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The Inverse Problem of Potential Scattering According to the Klein-Gordon Equation

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Abstract. The inverse problem of constructing a spherically symmetric potential from its scattering data is solved for the Klein-Gordon equation, following the approach of Marchenko for the Schrödinger equation. This theory is well suited for the application to actual scattering processes. The interaction potential can be calculated uniquely from the scattering phase shift and the bound state data.

1. Introduction

The inverse problem for the Klein-Gordon equation was first considered by Corinaldesi [1]. He developed a theory following the solution of the problem for the Schrödinger equation given by Gel'fand and Levitan [2]. Essentially the same results were obtained by Verde [3] using a dispersion technique. However, both these theories are formal and unfortunately contain many diverging expressions, so that, apart from the mathematical shortcomings, it is even not quite clear how the algorithm for the actual calculation of the potential from the scattering data looks. In addition, the connection between the scattering data and the quantities entering in the Gel'fand-Levitan theory is rather complicated. Hence, this theory is not well suited for the physical purpose.

An entirely different approach to the problem was given by Degasperis [4]. In this method one determines the derivatives of the potential at the origin $x = 0$ from the scattering data. This can only work if the potential is analytic, especially at $x = 0$. On the other hand, the potentials occuring in reality are always highly singular at $x = 0$. Then neither the method of Degasperis nor the Gel'fand-Levitan theory can be used. A theory which avoids these difficulties in the case of the Schrödinger equation is the theory of Marchenko [5]. Here, the origin $x = 0$ plays no essential role, so that the potential may be singular. Furthermore, the scattering data enter quite directly into the theory, consequently the algorithm for calculating the potential is quite manageable. Indeed, the Marchenko theory of the Schrödinger equation has been successfully applied to α-α and nucleon-nucleon scattering [6].

For these reasons, it seems to be quite desirable to develop an inverse theory of the Marchenko type for relativistic potential scattering as well. In this paper, this is done for the Klein-Gordon equation.—In the next section we prove a basic integral representation for the solutions of the Klein-Gordon equation. This representation contains the so-called orthogonalising kernels which are directly related to the potential. In section 3 we derive an integral equation (the relativistic Marchenko equation) from which these kernels can be calculated. Since the quantities in this integral equation depend directly on the scattering data, this completes the solution of the inverse problem. In Appendix I, the various properties of solutions of the Klein-Gordon equation are considered, which have been used in the body of the paper. Appendix II gives a high energy expansion for the solutions of the Klein-Gordon equation. This is a very important tool in the proof and may be also of independent interest.

2. The Integral Representation of the Functions $f^\sigma(k, x)$

We consider the Klein-Gordon equation for s-waves with a static potential

$$\psi''(E, x) + k^2\,\psi(E, x) = \big(2\,E\,V(x) - V^2(x)\big)\,\psi(E, x) \tag{2.1}$$

where $E = \pm\sqrt{k^2 + 1}$. Instead of the energy E we will use the momentum k as the basic variable, which we generally take to be complex ($k = k_1 + i\,k_2$). Since E is a double-valued function of k, the complex variable k varies on a two-fold Riemann surface, cut from $+\,i$ to $+\,i\,\infty$ and $-\,i$ to $-\,i\,\infty$. We distinguish the two sheets by the sign σ of $\operatorname{Re} E$.

Let us define a solution $f^\sigma(k, x)$ of equation (2.1) by the asymptotic condition

$$\lim_{x\to\infty}\big(e^{-ikx} f^\sigma(k, x)\big) = 1\,. \tag{2.2}$$

The equation (2.1) together with equation (2.2) is equivalent to the following integral equation

$$f^\sigma(k, x) = e^{ikx} + \frac{1}{k}\int_x^\infty dy\,\sin k(y - x)\,(2\,E\,V - V^2)\,f^\sigma(k, y)\,. \tag{2.3}$$

This equation is investigated in detail in Appendix I. If the potential satisfies

$$\int_0^\infty dy\,y^i\,|\,V(y)\,| < \infty, \quad i = 1, 2$$

$$\int_0^\infty dy\,y^k\,|\,V(y)\,|^2 < \infty, \quad k = 1, 2 \tag{2.4}$$

then equation (2.3) can be solved by iteration. The Born's series converges absolutely, defining the irregular solutions for all k with $k_2 \geq 0$. Furthermore, we have the following estimate

$$| f^\sigma(k, x) | \leq \text{const}\, e^{-k_2 x}, \; k_2 \geq 0 \,. \tag{2.5}$$

$f^\sigma(k, x)$ is an analytic function of k in the cut upper half-plane and on the real axis, except at $k = 0$. But even at $k = 0$ and $k = i$, $f^\sigma(k, x)$ is continuous in k. For real k we have

$$f^\sigma(-k, x) = \overline{f^\sigma(k, x)} \,, \tag{2.6}$$

and for $k \neq 0$, $f^\sigma(k, x)$ and $f^\sigma(-k, x)$ are two linearly independent solutions of equation (2.1). If, in addition to (2.4), we require that

$$\begin{aligned} &V(x),\ V'(x),\ V''(x) \quad \text{are bounded and continuous, and} \\ &V(x),\ V'(x),\ V''(x),\ V'''(x) \quad \text{are} \quad L^1(0, \infty) \quad \text{and tend to} \quad 0 \\ &\text{for} \quad x \to \infty \,, \end{aligned} \tag{2.7}$$

then the following high energy expansion holds (see Appendix II)

$$\begin{aligned} f^\sigma(k, x) = {} & e^{ikx}\, e^{i\varepsilon \int_x^\infty dy V} \left[1 + \frac{1}{k} \frac{\varepsilon V(x)}{2} + \frac{1}{k^2} \left(\frac{i\,\varepsilon}{2} \int_x^\infty dy\, V + \frac{i\,\varepsilon\, V'(x)}{4} + \frac{3}{8} V^2(x) \right) \right] \\ & + e^{ikx}\, 0 \left(\frac{1}{k^3} \right) \end{aligned} \tag{2.8}$$

with

$$\varepsilon = \text{sign} \frac{\text{Re}\, E}{\text{Re}\, k} \,.$$

It is the main point of the inverse theory, that there exists an integral representation for the irregular solution $f^\sigma(k, x)$ of the following form

$$f^\sigma(k, x) = q(x)\, e^{ikx} + \int_x^\infty dt \big(K_1(x, t) + E\, K_2(x, t) \big)\, e^{ikt}, \tag{2.9}$$

where the kernels K_1 and K_2 are independent of k and σ; the function $q(x)$ will be determined in a moment. The appearance of this factor $q(x)$ is due to the fact that for $|k| \to \infty$ the irregular solution $f^\sigma(k, x)$ does not approach e^{ikx} as in the case of the Schrödinger equation (compare equation (2.8)). Indeed, if the kernels $K_1(x, .)$, $K_2(x, .)$ are well-behaved, the integral on the right-hand side of equation (2.9) must vanish for $|k| \to \infty$ (k real); then the factor $q(x)$ must correct for the asymptotic behaviour of $f^\sigma(k, x)$.

In order to prove equation (2.9), let us consider the following Fourier integrals

$$2\,K_1(x,t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dk\,\tilde{K}_1(x,k)\,e^{-ikt} \overset{\text{def}}{=} \frac{1}{2\,\pi}\int_{-\infty}^{+\infty} dk\big(f^+(k,x)+f^-(k,x)-2\,q(x)\,e^{ikx}\big)\,e^{-ikt} \tag{2.10}$$

$$2\,K_2(x,t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dk\,\tilde{K}_2(x,k)\,e^{-ikt} \overset{\text{def}}{=} \frac{1}{2\,\pi}\int_{-\infty}^{+\infty} dk\,\frac{1}{E}\big(f^+(k,x)-f^-(k,x)\big)\,e^{-ikt}\,. \tag{2.11}$$

Obviously $\tilde{K}_1(x,k)$ and $\tilde{K}_2(x,k)$ are uniformly bounded in k. Inserting equation (2.8) and choosing

$$q(x) = \cos\int_x^{\infty} dy\,V(y) \tag{2.12}$$

we have the following asymptotic behaviour

$$2\,\tilde{K}_1(x,k) = \frac{e^{ikx}}{k}\,i\,V(x)\sin\int_x^{\infty} dy\,V$$
$$+\frac{e^{ikx}}{k^2}\left[\frac{3}{4}V^2(x)\cos\int_x^{\infty} dy\,V-\left(\int_x^{\infty} dy\,V+\frac{V'(x)}{2}\right)\sin\int_x^{\infty} dy\,V\right]+0\left(\frac{1}{k^3}\right), \tag{2.13}$$

$$2\,\tilde{K}_2(x,k) = \frac{e^{ikx}}{k}\,2\,i\sin\int_x^{\infty} dy\,V+\frac{e^{ikx}}{k^2}\,V(x)\cos\int_x^{\infty} dy\,V+0\left(\frac{1}{k^3}\right). \tag{2.14}$$

This shows that $\tilde{K}_1(x,k)$ and $\tilde{K}_2(x,k)$ are $L^2(-\infty,+\infty)$ in k. Since even $\tilde{K}_1(x,k)\sqrt{\log(|k|+2)}$ and $\tilde{K}_2(x,k)\sqrt{\log(|k|+2)}$ are $L^2(-\infty,+\infty)$ in k, we can apply Plancherel's theorem in its stronger form [7]. Hence, equations (2.10) and (2.11) define the kernels $K_1(x,t)$ and $K_2(x,t)$ for all x and almost all t. Both $K_1(x,.)$ and $K_2(x,.)$ are $L^2(-\infty,+\infty)$, and from equation (2.6) we conclude that they are real.

Now, let us consider $\tilde{K}_1(x,k)$ and $\tilde{K}_2(x,k)$ for complex k in the ordinary (uncut) k-plane. It follows from the definition (2.10) that $\tilde{K}_1(x,k)$ is analytic except possibly at $k=i$ (the branch point of the cut k-plane). But since $\tilde{K}_1(x,k)$ is bounded and single-valued, the theorem of Riemann ensures that it is analytic also at $k=i$. Using the estimate (I.9) of Appendix I, we see that the numerator in equation (2.11) vanishes as E for $E\to 0$. Hence $\tilde{K}_2(x,k)$ is bounded, single-valued and therefore, by Riemann's theorem, it is analytic in the whole upper halfplane. From the high energy expansion (2.8) we get for fixed x

$$\int_{-\infty}^{+\infty} dk_1 \, | \tilde{K}_1(x, k_1 + i\, k_2) |^2 = 0(e^{-2k_2 x}) \,, \tag{2.15}$$

and the same estimate holds for $\tilde{K}_2(x, k)$. In this situation, Titchmarsh's theorem applies [8], giving

$$K_1(x, t) = K_2(x, t) = 0 \quad \text{for} \quad t < x \,. \tag{2.16}$$

This proves the representation (2.9).

From the Fourier integrals (2.9) and (2.10) we can easily get more information about the kernels $K_1(x, t)$ and $K_2(x, t)$. Taking some fixed $\lambda > 0$, we modify equation (2.13) as follows

$$2\, \tilde{K}_1(x, k) = \frac{e^{ikx}}{k + i\,\lambda}\, i\, V(x) \sin \int_x^\infty dy\, V + \frac{e^{ikx}}{k^2 + \lambda^2}$$
$$\times \left[\frac{3}{4} V^2(x) \cos \int_x^\infty dy\, V - \left(\lambda\, V(x) + \int_x^\infty dy\, V + \frac{V'(x)}{2}\right) \sin \int_x^\infty dy\, V\right]$$
$$+ \tilde{M}_1(x, k)\,. \tag{2.13'}$$

Here, every term is bounded for all real k and $\tilde{M}_1(x, k) = 0(1/k^3)$ for large $| k |$. Hence we get by Fourier transformation

$$K_1(x, t) = \frac{1}{2} V(x) \sin \int_x^\infty dy\, V\, e^{-\lambda(t-x)} \theta(t - x)$$
$$+ \left[\frac{3}{4} V^2(x) \cos \int_x^\infty dy\, V - \left(\lambda\, V(x) + \int_x^\infty dy\, V + \frac{V'(x)}{2}\right) \sin \int_x^\infty dy\, V\right]$$
$$\times \frac{1}{4\,\lambda} e^{-\lambda |t-x|} + M_1(x, t) \tag{2.17}$$

where $\theta(z) = \begin{cases} 0 & \text{for} \quad z < 0 \\ 1 & \text{for} \quad z > 0\,. \end{cases}$

The remainder $M_1(x, t)$ and its derivative $(\partial/\partial t)\, M_1(x, t)$ are continuous and $L^2(-\infty, +\infty)$ in t [9]; furthermore even $(\partial^2/\partial t^2)\, M_1(x, t)$ is $L^2(-\infty, +\infty)$ in t. Hence $K_1(x, t)$ and $(\partial/\partial t)\, K_1(x, t)$ are continuous for $t > x$, $(\partial/\partial t)\, K_1(x, t)$ and $(\partial^2/\partial t^2)\, K_1(x, t)$ are $L^2(t_0, \infty)$ in t, $t_0 > x$. At $t = x$ $K_1(x, t)$ must jump from 0 to

$$K_1(x, x + 0) = \frac{1}{2} V(x) \sin \int_x^\infty dy\, V \tag{2.18}$$

because of equation (2.17). A similar calculation shows that $K_2(x, t)$ as a function of t has the same properties, but at $t = x$ jumps from 0 to

$$K_2(x, x + 0) = \sin \int_x^\infty dy\, V\,. \tag{2.19}$$

From equations (2.18) and (2.19) we get the important relation

$$V(x) = 2\, \frac{K_1(x, x + 0)}{K_2(x, x + 0)} \tag{2.20}$$

from which the potential can be calculated if the kernels $K_1(x, t)$ and $K_2(x, t)$ are known.

The representation (2.9) is valid even for complex k with $k_2 \geq 0$. In fact, $\tilde{K}_1(x, k)$ is analytic in the upper halfplane and vanishes for $|k| \to \infty$. Then using Cauchy's theorem in equation (2.10) we have for $k_2 \geq 0$

$$2\, K_1(x, t) = \frac{1}{2\,\pi} \int_{-\infty}^{+\infty} dk_1 \big(f^+(k, x) + f^-(k, x) - 2\, q(x)\, e^{ikx}\big)\, e^{-ikt}$$

and similarly for $K_2(x, t)$. Recalling equation (2.15), we can apply Plancherel's theorem again which proves equation (2.9) for all k in the upper halfplane.

3. The Relativistic Marchenko Equation

Let us now consider the regular solution $\varphi^\sigma(k, x)$ of equation (2.1), defined by the following boundary condition

$$\varphi^\sigma(k, 0) = 0\,, \left.\frac{\partial \varphi^\sigma(k, x)}{\partial x}\right|_{x=0} = 1\,. \tag{3.1}$$

Obviously $\varphi^\sigma(k, x)$ is real for real energies, and we have $\varphi^\sigma(-k, x) = \varphi^\sigma(k, x)$. Under the same conditions as for $f^\sigma(k, x)$, the function $\varphi^\sigma(k, x)$ is analytic in k for fixed x in the whole cut k-plane (see Appendix I). If, in addition, the conditions (2.7) hold, then we have the following high energy expansion (see Appendix II)

$$\varphi^\sigma(k, x) = \frac{1}{k} \sin \left(k\, x - \varepsilon \int_x^\infty dy\, V\right) \left(1 + \frac{\varepsilon}{2\,k} \big(V(0) + V(x)\big)\right)$$
$$+ e^{ikx}\, 0\left(\frac{1}{k^3}\right) + e^{-ikx}\, 0\left(\frac{1}{k^3}\right). \tag{3.2}$$

Defining the Jost function $f^\sigma(k)$ by

$$f^\sigma(k) \stackrel{\text{def}}{=} f^\sigma(k, 0) , \tag{3.3}$$

we have for real k

$$2\, i\, k\, \varphi^\sigma(k, x) = f^\sigma(-k)\, f^\sigma(k, x) - f^\sigma(k)\, f^\sigma(-k, x) . \tag{3.4}$$

The S-matrix element and the phase shift $\delta^\sigma(k)$ are given by

$$S^\sigma(k) = \frac{f^\sigma(-k)}{f^\sigma(k)} = e^{2i\delta^\sigma(k)} . \tag{3.5}$$

Using equation (2.8) for $x = 0$ we get the following high energy behaviour for $S^\sigma(k)$

$$S^\sigma(k) = e^{-2i\varepsilon\int\limits_0^\infty dy V} + 0\left(\frac{1}{k^2}\right) . \tag{3.6}$$

The fact that the high energy limit is not 1, as in the case of the Schrödinger equation, but

$$S^\sigma(\infty) = e^{-2i\varepsilon\int\limits_0^\infty dy V} \tag{3.6'}$$

causes considerable modifications of the Marchenko theory.

Generally bound states also will be present.It is a rather peculiar feature of the Klein-Gordon equation that complex binding energies may also occur. This can easily be illustrated in the case of a square-well potential [10]. On the other hand, if the potential is suitably restricted, one has no complex eigenvalues. Conditions for this have been given by Veselic [11]. We always consider this situation. The bound state energies correspond to the zeros k_n^σ of $f^\sigma(k)$ in the upper halfplane. It follows by standard arguments [12] that these zeros are located in the interval $(0, i]$, where the only possible limit point is i. For simplicity, we suppose that $k = i$ is no zero. As in the case of the Schrödinger equation [13], the derivative of the Jost function with respect to k can be expressed in the following way

$$\dot{f}^\sigma(k_n^\sigma) = \frac{-k_n^\sigma}{\sqrt{k_n^{\sigma 2} + 1}} f^{\sigma\prime}(k_n, 0)\, N_n^\sigma , \quad {}' = \frac{\partial}{\partial x} ,$$

$$N_n^\sigma = 2 \int\limits_0^\infty dx \left(\sqrt{k_n^{\sigma 2} + 1} - V(x)\right) \left(\varphi^\sigma(k_n, x)\right)^2 . \tag{3.7}$$

The normalization constants N_n^σ do not vanish, consequently the zeros are simple. If the function $f^\sigma(k)$ vanishes for $k = 0$, a virtual level is present; since the wave function $\varphi^\sigma(0, x)$ is not normalizable, $k = 0$ is not a bound state. It follows [13] from equation (3.7) that the derivative $\dot{f}^\sigma(0)$ is not zero, hence $f^\sigma(k)$ goes to zero proportional to k. Furthermore, from the results of Veselic it follows that i cannot be a limit point of eigenvalues, consequently the number of bound states is actually finite.

Now we are able to prove the relativistic Marchenko equation which completes the solution of the inverse problem. From equations (3.4) and (2.9) we get for any fixed $y > x > 0$ and all real k

$$2ik\frac{\varphi^\sigma(k,x)}{f^\sigma(k)}e^{iky} = S^\sigma(k)\,q(x)\,e^{ik(x+y)} - q(x)\,e^{-ik(x-y)} + S^\sigma(k)\int_x^\infty dt\big(K_1(x,t) + E\,K_2(x,t)\big)\,e^{ik(t+y)} - \int_x^\infty dt\big(K_1(x,t)+E\,K_2(x,t)\big)\,e^{-ik(t-y)}\,. \tag{3.8}$$

From this we compute the following integral:

$$\begin{aligned}\frac{1}{4\pi}\sum_{\sigma=\pm1}\int_{-\infty}^{+\infty}dk\frac{2ik}{E}\frac{\varphi^\sigma(k,x)}{f^\sigma(k)}e^{iky} &= \frac{1}{4\pi}q(x)\sum_\sigma\int_{-\infty}^{+\infty}dk\frac{S^\sigma(k)}{E}e^{ik(x+y)}\\ &+\frac{1}{4\pi}\sum_\sigma\int_{-\infty}^{+\infty}dk\frac{S^\sigma(k)}{E}\int_x^\infty dt\,K_1(x,t)\,e^{ik(t+y)}\\ &+\frac{1}{4\pi}\sum_\sigma\int_{-\infty}^{+\infty}dk\big(S^\sigma(k)-S^\sigma(\infty)\big)\int_x^\infty dt\,K_2(x,t)\,e^{ik(t+y)}\\ &-\frac{1}{4\pi}\sum_\sigma\int_{-\infty}^{+\infty}dk\,e^{iky}\int_x^\infty dt\,K_2(x,t)\,e^{-ikt}\\ &+\frac{1}{4\pi}\sum_\sigma S^\sigma(\infty)\int_{-\infty}^{+\infty}dk\,e^{iky}\int_x^\infty dt\,K_2(x,t)\,e^{ikt}\,.\end{aligned} \tag{3.9}$$

Recalling equation (3.6) we see that both the functions $S^\sigma(k)/E$ and $S^\sigma(k) - S^\sigma(\infty)$ are $L^2(-\infty, +\infty)$ in k. Therefore the functions $F_{s1}(z)$ and $F_{s2}(z)$ defined by

$$F_{s1}(z) = -\frac{1}{4\pi}\sum_\sigma\int_{-\infty}^{+\infty}dk\frac{S^\sigma(k)}{E}e^{ikz}\,, \tag{3.10}$$

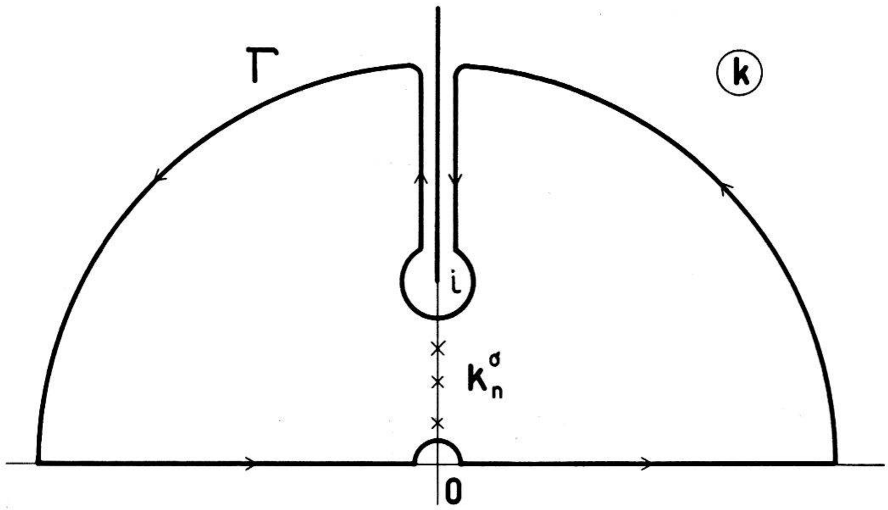
$$F_{s2}(z) = \frac{1}{4\pi}\sum_\sigma\int_{-\infty}^{+\infty}dk\big(S^\sigma(\infty)-S^\sigma(k)\big)\,e^{ikz} \tag{3.11}$$

are $L^2(-\infty, +\infty)$ in z. Using the definitions (3.10) and (3.11) in equation (3.9) and applying the convolution theorem and Plancherel's theorem we get

$$-\frac{1}{4\pi}\sum_{\sigma}\int_{-\infty}^{+\infty} dk \frac{2ik}{E}\frac{\varphi^{\sigma}(k,x)}{f^{\sigma}(k)} e^{iky} = q(x)\, F_{s1}(x+y)$$

$$+\int_{x}^{\infty} dt\, K_1(x,t)\, F_{s1}(t+y) + \int_{x}^{\infty} dt\, K_2(x,t)\, F_{s2}(t+y) + K_2(x,y)\,. \qquad (3.9')$$

Let us now evaluate the integral on the left-hand side of equation (3.9′) by means of Cauchy's theorem. The integrand is meromorphic in the cut upper halfplane; its only singularities are simple poles corresponding to the simple zeros of the Jost function $f^{\sigma}(k)$. The corresponding residues can be obtained from equation (3.7). Then, we get for the integral along the contour Γ



(see the Fig.), summed over both sheets of the cut k-plane

$$J_1 \overset{\text{def}}{=} \frac{1}{4\pi}\sum_{\sigma}\int_{\Gamma} dk \frac{2ik}{E}\frac{\varphi^{\sigma}(k,x)}{f^{\sigma}(k)} e^{iky} = \sum_{n,\sigma} e^{ik_n^{\sigma}y}\frac{1}{M_n^{\sigma}} f^{\sigma}(k_n^{\sigma}, x) \qquad (3.12)$$

where

$$M_n^{\sigma} = 2\int_{0}^{\infty} dx\big(E_n - V(x)\big)\big(f^{\sigma}(k_n^{\sigma}, x)\big)^2 = N_n^{\sigma}\big(f^{\sigma\prime}(k_n^{\sigma}, 0)\big)^2\,.$$

Inserting the integral representation (2.9) we arrive at

$$J_1 = \sum_{n,\sigma}\frac{1}{M_n^{\sigma}}\left(q(x)\, e^{ik_n^{\sigma}(x+y)} + \int_{x}^{\infty} dt\big(K_1(x,t) + E_n\, K_2(x,t)\big)\, e^{ik_n^{\sigma}(t+y)}\right). \qquad (3.12')$$

On the other hand the contributions to the integral (3.12) come only from the real axis. The integrals along the cut cancel each other. The contributions from the circles around the branch point i and the semicircles around $k = 0$ tend to zero in the limit of vanishing radius. The integrals along the large quarter circles vanish in the limit of infinitely large radius because of (3.2). Hence, defining the functions

$$F_1(z) = F_{s1}(z) + \sum_{n,\sigma} \frac{1}{M_n^\sigma} e^{ik_n^\sigma z}, \tag{3.13}$$

$$F_2(z) = F_{s2}(z) + \sum_{n,\sigma} \frac{1}{M_n^\sigma} E_n e^{ik_n^\sigma z} \tag{3.14}$$

we get from equation (3.9')

$$q(x) F_1(x+y) + K_2(x, y) + \int_x^\infty dt\, K_1(x, t) F_1(t+y) + \int_x^\infty dt\, K_2(x, t) F_2(t+y) = 0\,. \tag{3.15}$$

This is one of the two relativistic Marchenko equations we are going to derive.

To get the second equation we start again with equation (3.8), but without dividing by E. First let us consider

$$\sum_\sigma E\, S^\sigma(k) = |E| \big(S^+(k) - S^-(k)\big)\,.$$

From equation (3.6) we have the following asymptotic behaviour

$$\sum_\sigma E\, S^\sigma(k) = k\big(S^+(\infty) - S^-(\infty)\big) + 0\left(\frac{1}{k}\right),$$

hence $\sum_\sigma E\, S^\sigma(k) - k\big(S^+(\infty) - S^-(\infty)\big)$ is $L^2(-\infty, +\infty)$ in k. Using the explicit values (3.6') we write equation (3.8) in the following form

$$\begin{aligned}
\sum_\sigma 2\,i\,k \frac{\varphi^\sigma(k, x)}{f^\sigma(k)} e^{iky} &= \sum_\sigma \big(S^\sigma(k) - S^\sigma(\infty)\big) e^{ik(x+y)} q(x) \\
&+ \sum_\sigma \big(S^\sigma(k) - S^\sigma(\infty)\big) \int_x^\infty dt\, K_1(x, t)\, e^{ik(t+y)} \\
&+ \Big[\sum_\sigma E\, S^\sigma(k) - k\big(S^+(\infty) - S^-(\infty)\big)\Big] \int_x^\infty dt\, K_2(x, t)\, e^{ik(t+y)} \\
&- \sum_\sigma \int_x^\infty dt\, K_1(x, t)\, e^{-ik(t-y)} + 2\cos 2 \int_x^\infty dz\, V\, e^{ik(x+y)} \\
&\times \cos \int_x^\infty dz\, V - 2\cos \int_x^\infty dz\, V\, e^{-ik(x-y)} \\
&+ 2\cos 2 \int_0^\infty dz\, V \int_x^\infty dt\, K_1(x, t)\, e^{ik(t+y)} \\
&- 2\,i\,k \sin 2 \int_0^\infty dz\, V \int_x^\infty dt\, K_2(x, t)\, e^{ik(t+y)}\,.
\end{aligned} \tag{3.16}$$

The last integral in equation (3.16) can be integrated by parts. Using equation (2.18) we get

$$2 \sin 2 \int_0^\infty dz\, V \sin \int_x^\infty dz\, V\, e^{ik(x+y)}$$

$$+ 2 \sin 2 \int_0^\infty dz\, V \int_x^\infty dt\, \frac{\partial K_2(x, t)}{\partial t}\, e^{ik(t+y)}\,. \tag{3.16'}$$

Let us write the left-hand side of equation (3.16) as

$$\sum_\sigma 2\, i\, k\, \frac{\varphi^\sigma(k, x)}{f^\sigma(k)}\, e^{iky} \overset{\text{def}}{=} 2 \cos \left(2 \int_0^\infty dz\, V - \int_x^\infty dz\, V\right) e^{ik(x+y)}$$

$$- 2 \cos \int_x^\infty dz\, V\, e^{-ik(x-y)} + h(x, y, k)\,, \tag{3.17}$$

where

$$h(x, y, k) = e^{ik(x+y)}\, 0\left(\frac{1}{k}\right) + e^{-ik(x-y)}\, 0\left(\frac{1}{k}\right)$$

for large k because of equation (3.2). We insert equations (3.16′) and (3.17) into equation (3.16) and integrate over k. Applying the convolution theorem and Plancherel's theorem again, we arrive at

$$- \frac{1}{4\pi} \int_{-\infty}^{+\infty} dk\, h(x, y, k) = q(x)\, F_{s3}(x + y) + \int_x^\infty dt\, K_1(x, t)\, F_{s2}(t + y)$$

$$+ \int_x^\infty dt\, K_2(x, t)\, F_{s3}(t + y) + K_1(x, y)\,, \tag{3.18}$$

where

$$F_{s3}(z) = - \frac{1}{4\pi} \int_{-\infty}^{+\infty} dk \left[\sum_\sigma E\, S^\sigma(k) - k\big(S^+(\infty) - S^-(\infty)\big)\right] e^{ikz}\,. \tag{3.19}$$

As before the integral on the left-hand side of equation (3.18) can be evaluated by means of Cauchy's theorem. By exactly the same arguments as above we see that this integral is equal to the integral along the contour Γ, or equal to the contributions of the poles of $h(x, y, k)$ in the upper halfplane. Since the first two terms on the right-hand side of equation (3.17) are holomorphic, we get finally

$$\frac{1}{4\pi}\int_{-\infty}^{+\infty} dk\, h(x, y, k)$$

$$=\sum_{n,\sigma}\frac{1}{M_n}\left[q(x)\, e^{ik_n^\sigma(x+y)}\, E_n + \int_x^\infty dt\,[E_n\, K_1(x,t) + E_n^2\, K_2(x,t)]\, e^{ik_n^\sigma(t+y)}\right].$$

Thus, defining

$$F_3(z) = F_{s3}(z) + \sum_{n,\sigma}\frac{1}{M_n^\sigma}\, E_n^2\, e^{ik_n^\sigma z}, \tag{3.20}$$

we obtain the desired equation

$$q(x)\, F_2(x+y) + K_1(x,y) + \int_x^\infty dt\, K_1(x,t)\, F_2(t+y) + \int_x^\infty dt\, K_2(x,t)\, F_3(t+y) = 0\,. \tag{3.21}$$

The two equations (3.15) and (3.21) determine the kernels K_1, K_2 in terms of the Fourier transforms F_1, F_2, F_3. They correspond to the nonrelativistic Marchenko equation and will therefore be called the 'relativistic Marchenko equation'. They have been derived under the assumption $0 < x < y$. However, a discussion similar to that for the kernels K_1 and K_2 shows that the Fourier transforms $F_1(z)$, $F_2(z)$ and $F_3(z)$ are continuous for $z > 0$. Since also the kernels $K_1(x, t)$ and $K_2(x, t)$ are continuous for $t \geq x$, we find that the relativistic Marchenko equation is valid even for $y = x$, if for $K_1(x, x)$ and $K_2(x, x)$ we take the limits from above $K_1(x, x+0)$, $K_2(x, x+0)$.

The solution of the inverse problem is now given by the following procedure: One first computes the Fourier transforms $F_i(z)$, $i = 1, 2, 3$, from the scattering data by means of (3.10), (3.11), (3.19) and (3.13), (3.14), (3.20). All these quantities are well defined. For any x where the function $q(x)$ does not vanish, the relativistic Marchenko equation can be used to calculate the quantities

$$\hat{K}_1(x,y) \overset{\text{def}}{=} \frac{K_1(x,y)}{q(x)}, \quad \hat{K}_2(x,y) \overset{\text{def}}{=} \frac{K_2(x,y)}{q(x)} \tag{3.22}$$

for $y \geq x$. Then the potential $V(x)$ is determined by one of the equations

$$V(x) = 2\,\frac{\hat{K}_1(x,x)}{\hat{K}_2(x,x)} \tag{3.23}$$

or

$$V(x) = -\frac{d}{dx}\,\operatorname{arc\,tg}\hat{K}_2(x,x) \tag{3.24}$$

which follow directly from equations (2.19) and (2.20). Since $\hat{K}_2(x, x)$ tends to zero in the limit of large x, equation (3.24) will then be more convenient. The values x_0 where $q(x_0) = 0$ present no difficulty: Here we have $\int_{x_0}^\infty dy\, V = \pi[n + (1/2)]$, and because the potential is continuous, it easily can be determined.

Appendix I

Properties of Solutions of the Klein-Gordon Equation

Although the methods in this section are standard, we will, for completeness, give the proofs of the various properties of solutions of the Klein-Gordon equation, which we have used in connection with the inverse problem.

At first, let us consider the Volterra integral equation

$$f^{\sigma}(k, x) = e^{ikx} + \frac{1}{k} \int_{x}^{\infty} dx' \sin k(x' - x) \left(2 E V(x') - V^2(x')\right) f^{\sigma}(k, x') \tag{I.1}$$

with complex $k = k_1 + i k_2$. We take one fixed sign of $\mathrm{Re} E$ and omit the index σ; all considerations hold simultaneously on both sheets of the cut k-plane. We solve equation (I.1) by iteration

$$f(k, x) = \sum_{n=0}^{\infty} f_n(k, x); \tag{I.2}$$

$$f_0(k, x) = e^{ikx},$$

$$f_n(k, x) = \frac{1}{k} \int_{x}^{\infty} dx' \sin k(x' - x) \left(2 E V(x') - V^2(x')\right) f_{n-1}(k, x') . \tag{I.3}$$

Using the estimate

$$| \sin k x | \leq C e^{|k_2| x} \frac{|k| x}{1 + |k| x} \tag{I.4}$$

one finds

$$| f_n(k, x) | \leq e^{-|k_2| x} \frac{C^n}{n!} A^n(k, x) \tag{I.5}$$

with

$$A(k, x) = \int_{x}^{\infty} dx' e^{(|k_2| - k_2) x'} \frac{x'}{1 + |k| x'} \left(2 | E | | V(x') | + | V(x') |^2\right) . \tag{I.6}$$

Hence, if the potential satisfies

$$\int_{0}^{\infty} dx\, x^n | V(x) |^i < \infty \tag{I.7}$$

for $n = 1$ and $i = 1, 2$, then $A < \infty$. Consequently, the Born series (I.2) converges absolutely, defining the irregular solution $f(k, x)$ for all k in the upper halfplane $k_2 \geq 0$. Since the convergence is uniformly in k in every bounded region in the upper halfplane, $f(k, x)$ is a continuous function of k for fixed x. We note in particular the continuity at $k = 0$ and $k = i$.

From (I.5) we get the following estimate

$$| f(k, x) | \leq e^{-|k_2| x} e^{AC} \tag{I.8}$$

where

$$A \leq \int_0^\infty dx\, x\big(2 | E | | V(x) | + | V(x) |^2\big) .$$

Furthermore, let us estimate the difference of the two values of $f^\sigma(k, x)$ in the neighbourhood of $k = i$. Iterating the equation

$$f^+(k, x) - f^-(k, x) = \frac{2E}{k} \int_x^\infty dx' \sin k(x' - x)\, V(x')\, [f^+(k, x') + f^-(k, x')]$$

$$- \frac{1}{k} \int_x^\infty dx' \sin k(x' - x)\, V^2(x')\, [f^+(k, x') - f^-(k, x')]$$

n times, we see that all terms containing $f^+ + f^-$ are proportional to E, but the single term with $f^+ - f^-$ tends to zero as $1/n!$ for $n \to \infty$. Hence

$$| f^+(k, x) - f^-(k, x) | = 0(| E |) , \quad E \to 0 . \tag{I.9}$$

One can proceed in the same manner as above with the equation

$$g(k, x) = 1 + \frac{1}{2ik} \int_x^\infty dx' (e^{2ik(x'-x)} - 1)\big(2 E\, V(x') - V^2(x')\big)\, g(k, x') \tag{I.10}$$

for

$$g(k, x) = e^{-ikx} f(k, x) \tag{I.11}$$

and finds that the corresponding Born series is absolutely converging under the same conditions.

Differentiating equation (I.2) with respect to k, we obtain a series for $\big(\partial f(k, x)/\partial k\big)$. This series converges uniformly in k in every bounded region in the upper halfplane excluding $k = 0$, if the condition (I.7) is fulfilled for $n = 1, 2$ and $i = 1, 2$. Then $f(k, x)$ is an analytic function of k in the cut upper halfplane, excluding $k = 0$.

Next we turn to the integral equation

$$\varphi(k, x) = \frac{1}{k} \sin k x + \frac{1}{k} \int_0^x dx' \sin k(x - x')\big(2 E\, V(x') - V^2(x')\big)\, \varphi(k, x')$$

for the regular solution φ. Setting up an iteration as above and using the same estimate (I.4), we find that the Born series converges absolutely, provided the integral

$$A'(k, x) = \int_0^x dx' \frac{x'}{1 + |k| x'} (2 |E| |V(x')| + |V(x')|^2) \tag{I.12}$$

is finite. Hence, if the condition (I.7) is fulfilled for $n = 1$, $i = 1, 2$, then the regular solution $\varphi(k, x)$ can be defined in this way for all k. Under the same conditions, the Born series corresponding to the equation

$$\chi(k, x) = \frac{1}{2 i k} (1 - e^{-2ikx}) + \frac{1}{2 i k} \int_0^x dx' (1 - e^{-2ik(x-x')}) \times \big(2 E V(x') - V^2(x')\big) \chi(k, x') \tag{I.13}$$

for the function

$$\chi(k, x) = e^{-ikx} \varphi(k, x) \tag{I.14}$$

is absolutely converging.

Usually the regular solution $\varphi(k, x)$ is defined by means of the boundary condition

$$\varphi(k, 0) = 0 , \quad \varphi'(k, 0) = 1 .$$

Since this boundary condition is independent of k or E, and since the energy E enters analytically in the Klein-Gordon equation, we conclude by a general theorem [14], that for fixed x, $\varphi(E, x)$ is an entire analytic function of E.

Appendix II

High Energy Expansion

To derive a high energy expansion for the irregular solution $f(k, x)$ and the regular solution $\varphi(k, x)$ we start with the integral equations (I.10) and (I.13) respectively. For a moment, let us consider a Volterra integral equation of the following general form

$$g(x) = u(x) + \int_a^x dx' [H_1(x') + H_2(x, x')] g(x') , \tag{II.1}$$

with the assumption that the Born series corresponding to this equation is absolutely convergent (see Appendix I).

It is our object to sum up the Born series to all orders in H_1 and successive orders in H_2. We get the following result

$$g(x) = [1 + A_0 + (A_{11} + A_{12} + A_{13} + A_{14}) + \cdots] u(x) \tag{II.2}$$

where the A's are integral operators arising in the following way:

$$A_0\, u = \int_a^x dx_1\, H_1(x_1)\, u(x_1) + \int_a^x dx_1\, H_1(x_1) \int_a^{x_1} dx_2\, H_1(x_2)\, u(x_2) + \cdots$$

$$= \int_a^x dx_1\, H_1(x_1)\, u(x_1) + \int_a^x dx_2\, H_1(x_2)\, u(x_2) \int_{x_2}^x dx_1\, H_1(x_1) + \cdots$$

$$= \int_a^x dx_1\, H_1(x_1) \exp \int_{x_1}^x dx_2\, H_1(x_2)\, u(x_1)$$

$$\overset{\text{def}}{=} \int_a^x dx_1\, A(x, x_1)\, u(x_1)\,, \qquad \text{(II.3)}$$

$$A_{11}\, u = \int_a^x dx_1\, H_2(x, x_1)\, u(x_1)\,,$$

$$A_{12}\, u = \int_a^x dx_1\, H_2(x, x_1) \int_a^x dx_2\, A(x_1, x_2)\, u(x_2)\,,$$

$$A_{13}\, u = \int_a^x dx_1\, A(x, x_1) \int_a^{x_1} dx_2\, H_2(x_1, x_2)\, u(x_2)\,,$$

$$A_{14}\, u = \int_a^x dx_1\, A(x, x_1) \int_a^{x_1} dx_2\, H_2(x_1, x_2) \int_a^{x_2} dx_3\, A(x_2\, x_3)\, u(x_3)\,.$$

To realize the higher orders, a symbolic notation is very useful: Let us write:

$$A_0 = (x\, x_1)\,,$$

$$A_{11} = (x - x_1)\,, \qquad A_{12} = (x - x_1\, x_2)\,,$$

$$A_{13} = (x\, x_1 - x_2)\,, \qquad A_{14} = (x\, x_1 - x_2\, x_3)\,.$$

Here a line $x_k - x_{k+1}$ represents a kernel $H_2(x_k, x_{k+1})$, two unconnected variables x_k, x_{k+1} stand for $A(x_k, x_{k+1})$, defined by equation (II.3). The expression arising in this way has to be integrated over all indexed variables

$$x \gtrless x_1 \gtrless x_2 \cdots \gtrless x_k \gtrless a \quad \text{for} \quad x \gtrless a$$

respectively. For example, the second order terms are

$$A_{21} = (x - x_1 - x_2)\,, \qquad A_{22} = (x - x_1 - x_2\, x_3)\,,$$

$$A_{23} = (x\, x_1 - x_2 - x_3)\,, \qquad A_{24} = (x\, x_1 - x_2 - x_3\, x_4)\,,$$

$$A_{25} = (x - x_1\, x_2 - x_3)\,, \qquad A_{26} = (x - x_1\, x_2 - x_3\, x_4)\,,$$

$$A_{27} = (x\, x_1 - x_2\, x_3 - x_4)\,, \qquad A_{28} = (x\, x_1 - x_2\, x_3 - x_4\, x_5)\,.$$

Finally, in the nth order all possible terms with n lines occur. Looking at the above examples, the rules for writing these terms down are evident.

Now let us return to the Klein-Gordon equation. The high energy expansion for $f(k, x)$ can be derived from equation (I.10). To zeroth order we get from equation (II.3)

$$(1 + A_0)\, 1 = 1 - \int_x^\infty dx_1\, W(x_1) \exp \int_x^{x_1} dx_2\, W(x_2) = \exp \int_x^\infty dx_1\, W(x_1) \tag{II.4}$$

with

$$W(x) = \frac{1}{2\, i\, k} \big(2\, E\, V(x) - V^2(x)\big) \tag{II.5}$$

and therefore

$$A(x, x') = W(x') \exp \int_x^{x'} dx_1\, W(x_1) \int_x^\infty dx'\, A(x, x') = 1 - \exp \int_x^\infty dx_1\, W(x_1)\,. \tag{II.6}$$

In the first and higher orders, integrals of the following form appear

$$\int_{x_1}^\infty dx_2\, H_2(x_1, x_2) \left[\int_{x_2}^\infty dx_3\, A(x_2, x_3) - 1 \right]$$

$$= - \int_{x_1}^\infty dx_2\, e^{2ik(x_2 - x_1)}\, W(x_2) \exp \int_{x_2}^\infty dx_3\, W(x_3)\,. \tag{II.7}$$

An integration by parts yields

$$\frac{1}{2\, i\, k} W(x_1) \exp \int_{x_1}^\infty dx_3\, W(x_3) - \frac{1}{2\, i\, k} \int_{x_1}^\infty dx_2\, e^{2ik(x_2 - x_1)}$$

$$\times \big(W'(x_2) - W^2(x_2)\big) \exp \int_{x_2}^\infty dx_3\, W(x_3)\,.$$

The remaining integral, if compared with (II.7), is of higher order in $1/k$. Let us assume that the potential $V(x)$ and its first N derivatives $V^{(n)}(x)$ are piecewise continuous and satisfy

$$V^{(n)}(x) \in L^1(0, \infty) \cap L^\infty(0, \infty) ,$$
$$V^{(n)}(x) \to 0,\ x \to \infty,\ n = 0, 1, \ldots, N . \tag{II.8}$$

Then, in all orders we can integrate by parts N times. The remaining integrals are $o(1/k^N)$ by means of (II.8). Summing up the integrated terms and expanding E in equation (II.5) in powers of $1/k$, we obtain

$$g(k, x) = f(k, x)\, e^{-ikx} = \exp i\, \varepsilon \int_x^\infty dx'\, V(x') \sum_{n=0}^{N} \frac{a_n(x)}{k^n} + o\left(\frac{1}{k^N}\right),$$
$$\varepsilon = \lim_{|k| \to \infty} \frac{E}{k}, \tag{II.9}$$

where the coefficients $a_n(x)$ are independent of k. The terms arising from the nth order in equation (II.2) are at least $o(1/k^n)$ because n exponentials have to be integrated by parts.

Having established the existence of the expansion (II.9), the actual values of the coefficients $a_n(x)$ are most easily found by substituting into the Klein-Gordon equation. This leads to the following recurrence relation

$$a'_{n+1} = -i\, \varepsilon\, V \sum_{l=1}^{[(n+1)/2]} \binom{\frac{1}{2}}{l} a_{n+1-2l} + \frac{\varepsilon}{2} V'\, a_n + \varepsilon\, V\, a'_n + \frac{i}{2} a''_n , \quad ' = \frac{d}{dx} .$$

Using the boundary conditions

$$a_0 = 1, \quad a_n(\infty) = 0, \quad n \geq 1 ,$$

the coefficients can be successively calculated. The values up to second order are

$$a_0 = 1, \quad a_1 = \frac{\varepsilon}{2} V(x), \quad a_2 = i \frac{\varepsilon}{2} \int_x^\infty dx'\, V(x') + \frac{3}{8} V^2(x) + i \frac{\varepsilon}{4} V'(x) .$$

This gives the following expansion for the irregular solution

$$f(k, x) = \exp i \left(k\, x + \varepsilon \int_x^\infty dz\, V\right)$$
$$\times \left[1 + \frac{1}{k} \frac{\varepsilon}{2} V(x) + \frac{1}{k^2} \left(i \frac{\varepsilon}{2} \int_x^\infty dz\, V + \frac{3}{8} V^2(x) + i \frac{\varepsilon}{4} V'(x)\right) + \cdots\right]. \tag{II.10}$$

From equation (II.10) further quantities of interest can easily be calculated, for instance

$$e^{2i\delta^\sigma(k)} = \frac{f^\sigma(-k)}{f^\sigma(k)} = \exp\left(-2\,i\,\sigma\int\limits_0^\infty dz\;V\right)\left[1-\frac{i\,\sigma}{k^2}\left(\int\limits_0^\infty dz\;V+\frac{1}{2}\,V'(0)\right)+\cdots\right],$$

$$\delta^\sigma(k) = -\,\sigma\int\limits_0^\infty dz\;V-\frac{\sigma}{k^2}\left(\frac{1}{2}\int\limits_0^\infty dz\;V+\frac{1}{4}\,V'(0)\right)+\cdots. \tag{II.12}$$

An expansion of this kind for $\delta(E)$ has also been obtained by Verde [15], but his result is not correct.

Now we turn to the high energy expansion for the regular solution $\varphi(k, x)$. In this case, we start from equation (I.13)

$$\chi(k, x) = \frac{1}{2\,i\,k}\,(1-e^{-2ikx})+\int^x dx'[H_1(x')+H_2(x, x')]\;\chi(k, x')\,.$$

The appearance of the exponential in the inhomogeneous term $u(x)$ causes a modification of the above procedure. The terms arising from $1/(2\,i\,k)$ in u (terms of the first kind) have a k-dependence as discussed above. The terms of the second kind arising from $[1/(2\,i\,k)]\;e^{-2ikx}$ are complementary to them in the following sense: If we substitute in a term of the first kind H_1 by H_2 and vice versa, and operate on $[1/(2\,i\,k)]\;e^{-2ikx}$ instead of $1/(2\,i\,k)$, we get a term of the same order of k; for instance, the term

$$-\int\limits_0^x dx_1\;H_2(x, x_1)\,\frac{1}{2\,i\,k}\,e^{-2ikx} = \frac{1}{2\,i\,k}\int\limits_0^x dx_1\,e^{-2ik(x-x_1)}\;W(x_1)\;e^{-2ikx_1}$$

$$= \frac{e^{-2ikx}}{2\,i\,k}\int\limits_0^x dx_1\;W(x_1)$$

contributes to the lowest order. The terms of the second kind can be partially summed up in successive orders of $1/k$ in a similar way as we have done above with the terms of the first kind. It is not necessary to carry this out in detail, because, if the existence of the expansion for complex k is established, we can use our results for $f(k, x)$ to write this expansion down explicitly: For real k we get from equation (II.10)

$$\varphi(k, x) = \frac{1}{2\,i\,k}\,[f(-k)\,f(k, x)+f(k)\,f(-k, x)] = \frac{1}{k}\sin\left(k\,x-\varepsilon\int\limits_0^x dz\;V\right)$$

$$\times\left[1+\frac{\varepsilon}{2\,k}\,\big(V(x)-V(0)\big)+\frac{1}{k^2}\left(\frac{3}{8}\,V^2(0)+\frac{3}{8}\,V^2(x)+\frac{1}{4}\,V(0)\;V(x)\right)+\cdots\right]$$

$$+\frac{1}{k}\cos\left(k\,x-\varepsilon\int\limits_0^x dz\;V\right)\left[\frac{1}{k^2}\,\frac{\varepsilon}{2}\left(-\int\limits_0^x dz\;V+\frac{1}{2}\,\big(V'(x)-V'(0)\big)\right)+\cdots\right]$$

and this must hold for complex k, too.

REFERENCES

[1] E. CORINALDESI, Nuovo Cim. *11*, 468 (1954).
[2] I. M. GEL'FAND and B. M. LEVITAN, Dokl. Akad. Nauk. SSSR *15*, 309 (1951).
[3] M. VERDE, Nucl. Phys. *9*, 255 (1958/59).
[4] A. DEGASPERIS, J. Math. Phys. *11*, 551 (1970).
[5] Z. S. AGRANOVICH and V. A. MARCHENKO, *The Inverse Problem of Scattering Theory* (1963).
[6] J. BENN and G. SCHARF, Nucl. Phys. *A 134*, 481 (1969).
[7] E. TITCHMARSH, *Theory of Fourier Integrals*, theorems 48 and 63 (1937).
[8] ibid., theorem 96.
[9] ibid., p. 174.
[10] L. SCHIFF, H. SNYDER and J. WEINBERG, Phys. Rev. *57*, 315 (1940).
[11] K. VESELIC, Nucl. Phys. *A 147*, 215 (1970).
[12] R. G. NEWTON, *Quantum Scattering Theory*, p. 178 (1963).
[13] ibid., p. 180.
[14] E. A. CODDINGTON and N. LEVINSON, *Theory of Ordinary Diff. Equations*, theorem (8.4) p. 36 (1955).
[15] M. VERDE, Nuovo Cim. *8*, 560 (1958).