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# Perturbations and Non-Normalizable Eigenvectors

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Abstract. A spectral representation of a self-adjoint operator acting in a Hilbert space is given by eigenvectors of an extension of the operator to a suitable space containing the original Hilbert space. A perturbation argument shows the extended operator has no eigenvalues that do not belong to the spectrum of the original operator. The abstract result is applied to Schrödinger operators  $-\Delta + V$ .

## 1. Introduction

The spectral theory of self-adjoint operators may be treated without ever mentioning non-normalizable eigenvectors. In fact, the spectral theorem may be stated as follows [1].

**Theorem.** Let A be a self-adjoint operator acting in the Hilbert space H. Then A is unitarily equivalent to a multiplication operator. That is, there is a Hilbert space  $L^2(M, \mu)$ , a real measurable function  $\alpha$  on M, and a unitary operator  $U: H \to L^2(M, \mu)$  such that f is in the domain of A if and only if  $\alpha U f$  is in  $L^2(M, \mu)$ , and such that  $U A f = \alpha U f$ .

Here  $\mu$  is a positive measure and  $L^2(M, \mu)$  is the Hilbert space of all measurable complex functions h on M such that  $\int |h(p)|^2 d\mu(p) < \infty$ . (Functions which are equal almost everywhere are identified.) Such a unitary equivalence of A with a multiplication operator is called a spectral representation of A. The spectral theorem asserts the existence but not the uniqueness of spectral representations.

Let  $\phi$  be a Borel measurable function defined on the real numbers. Then  $\phi(A)$  may be defined by the spectral theorem as the operator acting in H which is unitarily equivalent to multiplication by  $\phi(\alpha)$  [1]. This definition is independent of the spectral representation.

Non-normalizable eigenvectors enter the picture only when one attempts to describe the form of the unitary operator U. A suitable space  $K^*$  containing H is chosen. Vectors in  $K^*$  which are not in H are called non-normalizable vectors. (The norm under consideration is that of H, of course.) The self-adjoint operator A acting in H has an extension to an operator  $\tilde{A}$  acting in  $K^*$ . When A has continuous spectrum U is given in terms of non-normalizable eigenvectors of  $\tilde{A}$ .

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G. I. Kac has given an elegant criterion for showing that  $K^*$  is large enough to contain all the non-normalizable eigenvectors necessary for any spectral representation [2]. It is desirable to choose  $K^*$  as small as possible consistent with this requirement, since this allows the widest class of perturbations and gives the sharpest estimates on non-normalizable eigenvectors. Also, if  $K^*$  is reasonably small, the eigenvalues of  $\tilde{A}$  may give a good idea of the spectrum of A. (In general, since  $\tilde{A}$  acting in  $K^*$  is not self-adjoint, it may even have non-real eigenvalues.) Here an abstract perturbation theory is developed to show that  $K^*$  does not contain unwanted non-normalizable eigenvectors of  $\tilde{A}$ . This is applied to Schrödinger operators  $-\Delta + V$ . In this case the results may be interpreted as estimates on growth at infinity of eigenfunctions of the Schrödinger operator. (Such results have also been obtained by partial differential equations methods [3].) Much stronger assumptions on the interaction would be needed in order to apply the theory of wave operators and scattering.

# 2. Non-Normalizable Eigenvectors

Let *H* be a Hilbert space. The inner product of *g* and *f* in *H* will be denoted  $\langle g, f \rangle$ . The convention adopted here is that the inner product is conjugate linear in the first variable and linear in the second variable. The norm of *f* in *H* is  $||f|| = \langle f, f \rangle^{1/2}$ .

We wish to consider a situation where there is given another Hilbert space K which is a dense linear subspace of H. If f is an element of K, the norm of f as an element of K is written  $||f||_{\kappa}$ . We shall assume that the injection of K into H is continuous. Thus there is a constant c > 0 such that  $||f|| \leq c ||f||_{\kappa}$ .

Let  $K^*$  be the set of all bounded linear functions from K to the complex numbers.

Proposition 1. Let H be a Hilbert space. Let  $K \subset H$  be another Hilbert space. Assume that K is dense in H and that the injection is continuous. To each g in H associate the linear function  $f \to \langle g, f \rangle$  in  $K^*$ . Then this correspondence is injective, and H may be identified with a dense subspace of  $K^*$ , so that we have continuous inclusions of Hilbert spaces  $K \subset H \subset K^*$ .

**Proof:** If g is an element of H, then  $|\langle g, f \rangle| \leq ||g|| ||f|| \leq c ||g|| ||f||_K$ , so the function which assigns to f in K the inner product  $\langle g, f \rangle$  is in  $K^*$ . If the inner product  $\langle g, f \rangle = 0$  for all f in K, then g = 0, since K is dense in H. Thus each element g in H determines a unique element of  $K^*$ . We identify each g in H with the corresponding element of  $K^*$ .

We wish to give  $K^*$  the structure of a Hilbert space in such a way that the injection of H into  $K^*$  is linear. If  $\psi$  is in  $K^*$  and f is in K, we write  $\langle \psi, f \rangle$  for the value of  $\psi$  on f. If  $\psi_1$  and  $\psi_2$  are in  $K^*$ , we define  $\psi_1 + \psi_2$  by  $\langle \psi_1 + \psi_2, f \rangle = \langle \psi_1, f \rangle + \langle \psi_2, f \rangle$ . If  $\psi$  is in  $K^*$  and a is a complex number, it is convenient to define the product of a with  $\psi$  to be given by  $\langle a \psi, f \rangle = a^* \langle \psi, f \rangle$ . With this convention, if  $\psi$  happens to be in H this coincides with scalar multiplication in H. With the definition  $|| \psi ||_{K^*} = \sup\{|\langle \psi, f \rangle| : || f ||_{K} \leq 1\}$ ,  $K^*$  becomes a Hilbert space.

Note that we have  $||g||_{K^*} = \sup\{|\langle g, f \rangle| : ||f||_K \leq 1\} \leq c \sup\{|\langle g, f \rangle| : ||f|| \leq 1\}$ = c ||g||. Hence the inclusion of H into  $K^*$  is continuous. It is also not hard to see that H is dense in  $K^*$ . This completes the proof. Warning: Having identified  $H \subset K^*$ , it is no longer permissible to identify  $K^*$  with K.

Definition. Let H be a Hilbert space and let  $K \subset H$  be another Hilbert space. Assume that K is dense in H and that the inclusion is continuous. Then the triple  $K \subset H \subset K^*$  will be called a scale of Hilbert spaces.

From now on we shall assume that all Hilbert spaces under consideration have a countable orthonormal basis. (This is equivalent to their being separable metric spaces.) This allows us to consider only measure spaces which are  $\sigma$ -finite.

We now recall the theorem of Kac [2].

**Theorem 1.** Let A be a self-adjoint operator acting in H. Assume that  $U: H \to L^2(M, \mu)$ is a unitary operator which gives a spectral representation of A. Let  $K \subset H \subset K^*$  be a scale of Hilbert spaces. Assume that there is a Borel measurable function  $\beta$  which is bounded on the spectrum of A and which does not vanish on the spectrum of A such that  $\beta(A)$  is a Hilbert-Schmidt operator from K to H. Then there is a function  $\psi$  from M to K\* such that for every f in K, Uf  $(\phi) = \langle \psi(\phi), f \rangle$  for almost every  $\phi$  in M.

For the convenience of the reader we sketch a proof.

Proof:  $U \ \beta(A) = \beta(\alpha) \ U: K \to L^2(M, \mu)$  is a Hilbert-Schmidt operator. Represent K as a space  $L^2(N, \nu)$ . Then there is an s in  $L^2(M \times N, \mu \times \nu)$  such that for f in K,  $\beta(\alpha(p)) \ Uf(p) = \int s(p, q) \ f(q) \ d\nu(q)$  for almost every p [4]. By Fubini's theorem, s(p, q) is in  $L^2(N, \nu)$  as a function of q for almost every p. Thus for these p we may define  $\langle \psi(p), f \rangle = (1/\beta(\alpha(p))) \ \int s(p, q) \ f(q) \ d\nu(q)$ .  $\psi(p)$  is in  $K^*$  by the Schwarz inequality.

*Remark.* In practice the most useful choices of  $\beta$  are  $\beta(a) = (a - z)^{-k}$  for some integer  $k = 1, 2, 3, \ldots$  and z not in the spectrum of A. Another possibility is  $\beta(a) = 1$ . In this case the condition is simply that the injection of K into H is Hilbert-Schmidt.

Note. If f and g are in H, and f is in the domain of A, we have

$$\langle g, A f \rangle = \int Ug \ (p)^* \ lpha(p) \ Uf \ (p) \ d\mu(p) \ .$$

In particular, under the conditions of Theorem 1, if f and g are in K,  $\langle g, A f \rangle = \int \langle g, \psi(p) \rangle \alpha(p) \langle \psi(p), f \rangle d\mu(p)$ . (In keeping with the usage in physics, we have written  $\langle g, \psi(p) \rangle$  for  $\langle \psi(p), g \rangle^*$ .)

Definition. Let  $K \subset H \subset K^*$  be a scale. Let A be a self-adjoint operator acting in H with domain D. Let  $D_0 = \{f \text{ in } K \cap D : A f \text{ is in } K\}$ . Assume  $D_0$  is dense in K. Then the scale extension  $\tilde{A}$  of A is defined as the operator acting in  $K^*$  which is the adjoint of A restricted to  $D_0$ .

Explicitly, if g is in  $K^*$ , g is in the domain of  $\tilde{A}$  if there is an h in  $K^*$  with  $\langle h, f \rangle = \langle g, A f \rangle$  for all f in  $D_0$ . Since  $D_0$  is dense in K, h is uniquely determined and we set  $\tilde{A} g = h$ .  $\tilde{A}$  is clearly an extension of the original self-adjoint operator A.

**Theorem 2** [2]. Let A be a self-adjoint operator acting in H. Let  $K \subset H \subset K^*$ . Let U be a spectral representation of A and assume that there is a function  $\psi$  from M to  $K^*$  such that  $(Uf)(\phi) = \langle \psi(\phi), f \rangle$  for almost every  $\phi$  in M. Assume that the scale exten-

sion  $\tilde{A}$  of A is defined. Then  $\psi(p)$  is in the domain of definition of  $\tilde{A}$  and  $\tilde{A}\psi(p) = \alpha(p) \psi(p)$  for almost every p.

Proof: Let f be in  $D_0$ . Then  $\langle \psi(p), A f \rangle = UAf(p) = \alpha(p) Uf(p) = \langle \alpha(p) \psi(p), f \rangle$ for almost every p. Thus for any countable subset of  $D_0$ ,  $\langle \psi(p), A f \rangle = \langle \alpha(p) \psi(p), f \rangle$ for f in the subset for almost every p. Now the graph of the operator A restricted to  $D_0$ is a subspace of the direct sum of K with itself. Since a subspace of a separable metric space is separable, the graph is separable. Thus there is a countable subset of  $D_0$  such that for each f in  $D_0$ , there is a subsequence  $f_n$  in the subset with  $f_n \to f$  and  $A f_n \to A f$ , in the norm of K. We conclude that  $\langle \psi(p), A f \rangle = \langle \alpha(p) \psi(p), f \rangle$  for all f in  $D_0$  for almost every p. In other words, for these  $p, \tilde{A}\psi(p) = \alpha(p)\psi(p)$ .

#### 3. Perturbation Theory

**Theorem 3.** Let  $K \subset H \subset K^*$  be a scale of Hilbert spaces. Let A be a self-adjoint operator acting in H. Assume that  $(A - z)^{-1}$  is a Hilbert-Schmidt operator from K to H for some z not in the spectrum of A. Assume that B is a self-adjoint operator whose domain contains the domain of A and such that A + B is self-adjoint with the same domain as A. Then if z is also not in the spectrum of A + B,  $(A + B - z)^{-1}$  is a Hilbert-Schmidt operator from K to H.

*Proof*: The two resolvents are related by

 $(A + B - z)^{-1} = [1 - (A + B - z)^{-1}B]$   $(A - z)^{-1}$ . Let T be the closure of  $1 - (A - B - z)^{-1}B$ . Then its adjoint  $T^* = 1 - B(A + B - z^*)^{-1}$ . Since A + B has the same domain as A, T\* is defined on all of H. But any adjoint has a closed graph. So  $T^*$  is a bounded operator from H to H, by the closed graph theorem. Hence T is also bounded from H to H. The identity  $(A + B - z)^{-1} = T(A - z)^{-1}$  thus exhibits  $(A + B - z)^{-1}$  as a Hilbert-Schmidt operator from K to H followed by a bounded operator from H to H.

Proposition 2. Let A be a self-adjoint operator acting in the Hilbert space H. Let  $K \subset H \subset K^*$  be a scale of Hilbert spaces. Assume that the scale extension of A to an operator  $\tilde{A}$  acting in  $K^*$  exists. Let  $\lambda$  be a complex number which is not in the spectrum of A. Then if  $(A - \lambda)^{-1}$  sends K into K,  $\lambda$  is not an eigenvalue of  $\tilde{A}$ .

*Proof*: Assume that  $(\tilde{A} - \lambda) \psi = 0$  for some  $\psi$  in  $K^*$ . If  $(A - \lambda)^{-1}$  sends K into K, then the range of  $A - \lambda$  restricted to  $D_0$  is K. Hence  $\langle \psi, (A - \lambda) f \rangle = 0$  for f in  $D_0$  implies  $\psi = 0$ .

Definition. Let T be a positive self-adjoint operator acting in H with bounded inverse  $T^{-1}$ :  $H \to H$ . For  $0 \leq s < \infty$ , let  $K_s$  be the domain of  $T^s$  with the norm  $||f||_s = ||T^s f||$ . Then the family  $K_s \subset H \subset K_s^*$ ,  $0 \leq s < \infty$ , of scales is called an analytic scale.

There are interpolation theorems which apply to analytic scales. We shall need only the following special case [5].

Proposition 3. Let the spaces  $K_s \subset H$ ,  $0 \leq s < \infty$ , define an analytic scale. Let  $R: H \to H$  be a bounded operator and assume that it has a bounded restriction  $R: K_a \to K_a$ . Then  $R: K_s \to K_s$  is bounded for  $0 \leq s \leq a$ .

**Theorem 4.** Let H be a Hilbert space. Let A be a self-adjoint operator acting in H with domain D. Let B be a self-adjoint operator acting in H whose domain contains D. Assume that A + B is self-adjoint on D. Let the spaces  $K_s \,\subset \, H, \, 0 \leq s < \infty$ , determine an analytic scale and set  $K_1 = K$ . Assume that for some  $\varepsilon > 0$  and all s,  $0 \leq s \leq 1, B: D \cap K_s \to K_{s+\varepsilon}$  is bounded. Then if  $\lambda$  is not in the spectrum of A or of A + B, and if the restriction  $(A - \lambda)^{-1}: K \to K$  is bounded, then the restriction  $(A + B - \lambda)^{-1}: K \to K$  is bounded.

*Proof*: We have  $K = K_1 \subset K_s \subset K_0 = H$  for  $0 \leq s \leq 1$ . The space  $D \cap K_s$  may be given the norm  $(||A f||^2 + ||f||_s^2)^{1/2}$ , where  $||f||_s$  is the norm on  $K_s$ .

Write 
$$(A + B - \lambda)^{-1} = \sum_{n=0}^{r-1} (-1)^n ((A - \lambda)^{-1} B)^n (A - \lambda)^{-1} + (-1)^r ((A - \lambda)^{-1} B)^r (A + B - \lambda)^{-1}.$$

Consider the first r terms in the sum. Since  $(A - \lambda)^{-1}$ :  $K \to D \cap K$  and  $B: K \cap D \to K$  are bounded, each term is bounded from K to  $D \cap K \subset K$ .

To treat the final term in the sum, we use interpolation. Since  $(A - \lambda)^{-1}: H \to H$ and  $(A - \lambda)^{-1}: K \to K$  are bounded, it follows from Proposition 3 that  $(A - \lambda)^{-1}: K_s \to K_s$  is bounded for  $0 \leq s \leq 1$ . Take r so large that  $1/r < \varepsilon$ . Then  $B: D \cap K_{(n-1)/r} \to K_{n/r}$  is bounded,  $n = 1, 2, 3, \ldots, r$ . Since  $(A - \lambda)^{-1}: K_{n/r} \to D \cap K_{n/r}$ and  $(A + B - \lambda)^{-1}: H \to D$  are bounded, the final term is bounded from  $H \supset K$  to  $D \cap K \subset K$ .

### 4. Schrödinger Operators

Let  $H = L^2(\mathbb{R}^3, dx)$ . Let  $\varrho \ge 0$  be a real function on  $\mathbb{R}^3$  which is bounded and never zero. Let  $K = L^2(\mathbb{R}^3, \varrho(x)^{-1} dx)$ . Then  $K \subset H$ , the injection is continuous, and Kis dense in H. So  $K \subset H \subset K^*$  is a scale of Hilbert spaces. The nice feature of this case is that  $K^*$  has a natural realization as a space of functions. It should be considered as  $K^* = L^2(\mathbb{R}^3, \varrho(x) dx)$ . Then if g is in  $K^*$  and f is in K,  $\langle g, f \rangle = \int g(x)^* f(x) dx$  and  $|\langle g, f \rangle| \leq ||g||_{K^*} ||f||_{K}$ .

In the following we shall require that  $\varrho$  be bounded away from zero on compact sets. This ensures that the K, H, and  $K^*$  norms are equivalent on any set of f with fixed compact support.

*Example.* Consider the Laplace operator  $\Delta$ .  $\Delta$  is a self-adjoint operator acting in H and one spectral representation is given by the Fourier transform  $F: H \to L^2(\mathbb{R}^3, (2\pi)^{-3} dk)$ .  $(F \Delta f)(k) = -k^2 Ff(k)$ , so  $\Delta$  is isomorphic to multiplication by  $\alpha(k) = -k^2$ . Assume now that  $\varrho$  is in  $L^1(\mathbb{R}^3, dx)$ . Notice that for f in K,  $(F f)(k) = \int \exp(-i k x) f(x) dx = \langle \psi(k), f \rangle$ , where  $\psi(k)$  is the function  $\exp(i k x)$  in  $K^*$ . The  $\psi(k)$  are non-normalizable eigenvectors:  $\Delta \exp(i k x) = -k^2 \exp(i k x)$ .

Definition. Let V be a real function on  $\mathbb{R}^3$  such that  $V = V_1 + V_2$ , where  $V_1$  is in  $L^{\infty}(\mathbb{R}^3, dx)$  and  $V_2$  is in  $L^2(\mathbb{R}^3, dx)$ . Then V will be said to satisfy the Kato condition.

Proposition 4 [6]. Let  $H = L^2(\mathbb{R}^3, dx)$ . Assume that V is a real function on  $\mathbb{R}^3$  which satisfies the Kato condition. Then  $-\Delta + V$  is a self-adjoint operator acting in H with the same domain as that of  $-\Delta$ .

**Theorem 5.** Let  $\varrho \ge 0$  be a function on  $\mathbb{R}^3$  which is bounded on  $\mathbb{R}^3$  and bounded away from zero on compact subsets of  $\mathbb{R}^3$ . Assume that  $\varrho$  is in  $L^1(\mathbb{R}^3, dx)$ . Let  $K \subset H \subset K^*$ be the scale  $L^2(\mathbb{R}^3, \varrho(x)^{-1} dx) \subset L^2(\mathbb{R}^3, dx) \subset L^2(\mathbb{R}^3, \varrho(x) dx)$ . Let V be a real function on  $\mathbb{R}^3$  which satisfies the Kato condition. Then for any spectral representation  $U: H \to L^2(M, \mu)$  of  $-\Delta + V$  there is a function  $\psi$  from M to  $K^*$  such that for each f in K,  $(Uf)(\phi) = \langle \psi(\phi), f \rangle$  for almost every  $\phi$  in M. The  $\psi(\phi)$  are (possibly nonnormalizable) eigenvectors of the scale extension of  $-\Delta + V$  for almost every  $\phi$ in M.

*Proof*: If z > 0,  $(z - \Delta)^{-1}$  is an integral operator acting in H with kernel  $(4 \pi | x - y |)^{-1} \exp(-z^{1/2} | x - y |)$ . Now  $(4 \pi | x - y |)^{-1} \exp(-z^{1/2} | x - y |) \varrho(y)^{1/2}$  is in  $L^2(\mathbb{R}^6, dx dy)$ . Hence it is the kernel of a Hilbert-Schmidt operator from H to H. Next note that multiplication by  $\varrho^{-1/2}$  is an isomorphism from K to H. It follows that  $(z - \Delta)^{-1}$  is a Hilbert-Schmidt operator from K to H.

If z > 0 is sufficiently positive, then -z will not be in the spectrum of  $-\Delta + V$ [6]. Hence Theorem 3 implies that  $(z - \Delta + V)^{-1}$  is Hilbert-Schmidt from K to H. Thus Theorem 1 applies to  $-\Delta + V$ .

We now show that  $-\Delta + V$  has a scale extension. First note that the  $L^2(\mathbb{R}^3, dy)$ norm of  $(4 \pi | x - y |)^{-1} \exp(-z^{1/2} | x - y |)$  is finite and independent of x. It follows that  $(z - \Delta)^{-1}g$  is in  $L^{\infty}(\mathbb{R}^3, dx)$  for g in H. Hence the domain of definition of  $\Delta$  is contained in  $L^{\infty}(\mathbb{R}^3, dx)$ . Now consider the space  $D_1$  of functions in the domain of  $\Delta$  which have compact support. Clearly  $D_1$  is dense in K.  $-\Delta$  sends  $D_1$  into K. Multiplication by  $V_1$  leaves K invariant. On the other hand, since the domain of  $\Delta$  is contained in  $L^{\infty}(\mathbb{R}^3, dx)$ , multiplication by  $V_2$  sends  $D_1$  into K. Thus  $-\Delta + V$  sends the dense set  $D_1$  into K. This implies that the scale extension of  $-\Delta + V$  exists and hence that Theorem 2 applies.

Surprisingly, Theorem 5 does not imply that the non-normalizable eigenfunctions are bounded. The exceptional set of p in M for which the  $\psi(p)$  are not eigenvectors in  $K^*$  will depend in general on the choice of the scale. Maslov [7] has given an example of a bounded continuous V for which the non-normalizable eigenfunctions for some interval of energy are unbounded at infinity. Berezanskii [8] has given estimates on their rate of increase.

In the following we write r = |x|.

Definition. Let V be a real measurable function on  $\mathbb{R}^3$ . Assume that  $V = V_1 + V_2$ , where  $|V_1(x)| \leq c(1 + r^2)^{-\epsilon/2}$  for some  $\epsilon > 0$  and some c, and  $V_2$  is in  $L^2(\mathbb{R}^3, dx)$  and has compact support. Then V will be said to satisfy the condition of slight decrease.

Note that a function V of slight decrease satisfies the Kato condition. In addition it is a relatively compact perturbation of  $-\Delta$ , so that the essential spectrum of  $-\Delta + V$  is  $[0, \infty)$  [9]. In particular, the spectrum of  $-\Delta + V$  contains the spectrum of  $-\Delta$ .

A particularly convenient choice of the function  $\rho$  defining the scale is  $\rho(x) = (1 + r^2)^{-s/2}$ ,  $s \ge 0$ . The condition that  $\rho$  is in  $L^1(\mathbb{R}^3, dx)$  is satisfied provided s > 3. **Theorem 6.** Fix  $s, 0 \le s < \infty$ . Let  $K_s \subset H \subset K_s^*$  be the scale  $L^2(\mathbb{R}^3, (1 + r^2)^{s/2} dx) \subset \mathbb{R}^3$ .

**Theorem 6.** Fix  $s, 0 \leq s < \infty$ . Let  $K_s \subset H \subset K_s^*$  be the scale  $L^2(\mathbb{R}^3, (1 + r^2)^{s/2} dx) \subset CL^2(\mathbb{R}^3, dx) \subset L^2(\mathbb{R}^3, (1 + r^2)^{-s/2} dx)$ . Let V be a real function on  $\mathbb{R}^3$  which satisfies the condition of slight decrease. Then the scale extension of  $-\Delta + V$  to an operator in  $K_s^*$  has the complex number  $\lambda$  as an eigenvalue only if  $\lambda$  is real and in the spectrum of the self-adjoint operator  $-\Delta + V$  acting in H.

*Proof*: Let T be the operator given by multiplication by  $(1 + r^2)^{1/4}$ . Then the scale in the theorem is the analytic scale associated with T;  $K_s$  is the domain of  $T^s$ ,  $s \ge 0$ .

In order to apply Theorem 4 we first show that for z > 0 or z not real,  $(z + \Delta)^{-1}$ :  $K_s \to K_s$  is continuous,  $s \ge 0$ . Start with the case when s is an integer multiple of 4, s = 4 k. By taking Fourier transforms we see that this is equivalent to showing that  $(z - k^2)^{-1}$  is a continuous multiplication operator from  $D(\Delta^k)$  to  $D(\Delta^k)$ . For f in  $D(\Delta^k)$ , expand  $\Delta^k((z - k^2)^{-1}f)$  as a sum of products of partial derivatives of  $(z - k^2)^{-1}$ and of f. The partial derivatives of f can be estimated in  $L^2$  norm in terms of  $||\Delta^k f||_2$ and  $||f||_2$ . On the other hand, the partial derivatives of  $(z - k^2)^{-1}$  are all bounded functions (since they are Fourier transforms of integrable functions). Thus we have an estimate on  $||\Delta^k((z - k^2)^{-1}f)||_2$ , which disposes of the case when k is an integer. The general case now follows from Proposition 3.

The other hypothesis of Theorem 4 follows from the assumption of slight decrease. Multiplication by  $V_1$  is bounded from  $K_s$  to  $K_{s+\varepsilon}$  for some  $\varepsilon > 0$ . On the other hand,  $V_2$  is bounded from D to  $K_s$  for all  $s \ge 0$ , since  $D \subset L^{\infty}(\mathbb{R}^3, dx)$  and  $V_2$  has compact support. Thus  $V: K_s \cap D \to K_{s+\varepsilon}$  is bounded.

So if  $\lambda$  is not in the spectrum of  $-\Delta + V$ ,  $(-\Delta + V - \lambda)^{-1}: K_s \to K_s$  is bounded. The theorem then follows from Proposition 2.

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- [8] See Reference [2], Chap. VI.
- [9] See Reference [4], Chap. V, § 5.3.