$. Defining the function $\tilde{h}_n(l, m)$ analogously to definition $(5.18)_{lm}$ we clearly have

$$\int_{\xi/2}^\infty |\tilde{h}(l, m)(\eta)|^2\, d\eta = 0(1) \left(\frac{1}{1+\xi}\right)^{n\varepsilon} \|\tilde{h}_n(l, m)\| . \quad (\text{A-}22)_n$$

As a first consequence of this estimate and of conclusion (A-2) we obtain

$$\left.\begin{aligned}\int_{\xi/2}^{\infty}(\nu+1)\left|C(\phi_\nu)\,n\left(\frac{\sqrt{-\omega}}{-j\,\nu}\,\xi\right)\right|^2\cdot\int_{\xi/2}^{\infty}|\,\tilde{h}(l,m)\,(\eta)\,|^2\,d\eta\\ \cdot\tilde{p}(\xi)\,(1+\xi)^{(n+1)}\,d\xi=0(1)\,\|\,\tilde{h}_n(l,m)\,\|^2\,.\end{aligned}\right\}\quad(\text{A-23})_n$$

Recall that estimate $(\text{A-15})_1$ holds for each positive integer k. At the same time we see from conclusion (A-2) that for k greater than $n\varepsilon$

$$\int_0^{\infty}(\nu+1)\left|C(\phi_\nu)\,n\left(\frac{\sqrt{-\omega}}{-i\,\nu}\,\xi\right)\right|^2\cdot\tilde{p}(\xi)\left(\frac{1}{1+\xi}\right)^k(1+\xi)^{(n+1)}\,d\xi=0(1)\,.\quad(\text{A-18})_n$$

Estimates $(\text{A-18})_n$ and $(\text{A-23})_n$ together imply

$$\int_0^{\infty}s_1^2(\xi)\cdot\tilde{p}(\xi)\,(1+\xi)^{(n+1)\varepsilon}\,d\xi=0(1)\,\|\,\tilde{h}_n\,\|^2\,,\quad(\text{A-24})_n$$

similarly to the way estimates $(\text{A-18})_1$, $(\text{A-23})_1$ and (A-17) did imply estimate $(\text{A-24})_1$. As another consequence of estimate $(\text{A-22})_n$ we obtain

$$s_2^2(\xi)\,(1+\xi)^{(n=1)\varepsilon}=0(1)\sum_{l\notin\mathfrak{J}^1(\xi),\,m}(\nu+1)\left|C(\phi_\nu)\,n\left(\frac{\sqrt{-\omega}}{-i\,\nu}\,\xi\right)\right|^2\|\,\tilde{h}_n(l,m)\,\|^2\,,\quad(\text{A-27})_n$$

if we remember the way estimates (A-16) and $(\text{A-22})_1$ did imply estimate $(\text{A-27})_1$. Estimate $(\text{A-27})_n$, in turn, implies

$$\int_0^{\infty}s_2^2(\xi)\cdot\tilde{p}(\xi)\,(1+\xi)^{(n+2)\varepsilon}\,d\xi=0(1)\sum_{l=0,\,m}^{l=\infty}\|\,\tilde{h}_n(l,m)\,\|^2\,,\quad(\text{A-28})_n$$

similarly to the way estimate $(\text{A-27})_1$ did imply $(\text{A-28})_1$. Finally combining estimates $(\text{A-24})_n$ and $(\text{A-28})_n$ and remembering relation (5–16) and estimate $(\text{A-21})_n$ we arrive at the validity of conclusion $(\text{A-1})_{n+1}$. This completes the proof of Lemma A-1.

We shall use this lemma in the proof of Theorem 5.1 via the lemma that follows.

Lemma A-2. *The exceptional value* ω *and exceptional vector* h *of Theorem* 5.1 *is such that* $\mathrm{R}_0^{\pm}(\omega)\,\mathrm{h}(\xi)$, *the* $\mathfrak{L}_2(\mathcal{S}_{d-1})$*-valued function of the variable* ξ, *is square integrable. That is to say*

$$R_0^{\pm}(\omega)\,h\in\mathfrak{L}_2((0,\infty),\,\mathfrak{L}_2(\mathcal{S}_{d-1}))\,.\quad(\text{A-29})$$

To establish this lemma we need an elementary estimate observed elsewhere [B.19.d]. It says that

$$|\,C(\phi_\nu)\,n(z)\,|=0(1)\left(\frac{1}{\nu+1}\right)^{1/3}\min(1,\,|\,z^{1/2}\,|)\,,$$

which implies

$$\left| C(\phi_\nu)\, n\left(\frac{\sqrt{-\omega}}{-i\nu}\,\xi\right)\right| = 0(1)\left(\frac{1}{\nu+1}\right)^{2/3}\cdot\frac{\xi}{\nu+1}\,. \tag{A-30}$$

First inserting estimate (A-30) in estimate $(\text{A-27})_n$ we obtain for each positive integer n

$$s_2^2(\xi)\,(1+\xi)^{n\varepsilon} = 0(\xi)\,, \tag{A-31$_1$}$$

if we remember estimate $(\text{A-21})_n$. Second inserting estimate (A-30) and $(\text{A-22})_n$ in estimate (A-16) we obtain for each positive integer n,

$$s_1^2(\xi)\,(1+\xi)^{n\varepsilon} = 0(\xi)\,, \tag{A-31$_2$}$$

if we use estimate $(\text{A-21})_n$ again. Finally combining estimates $(\text{A-31})_1$ and $(\text{A-31})_2$ and remembering relation (5.16) we arrive at the validity of conclusion (A-29). This completes the proof of Lemma A-2.

Having established Lemma A-2 we can derive Theorem 5.1 from it for the present class of potentials similarly to the way we derived the original version from Lemma 5.2. This extended version of Theorem 5.1 implies Theorem A similarly to the way the original version did imply Theorem 2.2. For brevity we omit the proofs.

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