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Coulomb Corrections to Low Energy Elastic and Charge Exchange πN Scattering

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Summary. We consider the problem of πN scattering in the simultaneous presence of the short range nuclear potential and the long range Coulomb potential. The existing treatment of $\pi^+ p$ scattering is outlined and similar methods are then used to derive the corresponding results for the coupled channel processes $\pi^- p \to \pi^- p$ and $\pi^- p \to \pi^0 n$. Finally we show how the Coulomb corrections so obtained can be calculated to first order in the Coulomb parameter.

1. Introduction

Most pion-nucleon scattering data are obtained from experiments having $\pi^+ p$ or $\pi^- p$ initial states. Since in either case both particles are charged, it is necessary to include both electromagnetic and strong interactions in any complete analysis of the data. This is particularly important in the case of phase shift analyses of scattering data for the elastic process

$$\pi^+ p \to \pi^+ p$$

and for the coupled processes

$$\pi^- p \to \pi^- p ,$$
$$\pi^- p \to \pi^0 n$$

where one is interested in isolating the purely nuclear interaction.

In the usual treatment of this problem, it is assumed that the total scattering amplitude is the sum of a pure Coulomb scattering amplitude and the appropriate combination of charge independent, purely nuclear scattering amplitudes suitably modified by certain Coulomb phase factors [1, 2, 3]. It has long been realised that nuclear phases extracted in this way still contain some effects of the Coulomb interaction, the separation of which is a model dependent process. Van Hove [4] first treated this problem by assuming that the Coulomb interaction vanished within the nuclear interaction region. The corrections derived in this way, now called outer Coulomb corrections, depend strongly on the value chosen for the nuclear interaction radius. It was later pointed out by Hamilton and Woolcock [5] that this dependence was almost eliminated if the Coulomb interaction was also included within the nuclear interaction region, thus giving additional inner Coulomb corrections.

In his treatment of the outer Coulomb corrections, Van Hove dealt both with the process $\pi^+ p \to \pi^+ p$ and with the coupled processes $\pi^- p \to \pi^- p$ and $\pi^- p \to \pi^0 n$. In their calculation of the inner Coulomb corrections, Hamilton and Woolcock dealt first with the process $\pi^+ p \to \pi^+ p$ using a real potential and then introduced a complex potential to describe the process $\pi^- p \to \pi^- p$. The treatment of inner Coulomb corrections was later extended by Schnitzer [6] but without inclusion of the important effect of the coupling between the $\pi^- p$ elastic and charge exchange channels.

The πN interaction can most easily be studied when data on the $\pi^+ p$ elastic and $\pi^- p$ elastic and charge exchange channels are all available at the same energy. In the treatment of this data, the present series of large scale phase shift analyses ignore the inner and outer Coulomb corrections [8]. While this may be a good approximation at higher energies, the neglect can be dangerous in the low energy region. This is particularly true when data with errors of only a few percent become available below 300 MeV thus allowing the determination of coupling constants and scattering lengths to a high accuracy. For this reason, we have extended Van Hove's coupled channel treatment of the $\pi^- p$ elastic and charge exchange outer Coulomb corrections to provide a systematic coupled channel treatment of both inner and outer Coulomb corrections. Using these results, the simultaneous analysis of low energy $\pi^- p \to \pi^- p$ and $\pi^- p \to \pi^0 n$ data with the inclusion of inner and outer Coulomb corrections now becomes possible. This enables a better determination of the low energy purely nuclear scattering amplitudes and also makes clearer the difficulties of formulating tests of charge independence [9].

In Section 2, for completeness and as an introduction to the methods used, we outline the treatment of Coulomb corrections for the $\pi^+ p$ elastic channel. Details have been published elsewhere [10, 11] and the results agree with those of Schnitzer [6] although the method avoids the cancelling divergences of his treatment. In Section 3, we introduce the coupled channel formalism for the processes $\pi^- p \to \pi^- p$ and $\pi^- p \to \pi^0 n$ and we show how the idea of an S-matrix has to be modified when Coulomb forces are present in one of the channels. In Section 4, we obtain expressions for the two channel S-matrix and in Section 5 we show how the inner and outer Coulomb corrections can be calculated to first order in the Coulomb interaction without having to solve the coupled radial wave equations numerically. In two appendices we show that our results go over into those of Van Hove when we neglect the inner part of the Coulomb potential and we show that our explicit form for the S-matrix when full Coulomb corrections are included is symmetric and unitary.

2. Coulomb Corrections for $\pi^+ p$ Elastic Scattering

We assume that the πN system in the low energy region can be described by a nonrelativistic potential model where the nuclear potential can depend on energy as well as on the total angular momentum, the orbital angular momentum and the total isospin. We further assume that the most important electromagnetic corrections are obtained by introducing the additional Coulomb potential into this model. This assumption of non-relativistic potential scattering is common to all treatments of low energy Coulomb corrections with the exception of the work of Sauter [12]. He has applied the techniques of Dashen and Frautschi [13] to the problem of Coulomb corrections for $\pi^+ p$ elastic scattering and in the low energy region he obtains agreement with the results of Ref. [10].

a) *Purely Nuclear Scattering*

In the absence of Coulomb forces, the states of the πN system are specified uniquely in the c.m. frame by the total energy and by the quantum numbers J, J_3, T, T_3 and P. Here $\boldsymbol{J}$ denotes the total angular momentum, $\boldsymbol{T}$ the total isospin and P the parity. The orbital angular momentum quantum number l can be used instead of P and it is customary to write the eigenstates of these conserved quantities in the form

$$\psi_{l\pm}^{(T)} = \frac{1}{r} R_{l\pm}^{(T)}(\varrho)\, \Omega_{l\pm,s}(\theta, \phi)\, | T \rangle \tag{1}$$

where $l\pm$ denotes states with $J = l \pm 1/2$. r, θ and ϕ are the relative spherical polar coordinates of the πN system in the c.m. frame, the incoming pion direction being taken as the polar axis and the outgoing pion direction being specified by θ and ϕ. $\Omega_{l\pm,s}$ is the eigenfunction of total angular momentum and J_3 with eigenvalues $l \pm 1/2$ and s respectively; it is defined at the end of section 3. ϱ, the argument of the radial wave function $R_{l\pm}^{(T)}$, is given by

$$\varrho = k\, r$$

where $k = |\tilde{p}|/\hbar$ is the wave number corresponding to the relative momentum p. $|T\rangle$ is the isospin part of the state vector, the quantum number T_3 being suppressed. When Coulomb forces are present T_3 is still conserved by virtue of charge conservation and we will still not include it in our notation. In this section, we consider the single channel $\pi^+ p \to \pi^+ p$ where we will be dealing with the $T = 3/2$, $T_3 = 3/2$ state only. In the subsequent sections, we deal with the coupled processes $\pi^- p \to \pi^- p$ and $\pi^- p \to \pi^0 n$ where we have $T = 3/2$, $T_3 = -1/2$ and $T = 1/2$, $T_3 = -1/2$ states.

The purely nuclear $\pi^+ p$ radial wave function obeys the differential equation

$$\frac{d^2 R_{l\pm}^{(3/2)}}{d\varrho^2} + \left(1 - \tilde{U}_{l\pm}^{(3/2)} - \frac{l\,(l+1)}{\varrho^2}\right) R_{l\pm}^{(3/2)} = 0 \tag{2}$$

where $\tilde{U}_{l\pm}^{(3/2)}(\varrho)$ is related to the charge independent $T = 3/2$ nuclear potential $U_{l\pm}^{(3/2)}(r)$ by

$$\tilde{U}_{l\pm}^{(3/2)}(\varrho) = \frac{2\,m}{\hbar^2} \cdot \frac{1}{k^2}\, U_{l\pm}^{(3/2)}(\varrho/k)\,, \tag{3}$$

m being the reduced mass of the πN system.

For simplicity, we now deal only with the s-wave state although where appropriate we give results for general l. We also simplify the notation so that

$$R_3^N \equiv R_{0+}^{(3/2)} ,$$

$$\tilde{U}_3 \equiv \tilde{U}_{0+}^{(3/2)} .$$

Assuming that the nuclear potential $\tilde{U}_3$ is of finite range ϱ_N the radial wave equation (2) becomes

$$\frac{d^2 R_3^N}{d\varrho^2} + (1 - \tilde{U}_3)\, R_3^N = 0 , \qquad \varrho \leqslant \varrho_N , \tag{4}$$

$$\frac{d^2 R_3^N}{d\varrho^2} + R_3^N = 0 , \qquad \varrho \geqslant \varrho_N , \tag{5}$$

We introduce a solution of equation (5), $R_{3\alpha}^N$, normalised so that

$$R_{3\alpha}^N(\varrho) = \operatorname{Sin}(\varrho + \delta_3) , \qquad \varrho \geqslant \varrho_N , \tag{6}$$

$\delta_3 \equiv \delta_{0+}^{(3/2)}$ being the $T = 3/2$ s-wave purely nuclear phase shift. Solving equation (4) with the boundary condition that $R_{3\alpha}^N$ is regular at the origin, i.e.

$$R_{3\alpha}^N(0) = 0 ,$$

then specifies δ_3 in terms of $\tilde{U}_3$.

We can write the most general regular solution of equations (4) and (5) in the form

$$R_3^N(\varrho) = A\, R_{3\alpha}^N(\varrho)$$

where A is an arbitrary constant. Asymptotically this has the form

$$R_3^N(\varrho) \underset{\varrho \to \infty}{\sim} \frac{A\, e^{i\delta_3}}{2\, i}\, e^{i\varrho} - \frac{A\, e^{-i\delta_3}}{2\, i}\, e^{-i\varrho} . \tag{7}$$

If we write the asymptotic form as

$$R_3^N(\varrho) \underset{\varrho \to \infty}{\sim} B\, e^{-i\varrho} + C\, e^{i\varrho} , \tag{8}$$

then C, the amplitude for the outgoing radial wave, is related to B, the amplitude for the incoming radial wave, by

$$C = - S_N^{++}\, B , \tag{9}$$

S_N^{++} being the $\pi^+ p$ s-wave purely nuclear S-matrix element. Comparing equations (7) and (8) we obtain the expected result,

$$S_N^{++} = e^{2 i \delta_3} . \tag{10}$$

b) *Inner Coulomb Corrections*

We now introduce the Coulomb interaction via the electromagnetic potential $V_c(r)$. We define the reduced potential

$$\tilde{V}_c(\varrho) = \frac{2m}{\hbar^2}\frac{1}{k^2} V_c(\varrho/k)$$

and, to allow for the possibility of finite charge distribution effects, we assume that $\tilde{V}_c$ deviates from the corresponding point charge potential $\tilde{V}_{c_p}$ over a finite range so that

$$\tilde{V}_c = \tilde{V}_{c_p}. \qquad \varrho \geqslant \varrho_c . \tag{11}$$

Here the reduced point charge potential is given by

$$\tilde{V}_{c_p} = \frac{2\eta}{\varrho}, \tag{12}$$

the Coulomb parameter η being given by

$$\eta = \frac{e^2 m}{\hbar^2 k} \qquad \left(\frac{e^2}{\hbar c} = \alpha = 137.0388^{-1}\right). \tag{13}$$

Defining

$$\varrho_0 = \max(\varrho_c, \varrho_N)$$

we first consider the effect of the inner Coulomb potential

$$\tilde{V}_c^{IN} = \tilde{V}_c, \qquad \varrho \leqslant \varrho_0, \qquad \tilde{V}_C^{IN} = 0, \qquad \varrho \geqslant \varrho_0$$

The $\pi^+ p$ radial wave function in the simultaneous presence of the nuclear potential and this inner part of the Coulomb potential, R_+^{IN}, then satisfies the equations

$$\frac{d^2 R_+^{IN}}{d\varrho^2} + (1 - \tilde{U}_3 - \tilde{V}_c^{IN})\, R_+^{IN} = 0, \qquad \varrho \leqslant \varrho_0, \tag{14}$$

$$\frac{d^2 R_+^{IN}}{d\varrho^2} + R_+^{IN} = 0 \qquad \varrho \geqslant \varrho_0 \tag{15}$$

We introduce a solution of equation (15), $R_{+\alpha}^{IN}$, normalised, so that

$$R_{+\alpha}^{IN}(\varrho) = \mathrm{Sin}(\varrho + \tilde{\tau}_+), \qquad \varrho \geqslant \varrho_0 \tag{16}$$

and note that solving equation (14) subject to the regularity condition

$$R_{+\alpha}^{IN}(0) = 0$$

then specifies the phase $\tilde{\tau}_+$ in terms of $\tilde{U}_3$ and $\tilde{V}_c^{IN}$.

The most general regular solution of equations (14) and (15) then has the form

$$R_+^{IN}(\varrho) = A\, R_{+\alpha}^{IN}(\varrho)$$

and similar arguments to those used above show that the S-matrix element in the presence of the nuclear and the inner part of the Coulomb potential is given by

$$S_{IN}^{++} = e^{2i\tilde{\tau}_+}. \tag{17}$$

We see that the addition of the inner part of the Coulomb potential changes the $\pi^+ p$ phase shift from δ_3 to $\tilde{\tau}_+$. In order to relate $\tilde{\tau}_+$ to δ_3 we note from equations (4) and (14) that

$$\frac{d}{d\varrho}\left[R^N_{3\alpha}\frac{d}{d\varrho}R^{IN}_{+\alpha} - R^{IN}_{+\alpha}\frac{d}{d\varrho}R^N_{3\alpha}\right] = \tilde{V}^{IN}_c R^N_{3\alpha} R^{IN}_{+\alpha}$$

and so using the fact that both $R^N_{3\alpha}$ and $R^{IN}_{+\alpha}$ are regular at the origin we obtain

$$\left[R^N_{3\alpha}\frac{d}{d\varrho}R^{IN}_{+\alpha} - R^{IN}_{+\alpha}\frac{d}{d\varrho}R^N_{3\alpha}\right]_{\varrho=\varrho_0} = \int_0^{\varrho_0} \tilde{V}^{IN}_c R^N_{3\alpha} R^{IN}_{+\alpha}\, d\varrho \,.$$

But at ϱ_0 we can use equations (6) and (16) to obtain the well known result

$$\operatorname{Sin}(\delta_3 - \tilde{\tau}_+) = \int_0^{\varrho_0} \tilde{V}^{IN}_c R^N_{3\alpha} R^{IN}_{+\alpha}\, d\varrho \,. \tag{18}$$

This result is of particular use since, by approximating $R^{IN}_{+\alpha}$ by $R^N_{3\alpha}$, a first order value of $(\delta_3 - \tilde{\tau}_+)$ can be obtained. A similar result is obtained for general l, i.e.

$$\operatorname{Sin}(\delta^{(3/2)}_{l\pm} - \tilde{\tau}^{(+)}_{l\pm}) = \int_0^{\varrho_0} \tilde{V}^{IN}_c R^{(3/2)N}_{\alpha,l\pm} R^{(+)IN}_{\alpha,l\pm}\, d\varrho \tag{19}$$

where $R^{(3/2)N}_{\alpha,l\pm}$ is the purely nuclear radial wave function normalised, so that

$$R^{(3/2)N}_{\alpha,l\pm} = \operatorname{Cos}\delta^{(3/2)}_{l\pm}\,\varrho\, j_l(\varrho) - \operatorname{Sin}\delta^{(3/2)}_{l\pm}\,\varrho\, n_l(\varrho)\,, \quad \varrho \geqslant \varrho_N\,, \tag{20}$$

and $R^{(+)IN}_{\alpha,l\pm}$ is the corresponding $\pi^+ p$ radial wave function when the inner part of the Coulomb potential is included, its normalisation being such that

$$R^{(+)IN}_{\alpha,l\pm} = \operatorname{Cos}\tilde{\tau}^{(+)}_{l\pm}\,\varrho\, j_l(\varrho) - \operatorname{Sin}\tilde{\tau}^{(+)}_{l\pm}\,\varrho\, n_l(\varrho)\,, \quad \varrho \geqslant \varrho_0\,. \tag{21}$$

Here $j_l(\varrho)$ and $n_l(\varrho)$ are the usual regular and irregular spherical Bessel functions.

c) *Full Coulomb Corrections*

Finally we consider the case when, in addition to the nuclear potential, the full Coulomb potential is present. The $\pi^+ p$ radial wave function, R^{TOT}_+, now satisfies the equations

$$\frac{d^2 R^{TOT}_+}{d\varrho^2} + (1 - \tilde{U}_3 - \tilde{V}^{IN}_c)\, R^{TOT}_+ = 0\,, \quad \varrho \leqslant \varrho_0\,, \tag{22}$$

$$\frac{d^2 R^{TOT}_+}{d\varrho^2} + (1 - \tilde{V}_{c_p})\, R^{TOT}_+ = 0\,, \quad \varrho \geqslant \varrho_0\,. \tag{23}$$

We introduce a solution of equation (23) $R^{TOT}_{+\alpha}$ normalised, so that

$$R^{TOT}_{+\alpha}(\varrho) = \operatorname{Cos}\tau_+\, F_0(\varrho) + \operatorname{Sin}\tau_+\, G_0(\varrho)\,, \quad \varrho \geqslant \varrho_0 \tag{24}$$

where $F_0(\varrho)$ and $G_0(\varrho)$ are the zeroth order regular and irregular Coulomb wave functions. We further note that, in addition to the asymptotic behaviour of equation

(24), $R^{TOT}_{+\alpha}$ must also be a regular solution of equation (22). Comparing this equation with equation (14) we see that $R^{TOT}_{+\alpha}$ and $R^{IN}_{+\alpha}$ are regular solutions of the same differential equation for $0 \leqslant \varrho \leqslant \varrho_0$. Thus they can only differ by a multiplicative constant in this range of ϱ and so, fitting value and derivative at ϱ_0, we obtain the relation between $\tilde{\tau}_+$ and τ_+

$$\operatorname{Tan}(\varrho_0 + \tilde{\tau}_+) = \frac{F_0(\varrho_0) + \operatorname{Tan}\tau_+ \, G_0(\varrho_0)}{F'_0(\varrho_0) + \operatorname{Tan}\tau_+ \, G'_0(\varrho_0)} \; . \tag{25}$$

The most general regular solution of equations (22) and (23) has the form

$$R^{TOT}_{+}(\varrho) = A \; R^{TOT}_{+\alpha}(\varrho)$$

where A is an arbitrary constant. We see from equation (24) that this general regular solution has the asymptotic form

$$R^{TOT}_{+}(\varrho) \underset{\varrho\to\infty}{\sim} \frac{A\, e^{i\tau_+}}{2\,i}\, e^{i(\varrho+\sigma_0-\eta\ln 2\varrho)} - \frac{A\, e^{-i\tau_+}}{2\,i}\, e^{-i(\varrho+\sigma_0-\eta\ln 2\varrho)} \tag{26}$$

where σ_0 is the s-wave Coulomb phase, the values of these phases being given by

$$\sigma_l = \arg \Gamma\,(l + 1 + i\,\eta) \; .$$

The S-matrix in the presence of the long range Coulomb potential is defined by writing the asymptotic form as

$$R^{TOT}_{+}(\varrho) \sim B\, e^{-i(\varrho+\sigma_0-\eta\ln 2\varrho)} + C\, e^{i(\varrho+\sigma_0-\eta\ln 2\varrho)} \tag{27}$$

where C, the amplitude of the outgoing 'radial Coulomb wave' is related to B, the amplitude of the incoming 'radial Coulomb wave' by

$$C = - S^{++}_{TOT}\, B \; .$$

Comparing equations (26) and (27) we see that the $\pi^+ p$ s-wave S-matrix element in the simultaneous presence of the nuclear and full Coulomb potential is given by

$$S^{++}_{TOT} = e^{2i\tau_+} \; . \tag{28}$$

Thus, we see from equations (10) and (28) that, in addition to adding the pure Coulomb scattering amplitude, the effect of the Coulomb potential on $\pi^+ p$ elastic scattering is to modify each phase $\delta^{(3/2)}_{l\pm}$ to a new value $\tau^{(+)}_{l\pm}$ [14]. These Coulomb phase corrections can be calculated to first order in η in two stages. In the s-wave case the inner Coulomb correction is first obtained from equation (18) and then knowing $\tilde{\tau}_+$ the outer Coulomb correction is obtained by calculating τ_+ from equation (25). A similar procedure is followed for general l, the inner Coulomb correction being obtained from equation (19). Then knowing $\tilde{\tau}^{(+)}_{l\pm}$, the outer Coulomb correction is obtained by calculating $\tau^{(+)}_{l\pm}$ from the corresponding general l form of equation (25),

$$\left.\frac{\varrho\, j_l(\varrho) - \operatorname{Tan}\tilde{\tau}^{(+)}_{l\pm}\, \varrho\, n_l(\varrho)}{(\varrho\, j_l(\varrho))' - \operatorname{Tan}\tilde{\tau}^{(+)}_{l\pm}\, (\varrho\, n_l(\varrho))'}\right|_{\varrho=\varrho_0} = \left.\frac{F_l(\varrho) + \operatorname{Tan}\tau^{(+)}_{l\pm}\, G_l(\varrho)}{F'_l(\varrho) + \operatorname{Tan}\tau^{(+)}_{l\pm}\, G_l(\varrho)}\right|_{\varrho=\varrho_0} . \tag{29}$$

In Appendix I we show that when $V^{IN}_c = 0$ our results go over into the outer $\pi^+ p$ Coulomb corrections of Van Hove [4].

3. Coupled Channel Formalism

We now consider the coupled channels $\pi^- p \to \pi^- p$ and $\pi^- p \to \pi^0 n$ where we have to deal with linear combinations of the $T = 3/2$, $T_3 = -1/2$ and $T = 1/2$, $T_3 = -1/2$ states. In this and the following sections, we will ignore the $n - p$ mass difference and the $\pi^\pm - \pi^0$ mass difference. These mass differences give rise to similar effects to those to be described below and they, as well as the influence of the radiative capture process $\pi^- p \to \gamma n$, will be treated in a subsequent paper.

In the absence of Coulomb forces and assuming charge independence, the states $\pi^- p$ and $\pi^0 n$ are described by linear combinations of $\psi_{l\pm}^{(3/2)}$ and $\psi_{l\pm}^{(1/2)}$. Using an ansatz of the form given by equation (1), the $T = 3/2$ radial wave function obeys equation (2) while the $T = 1/2$ radial wave function obeys the similar equation

$$\frac{d^2 R_{l\pm}^{(1/2)}}{d\varrho^2} + \left(1 - \tilde{U}_{l\pm}^{(1/2)} - \frac{l\,(l+1)}{\varrho^2}\right) R_{l\pm}^{(1/2)} = 0 \tag{30}$$

where $\tilde{U}_{l\pm}^{(1/2)}$ is related to the charge independent $T = 1/2$ nuclear potential $U_{l\pm}^{(1/2)}(r)$ by an equation similar to (3).

Assuming the nuclear potentials are of short range, the solutions of equations (2) and (30) which are regular at the origin can be chosen to have the asymptotic behaviour

$$R_{l\pm}^{(T)}\ (\varrho) \underset{\varrho \to \infty}{\sim} \operatorname{Sin}\left(\varrho - \frac{l\pi}{2} + \delta_{l\pm}^{(T)}\right) . \tag{31}$$

The S-matrix in the subspace of fixed l and J is defined as usual by

$$C_{l\pm}^{(T')} = - \sum_{T = 3/2, 1/2} S_{l\pm}^{T'T}\ B_{l\pm}^{(T)} \tag{32}$$

where $B_{l\pm}^{(T)}$ is the amplitude of the incoming spherical wave and $C_{l\pm}^{(T')}$ is the amplitude of the outgoing spherical wave. We see from equation (31) that

$$S_{l\pm}^{T'T} = \delta_{T'T}\, e^{2i\delta_{l\pm}^{(T)}} , \tag{33}$$

the diagonality in T and T' representing the conservation of total isospin.

Instead of the states $|T\rangle$ we can equally well use the charge states $\pi^- p$ and $\pi^0 n$ as a basis for calculating the S-matrix elements. If we denote the $\pi^- p$ state by $|-\rangle$ and the $\pi^0 n$ state by $|0\rangle$ then we have the relations

$$\begin{aligned} |-\rangle &= \sqrt{\frac{1}{3}}\ |3/2\rangle - \sqrt{\frac{2}{3}}\ |1/2\rangle , \\ |0\rangle &= \sqrt{\frac{2}{3}}\ |3/2\rangle + \sqrt{\frac{1}{3}}\ |1/2\rangle \end{aligned} \tag{34}$$

with the corresponding inverse relations

$$\begin{aligned} |3/2\rangle &= \sqrt{\frac{1}{3}}\ |-\rangle + \sqrt{\frac{2}{3}}\ |0\rangle , \\ |1/2\rangle &= -\sqrt{\frac{2}{3}}\ |-\rangle + \sqrt{\frac{1}{3}}\ |0\rangle . \end{aligned} \tag{35}$$

In this basis we define the S-matrix in the subspace of fixed l and J by

$$C_{l\pm}^{(c')} = -\sum_{c=-,0} S_{l\pm}^{c'c} B_{l\pm}^{(c)} \tag{36}$$

where $B_{l\pm}^{(c)}$ and $C_{l\pm}^{(c')}$ are again the amplitudes of the incoming and outgoing spherical waves. The S-matrix elements defined by equation (36) are related to those of equation (32) by a unitary transformation corresponding to equations (34) and (35). Suppressing the subscripts $l\pm$, these relations are

$$S^{--} = \frac{1}{3} S^{3/2\,3/2} + \frac{2}{3} S^{1/2\,1/2} - \frac{\sqrt{2}}{3} (S^{3/2\,1/2} + S^{1/2\,3/2}) ,$$

$$S^{-0} = \frac{\sqrt{2}}{3} (S^{3/2\,3/2} - S^{1/2\,1/2}) + \frac{1}{3} S^{3/2\,1/2} - \frac{2}{3} S^{1/2\,3/2} ,$$

$$S^{0-} = \frac{\sqrt{2}}{3} (S^{3/2\,3/2} - S^{1/2\,1/2}) - \frac{2}{3} S^{3/2\,1/2} + \frac{1}{3} S^{1/2\,3/2} ,$$

$$S^{00} = \frac{2}{3} S^{3/2\,3/2} + \frac{1}{3} S^{1/2\,1/2} + \frac{\sqrt{2}}{3} (S^{3/2\,1/2} + S^{1/2\,3/2}) \tag{37}$$

with the corresponding inverse relations

$$S^{3/2\,3/2} = \frac{1}{3} S^{--} + \frac{2}{3} S^{00} + \frac{\sqrt{2}}{3} (S^{-0} + S^{0-}) ,$$

$$S^{3/2\,1/2} = -\frac{\sqrt{2}}{3} (S^{--} - S^{00}) + \frac{1}{3} S^{-0} - \frac{2}{3} S^{0-} ,$$

$$S^{1/2\,3/2} = -\frac{\sqrt{2}}{3} (S^{--} - S^{00}) - \frac{2}{3} S^{-0} + \frac{1}{3} S^{0-} ,$$

$$S^{1/2\,1/2} = \frac{2}{3} S^{--} + \frac{1}{3} S^{00} - \frac{\sqrt{2}}{3} (S^{-0} + S^{0-}) . \tag{38}$$

In the case when only the charge independent nuclear force is present, we have seen that $S_{l\pm}^{T'T}$ has the diagonal form given by equation (33) and then equation (37) simplifies to give the purely nuclear charge independent values of $S_{l\pm}^{c'c}$

$$S_{l\pm}^{--} = \frac{1}{3} e^{2i\delta_{l\pm}^{(3/2)}} + \frac{2}{3} e^{2i\delta_{l\pm}^{(1/2)}} \qquad S_{l\pm}^{-0} = S_{l\pm}^{0-} = \frac{\sqrt{2}}{3} (e^{2i\delta_{l\pm}^{(3/2)}} - e^{2i\delta_{l\pm}^{(1/2)}}) ,$$

$$S_{l\pm}^{00} = \frac{2}{3} e^{2i\delta_{l\pm}^{(3/2)}} + \frac{1}{3} e^{2i\delta_{l\pm}^{(1/2)}} . \tag{39}$$

Once the Coulomb potential is introduced total isospin is no longer conserved and the diagonality property of $S_{l\pm}^{T'T}$ is lost. However, even in the general case conservation of probability and time reversal invariance require that the S-matrix is unitary and symmetric so that the relations (37) and (38) simplify a little by making use of the symmetry property

$$S_{l\pm}^{3/2\,1/2} = S_{l\pm}^{1/2\,3/2}, \qquad S_{l\pm}^{-0} = S_{l\pm}^{0-}.$$

We now take into account the Coulomb force between the charged particles. The reduced Coulomb potential can be written as an operator in isospin space in the form

$$\tilde{V}_{op}(\varrho) = \tilde{V}_c(\varrho) \cdot t_3 \cdot \frac{1}{2}\,(1 + \tau_3)$$

where t_3 and $1/2\,\tau_3$ are the 3rd component isospin operators for the pion and the nucleon and where $\tilde{V}_c(\varrho)$ is the reduced Coulomb potential introduced in Section 2.

Operating on the states $|c\rangle$ we obtain

$$\tilde{V}_{op}\,|-\rangle = -\tilde{V}_c(\varrho)\,|-\rangle\,,$$

$$\tilde{V}_{op}\,|\,0\rangle = 0 \tag{40}$$

while on the states $|\,T\rangle$ we obtain

$$\tilde{V}_{op}\,|\,3/2\rangle = -\frac{1}{3}\,\tilde{V}_c(\varrho)\,(|\,3/2\rangle - \sqrt{2}\,|\,1/2\rangle)\,,$$

$$\tilde{V}_{op}\,|\,1/2\rangle = \frac{\sqrt{2}}{3}\,\tilde{V}_c(\varrho)\,(|\,3/2\rangle - \sqrt{2}\,|\,1/2\rangle)\,. \tag{41}$$

We see from equation (41) that the Coulomb interaction mixes the two states of different total isospin. In this case, the simple form of equation (1) no longer separates the Schrödinger equation and we have to introduce the more general ansatz

$$\psi_{l\pm} = [R_{l\pm}^{(3/2)}\,(\varrho)\,|\,3/2\rangle + R_{l\pm}^{(1/2)}\,|\,1/2\rangle]\,\frac{1}{r}\,\Omega_{l\pm,s}(\theta,\phi)\,\cdot \tag{42}$$

The radial wave functions $R_{l\pm}^{(T)}$ then satisfy the coupled equations

$$\frac{d^2 R_{l\pm}^{(3/2)}}{d\varrho^2} + \left(1 - \tilde{U}_{l\pm}^{(3/2)} - \frac{l\,(l+1)}{\varrho^2}\right) R_{l\pm}^{(3/2)} = -\,\tilde{V}_c(\varrho)\left(\frac{1}{3}\,R_{l\pm}^{(3/2)} - \frac{\sqrt{2}}{3}\,R_{l\pm}^{(1/2)}\right),$$

$$\frac{d^2 R_{l\pm}^{(1/2)}}{d\varrho^2} + \left(1 - \tilde{U}_{l\pm}^{(1/2)} - \frac{l\,(l+1)}{\varrho^2}\right) R_{l\pm}^{(1/2)} = \sqrt{2}\,\tilde{V}_c(\varrho)\left(\frac{1}{3}\,R_{l\pm}^{(3/2)} - \frac{\sqrt{2}}{3}\,R_{l\pm}^{(1/2)}\right). \tag{43}$$

Alternatively, we can work in the $-$, 0 basis using the ansatz

$$\psi_{l\pm} = [R_{l\pm}^{(-)}(\varrho)\,|\,-\rangle + R_{l\pm}^{(0)}(\varrho)\,|\,0\rangle]\,\frac{1}{r}\,\Omega_{l\pm,s}(\theta,\phi)$$

in which case the $R_{l\pm}^{(c)}$ satisfy the coupled equations

$$\frac{d^2 R_{l\pm}^{(-)}}{d\varrho^2} + \left(1 - \frac{1}{3}\,\tilde{U}_{l\pm}^{(3/2)} - \frac{2}{3}\,\tilde{U}_{l\pm}^{(1/2} + \tilde{V}_c - \frac{l\,(l+1)}{\varrho^2}\right) R_{l\pm}^{(-)}$$

$$= \frac{\sqrt{2}}{3}\,(\tilde{U}_{l\pm}^{(3/2)} - \tilde{U}_{l\pm}^{(1/2)})\,R_{l\pm}^{(0)}$$

$$\frac{d^2 R_{l\pm}^{(0)}}{d\varrho^2} + \left(1 - \frac{2}{3}\,\tilde{U}_{l\pm}^{(3/2)} - \frac{1}{3}\,\tilde{U}_{l\pm}^{(1/2)} - \frac{l\,(l+1)}{\varrho^2}\right) R_{l\pm}^{(0)} = \frac{\sqrt{2}}{3}\,(\tilde{U}_{l\pm}^{(3/2)} - \tilde{U}_{l\pm}^{(1/2)})\,R_{l\pm}^{(-)}\,. \tag{44}$$

An important feature of the coupled system can be seen from equation (44) where, making use of the fact that the nuclear potentials $\tilde{U}_{l\pm}^{(T)}$ are of short range, we see that these coupled equations reduce asymptotically to the form

$$\left.\begin{aligned} &\frac{d^2 R_{l\pm}^{(-)}}{d\varrho^2} + \left(1 + \tilde{V}_c(\varrho) - \frac{l(l+1)}{\varrho^2}\right) R_{l\pm}^{(-)} = 0 \\ &\frac{d^2 R_{l\pm}^{(0)}}{d\varrho^2} + R_{l\pm}^{(0)} - \frac{l(l+1)}{\varrho^2}\,R_{l\pm}^{(0)} = 0 \end{aligned}\right\}\ \varrho \to \infty\,. \tag{45}$$

Equation (45) shows that, as one would expect, $R_{l\pm}^{(-)}$ has the asymptotic behaviour of a radial Coulomb wave with Coulomb parameter $-\eta$ and $R_{l\pm}^{(0)}$ has the asymptotic behaviour of a radial free wave, i.e.

$$\begin{aligned} R_{l\pm}^{(-)}(\varrho) &\underset{\varrho\to\infty}{\sim} B_{l\pm}^{(-)}\,e^{-i(\varrho - l\pi/2 - \sigma_l + \eta \ln 2\varrho)} + C_{l\pm}^{(-)}\,e^{i(\varrho - l\pi/2 - \sigma_l + \eta \ln 2\varrho)}\,, \\ R_{l\pm}^{(0)}(\varrho) &\underset{\varrho\to\infty}{\sim} B_{l\pm}^{(0)}\,e^{-i(\varrho - l\pi/2)} + C_{l\pm}^{(0)}\,e^{i(\varrho - l\pi/2)}\,. \end{aligned} \tag{46}$$

Since the outgoing amplitudes $C_{l\pm}^{(c')}$ are linear functions of the incoming amplitudes $B_{l\pm}^{(c)}$ we can write

$$C_{l\pm}^{(c')} = -\sum_c S_{l\pm}^{c'c}\,B_{l\pm}^{(c)} \qquad c, c' = -, 0 \tag{47}$$

thus defining the S-matrix in the isospin sub-space in this case when the long range Coulomb force is present in one of the channels.

We now discuss certain general properties of solutions of the coupled set of equations (44). This treatment is similar to that made when tensor forces are included in nucleon-nucleon scattering [15], except that the 'mixing' occurs in isospin space [16] and the long range Coulomb force is present only in the (–) channel. As stated earlier, probability conservation and time reversal invariance require that $S_{l\pm}^{c'c}$ be unitary and symmetric. Such a 2×2 matrix can always be parameterised by three real quantities, two of which are eigenphases, the third being called the mixing parameter. It should be noted that, while the eigenphases do not depend on the representation used, the mixing parameter does depend on the choice of representation. This we fix by writing in the notation of Ref. [15]

$$\begin{aligned} S^{--} &= e^{2i\tau^{(3/2)}}\,\mathrm{Cos}^2\varepsilon + e^{2i\tau^{(1/2)}}\,\mathrm{Sin}^2\varepsilon\,, \\ S^{-0} &= S^{0-} = \mathrm{Sin}\,\varepsilon\,\mathrm{Cos}\,\varepsilon\,(e^{2i\tau^{(3/2)}} - e^{2i\tau^{(1/2)}})\,, \\ S^{00} &= e^{2i\tau^{(3/2)}}\,\mathrm{Sin}^2\varepsilon + e^{2i\tau^{(1/2)}}\,\mathrm{Cos}^2\varepsilon \end{aligned} \tag{48}$$

where we have suppressed the $l\pm$ subscript. Comparing equations (39) and (48) we see that the charge independent limits of the eigenphases are

$$\underset{\eta\to 0}{Lt}\ \tau_{l\pm}^{(3/2)} = \delta_{l\pm}^{(3/2)}\,, \qquad \underset{\eta\to 0}{Lt}\ \tau_{l\pm}^{(1/2)} = \delta_{l\pm}^{(1/2)} \tag{49}$$

while the charge independent limit of the mixing parameter is such that

$$\underset{\eta\to 0}{Lt}\ \mathrm{Sin}\,\varepsilon = \sqrt{\frac{2}{3}}\ , \qquad \underset{\eta\to 0}{Lt}\ \mathrm{Cos}\,\varepsilon = \sqrt{\frac{1}{3}}\ . \tag{50}$$

Returning now to the asymptotic behaviour given by equation (46), we can choose the $B^{(c)}$'s arbitrarily so as to construct two independent solutions of the coupled equations (44). If we choose

$$\begin{aligned} B_{\alpha}^{(-)} &= \mathrm{Cos}\,\varepsilon\,, & B_{\alpha}^{(0)} &= \mathrm{Sin}\,\varepsilon\,, \\ B_{\beta}^{(-)} &= -\ \mathrm{Sin}\,\varepsilon\,, & B_{\beta}^{(0)} &= \mathrm{Cos}\,\varepsilon \end{aligned} \tag{51}$$

where we have again suppressed the $l\pm$ subscript, then we can obtain the corresponding $C^{(c)}$'s by use of equations (47) and (48). After suitable normalisation, the α and β solutions of equation (44) then have the asymptotic form

$$\begin{aligned} R_{\alpha}^{(-)}(\varrho) &\underset{\varrho\to\infty}{\sim} \mathrm{Sin}\left(\varrho - \frac{l\pi}{2} - \sigma_l + \eta\ln 2\varrho + \tau^{(3/2)}\right), \\ R_{\alpha}^{(0)}(\varrho) &\underset{\varrho\to\infty}{\sim} \mathrm{Tan}\,\varepsilon\ \mathrm{Sin}\left(\varrho - \frac{l\pi}{2} + \tau^{(3/2)}\right), \end{aligned} \tag{52}$$

$$\begin{aligned} R_{\beta}^{(-)}(\varrho) &\underset{\varrho\to\infty}{\sim} -\ \mathrm{Tan}\,\varepsilon\ \mathrm{Sin}\left(\varrho - \frac{l\pi}{2} - \sigma_l + \eta\ln 2\varrho + \tau^{(1/2)}\right), \\ R_{\beta}^{(0)}(\varrho) &\underset{\varrho\to\infty}{\sim} \mathrm{Sin}\left(\varrho - \frac{l\pi}{2} + \tau^{(1/2)}\right). \end{aligned} \tag{53}$$

Finally in this section we summarise the derivation of the elastic and charge exchange scattering amplitudes from the corresponding S-matrix elements defined by equation (47). As a first step we consider a single $\pi^- p$ partial wave when only the Coulomb force is present. The asymptotic form of the radial and isotopic part of the wave function is

$$-\frac{1}{2i}\left[e^{-i(\varrho - l\pi/2 - \sigma_l + \eta\ln 2\varrho)} - e^{i(\varrho - l\pi/2 - \sigma_l + \eta\ln 2\varrho)}\right]|-\rangle\,,$$

from which we see that the ingoing amplitudes are given by

$$B_{l\pm}^{(-)} = -\frac{1}{2i}\,, \qquad B_{l\pm}^{(0)} = 0\,.$$

When the nuclear force is also present, only the outgoing amplitudes change from the values implied by the above asymptotic form, the new values being given by equation (47)

$$C_{l\pm}^{(-)} = \frac{1}{2i}\,S_{l\pm}^{--}\,, \qquad C_{l\pm}^{(0)} = \frac{1}{2i}\,S_{l\pm}^{0-}\,.$$

Thus, the asymptotic form for the radial and isotopic part of the wave function in the presence of both Coulomb and nuclear forces is

$$\left[[\text{Sin}\left(\varrho - \frac{l\pi}{2} - \sigma_l + \eta \ln 2\,\varrho\right) + \frac{1}{2\,i} (S_{l\pm}^{--} - 1)\, e^{i(\varrho - l\pi/2 - \sigma_l + \eta \ln 2\varrho)} \right] |-\rangle$$

$$+ \frac{1}{2\,i} S_{l\pm}^{0-}\, e^{i(\varrho - l\pi/2)}\, |0\rangle\,. \qquad (54)$$

Guided by this, we define the partial wave amplitudes

$$f_{l\pm}^{--} = \frac{1}{2\,i} (S_{l\pm}^{--} - 1)\, e^{-2\,i\sigma_l}\,, \qquad f_{l\pm}^{0-} = \frac{1}{2\,i} S_{l\pm}^{0-}\, e^{-i\sigma_l}\,. \qquad (55)$$

We now construct the incoming 'plane' Coulomb wave corresponding to relative momentum $\underset{\sim}{p}$ and nucleon spin component s in the direction of $\underset{\sim}{p}$. Using the same relative spherical polar coordinates as in equation (1) and letting χ_s denote the spin part of the wave function, we can take account of the nucleon spin by suitably modify the usual treatment of the spinless case in the presence of Coulomb forces [14] to obtain the asymptotic expansion

$$\psi_{\underset{\sim}{p},\,s}^{\text{pure Coulomb}} \underset{\varrho\to\infty}{\sim} \sum_l i^l\, (2\,l + 1)^{1/2}\, \sqrt{4\pi}\, e^{-i\sigma_l}\, \frac{\text{Sin}\left(\varrho - \frac{l\pi}{2} - \sigma_l + \eta \ln 2\,\varrho\right)}{\varrho}\, Y_l^0(\theta, \phi)\, \chi_s\,.$$

If we define

$$\Omega_{l\pm,s}(\theta, \phi) = \sum_{m,\,s'} \left(l\, \frac{1}{2}\, m\, s' \mid l \pm 1/2\, s\right) Y_l^m(\theta, \phi)\, \chi_{s'}$$

we can use the inverse relation

$$Y_l^m(\theta, \phi)\, \chi_s = \sum_{s'} \left[\left(l\, \frac{1}{2}\, m\, s \mid l + \frac{1}{2}\, s'\right) \Omega_{l+,s'}(\theta, \phi) + \left(l\, \frac{1}{2}\, m\, s \mid l - \frac{1}{2}\, s'\right) \Omega_{l-,s'}\,(\theta, \phi) \right]$$

to write

$$\psi_{\underset{\sim}{p},\,s}^{\text{pure Coulomb}} \underset{\varrho\to\infty}{\sim} \sum_l i^l\, (2\,l + 1)^{1/2}\, \sqrt{4\,\pi}\, e^{-i\sigma_l}\, \frac{\text{Sin}\left(\varrho - \frac{l\pi}{2} - \sigma_l + \eta \ln 2\,\varrho\right)}{\varrho}$$

$$\times [(l\,1/2\,0\,s \mid l + 1/2\,s)\, \Omega_{l+,s}\,(\theta, \phi) + (l\,1/2\,0\,s \mid l - 1/2\,s)\, \Omega_{l-,s}\,(\theta, \phi)]\,. \qquad (56)$$

Using this expansion together with the asymptotic behaviour given by equation (54) and the partial waves defined by equation (55), we can then write the asymptotic behaviour of the total wave function in the presence of both Coulomb and nuclear forces for a $\pi^- p$ incoming state with relative momentum p and nucleon spin component s along $\underset{\sim}{p}$ as

$$\psi_{\underset{\sim}{p},\,s}^{\text{TOTAL}} \underset{\varrho\to\infty}{\sim} \psi_{\underset{\sim}{p},\,s}^{\text{pure Coulomb}}\, |-\rangle + \sum_l (2\,l + 1)^{1/2}\, (4\,\pi)^{1/2}\, \frac{e^{i(\varrho + \eta \ln 2\varrho)}}{\varrho}$$

$$\times \left\{ f_{l+}^{--} \left(l\, \frac{1}{2}\, 0\, s \mid l + 1/2\, s\right) \Omega_{l+,s} + f_{l-}^{--} \left(l\, \frac{1}{2}\, 0\, s \mid l - 1/2\, s\right) \Omega_{l-,s} \right\} |-\rangle$$

$$+ \sum_l (2\,l+1)^{1/2}\,(4\,\pi)^{1/2}\,\frac{e^{i\varrho}}{\varrho}\left\{ f^{0-}_{l+}\left(l\,\frac{1}{2}\,0\,s\mid l+1/2\,s\right)\Omega_{l+,s} + f^{0-}_{l-}\left(l\,\frac{1}{2}\,0\,s\mid l-1/2\,s\right)\Omega_{l-,s}\right\}\,|0\rangle\,. \qquad (57)$$

Finally from this asymptotic form we have to extract the $\pi^- p$ elastic scattering amplitude F^{--} and the charge exchange scattering amplitude F^{0-}. To achieve this, the scattering amplitude F_{c_p} for the case of pure point charge Coulomb scattering has to be isolated from $\psi^{\text{pure Coulomb}}_{\tilde{p},\,s}$ and one has to make the usual treatment of the logarithmic term in the exponent of the outgoing $\pi^- p$ spherical wave [14]. In this way we obtain the amplitudes of the spherical outgoing waves for an incoming nucleon with spin component s as

$$F^{--}_s = F_{c_p} + \sum_l \frac{(2\,l+1)^{1/2}\,(4\,\pi)^{1/2}}{k}\left\{ f^{--}_{l+}\left(l\,\frac{1}{2}\,0\,s\mid l+\frac{1}{2}\,s\right)\Omega_{l+,s} + f^{--}_{l-}\left(l\,\frac{1}{2}\,0\,s\mid l-\frac{1}{2}\,s\right)\Omega_{l-,s}\right\},$$

$$F^{0-}_s = \sum_l \frac{(2\,l+1)^{1/2}\,(4\,\pi)^{1/2}}{k}\left\{ f^{0-}_{l+}\left(l\,\frac{1}{2}\,0\,s\mid l+\frac{1}{2}\,s\right)\Omega_{l+,s} + f^{0-}_{l-}\left(l\,\frac{1}{2}\,0\,s\mid l-\frac{1}{2}\,s\right)\Omega_{l-,s}\right\}. \qquad (58)$$

The treatment of F^{--}_s and F^{0-}_s now proceeds as in the usual derivation of the scattering amplitude for spin 1/2 – spin 0 scattering. The resulting scattering amplitude written as a matrix in spin space has the form

$$F^{c'-} = f^{c'-} + i\,\tilde{\sigma}\,.\,\tilde{\hat{n}}\,g^{c'-}\,,\qquad c' = -, 0 \qquad (59)$$

where

$$f^{c'-} = \delta_{c'-}\,F_{c_p} + 1/k \sum_l \left[(l+1)\,f^{c'-}_{l+} + l\,f^{c-}_{l-}\right] P_l\,(\mathrm{Cos}\,\theta)\,,$$

$$g^{c'-} = 1/k \sum_l \left[f^{c'-}_{l+} - f^{c'-}_{l-}\right] P^1_l\,(\mathrm{Cos}\,\theta)\,. \qquad (60)$$

and where $\tilde{\hat{n}} = \tilde{\hat{q}} \times \tilde{\hat{q}}'$ is the normal to the scattering plane, $\tilde{\hat{q}}$ being a unit vector along the incoming pion direction and $\tilde{\hat{q}}'$ being a unit vector along the outgoing pion direction. Slight changes in equation (60) occur when magnetic moment and relativistic effects are included. These are reviewed by Roper [2] and in particular it should be noted that Solmitz [17] terms give a pure electromagnetic contribution to the $\pi^\pm p \to \pi^\pm p$ spin flip amplitude.

4. Coulomb Corrections for $\pi^- p$ Elastic and Charge Exchange Scattering

a) *Purely Nuclear Scattering*

For simplicity, as in Section 2, we treat in detail the s-wave state although again we give general l results where appropriate. In the absence of Coulomb forces, charge independence implies that the radial wave function for the $\pi^- p$ system is given by

$$R_-^N(\varrho) = \sqrt{\frac{1}{3}}\, R_3^N(\varrho) - \sqrt{\frac{2}{3}}\, R_1^N(\varrho)$$

and that the radial wave function for the $\pi^0 n$ system is given by

$$R_0^N(\varrho) = \sqrt{\frac{2}{3}}\, R_3^N(\varrho) + \sqrt{\frac{1}{3}}\, R_1^N(\varrho)$$

where we are using the simplified notation

$$R_{2T}^N(\varrho) \equiv R_{0+}^{(T)}(\varrho)\ .$$

$R_3^N(\varrho)$ satisfies equations (4) and (5) and $R_1^N(\varrho)$ satisfies the corresponding isospin 1/2 equations

$$\frac{d^2 R_1^N(\varrho)}{d\varrho^2} + (1 - \tilde{U}_1(\varrho))\, R_1^N(\varrho) = 0 \quad \varrho \leqslant \varrho_N\ , \tag{61}$$

$$\frac{d^2 R_1^N(\varrho)}{d\varrho^2} + R_1^N(\varrho) = 0\ , \qquad \varrho \geqslant \varrho_N \tag{62}$$

where $\tilde{U}_1(\varrho) \equiv \tilde{U}_{0+}^{(1/2)}(\varrho)$ is the reduced s-wave isospin 1/2 nuclear potential which we also assume to be of finite range ϱ_N. In the case when the isospin 3/2 and 1/2 nuclear potentials are of different ranges we take ϱ_N as the maximum of the two values. We introduce a solution of equations (61) and (62), $R_{1\beta}^N$, normalised so that

$$R_{1\beta}^N(\varrho) = \operatorname{Sin}(\varrho + \delta_1)\ , \qquad \varrho \geqslant \varrho_N\ , \tag{63}$$

$\delta_1 \equiv \delta_{0+}^{(1/2)}$ being the $T = 1/2$ s-wave purely nuclear phase shift. Just as in the case of $R_{3\alpha}^N$, imposing the regularity condition $R_{1\beta}^N(0) = 0$ specifies δ_1 in terms of $\tilde{U}_1$.

In Section 3, we have seen that in the 3/2, 1/2 representation the s-wave purely nuclear S-matrix is diagonal with the elements

$$S_N^{3/2\,3/2} = e^{2i\delta_3}\ , \qquad S_N^{1/2\,1/2} = e^{2i\delta_1}\ .$$

In equation (37) we showed how a suitable unitary transformation gives the corresponding S-matrix elements in the $-$, 0 basis. However, in order to introduce the technique we will use when the Coulomb interaction is present, we will directly derive the purely nuclear S-matrix elements in the $-$, 0 basis.

We see from equations (44) and (45) that, in the absence of the Coulomb potential, $R_-^N(\varrho)$ and $R_0^N(\varrho)$ satisfy the coupled equations

$$\frac{d^2 R_-^N(\varrho)}{d\varrho^2} + \left(1 - \frac{1}{3}\,\tilde{U}_3 - \frac{2}{3}\,\tilde{U}_1\right) R_-^N(\varrho) = \frac{\sqrt{2}}{3}\,(\tilde{U}_3 - \tilde{U}_1)\, R_0^N(\varrho)\ ,$$

$$\varrho \leqslant \varrho_N$$

$$\frac{d^2 R_0^N(\varrho)}{d\varrho^2} + \left(1 - \frac{2}{3}\,\tilde{U}_3 - \frac{1}{3}\,\tilde{U}_1\right) R_0^N(\varrho) = \frac{\sqrt{2}}{3}\,(\tilde{U}_3 - \tilde{U}_1)\; R_-^N(\varrho) \tag{64}$$

$$\frac{d^2 R_-^N(\varrho)}{d\varrho^2} + R_-^N(\varrho) = 0\;, \qquad \frac{d^2 R_0^N(\varrho)}{d\varrho^2} + R_0^N(\varrho) = 0\;, \qquad \varrho \geqslant \varrho_N\,. \tag{65}$$

These coupled equations have two independent sets of regular solutions which we can construct from $R_{3\alpha}^N$ and $R_{1\beta}^N$. We choose the α solution to correspond to $R_3^N = R_{3\alpha}^N$ and $R_1^N = 0$ so that

$$R_{-\alpha}^N = \sqrt{\frac{1}{3}}\; R_{3\alpha}^N\,, \qquad R_{0\alpha}^N = \sqrt{\frac{2}{3}}\; R_{3\alpha}^N \tag{66}$$

and we choose the β solution to correspond to $R_3^N = 0$, $R_1^N = R_{1\beta}^N$, so that

$$R_{-\beta}^N = -\sqrt{\frac{2}{3}}\; R_{1\beta}^N\,, \qquad R_{0\beta}^N = \sqrt{\frac{1}{3}}\; R_{1\beta}^N\,. \tag{67}$$

The most general regular solution of equations (64) and (65) can be written as

$$\begin{Bmatrix} R_-^N(\varrho) \\ R_0^N(\varrho) \end{Bmatrix} = A_\alpha \begin{Bmatrix} R_{-\alpha}^N(\varrho) \\ R_{0\alpha}^N(\varrho) \end{Bmatrix} + A_\beta \begin{Bmatrix} R_{-\beta}^N(\varrho) \\ R_{0\beta}^N(\varrho) \end{Bmatrix}$$

and has the asymptotic form

$$\begin{Bmatrix} R_-^N(\varrho) \\ R_0^N(\varrho) \end{Bmatrix} \underset{\varrho\to\infty}{\sim} \begin{Bmatrix} \sqrt{\frac{1}{3}}\; A_\alpha\, e^{i\delta_3} - \sqrt{\frac{2}{3}}\; A_\beta\, e^{i\delta_1} \\ \sqrt{\frac{2}{3}}\; A_\alpha\, e^{i\delta_3} + \sqrt{\frac{1}{3}}\; A_\beta\, e^{i\delta_1} \end{Bmatrix} \frac{e^{i\varrho}}{2i}$$
$$- \begin{Bmatrix} \sqrt{\frac{1}{3}}\; A_\alpha\, e^{-i\delta_3} - \sqrt{\frac{2}{3}}\; A_\beta\, e^{-i\delta_1} \\ \sqrt{\frac{2}{3}}\; A_\alpha\, e^{-i\delta_3} + \sqrt{\frac{1}{3}}\; A_\beta\, e^{-i\delta_1} \end{Bmatrix} \frac{e^{-i\varrho}}{2i} \tag{68}$$

where A_α and A_β are arbitrary constants. However, we have seen in Section 3 that the purely nuclear S-matrix $S_{0+}^{c'c}$ relates the amplitudes of the incoming and outgoing spherical waves. Taking these amplitudes from equation (68) and using equation (36) we obtain again the relations given in equation (39)

$$S_N^{--} = \frac{1}{3}\; e^{2i\delta_3} + \frac{2}{3}\; e^{2i\delta_1} \qquad S_N^{-0} = S_N^{0-} = \frac{\sqrt{2}}{3}\;(e^{2i\delta_3} - e^{2i\delta_1})$$
$$S_N^{00} = \frac{2}{3}\; e^{2i\delta_3} + \frac{1}{3}\; e^{2i\delta_1}\,. \tag{69}$$

b) *Inner Coulomb Corrections*

We now introduce the reduced Coulomb potential $\tilde{V}_c(\varrho)$ and first consider the inner part of this potential $\tilde{V}_c^{IN}(\varrho)$. In this case it is convenient to work in the 3/2, 1/2 basis and we see from equation (43) that the s-wave radial wave functions in the presence of the nuclear and the inner Coulomb potentials satisfy the equations

$$\frac{d^2 R_3^{IN}(\varrho)}{d\varrho^2} + (1 - \tilde{U}_3(\varrho))\, R_3^{IN}(\varrho) = - \tilde{V}_c^{IN}(\varrho) \left(\frac{1}{3} R_3^{IN}(\varrho) - \frac{\sqrt{2}}{3} R_1^{IN}(\varrho) \right),$$

$$\varrho \leqslant \varrho_0 \qquad (70)$$

$$\frac{d^2 R_1^{IN}(\varrho)}{d\varrho^2} + (1 - \tilde{U}_1(\varrho))\, R_1^{IN}(\varrho) = \sqrt{2}\, \tilde{V}_c^{IN}(\varrho) \left(\frac{1}{3} R_3^{IN}(\varrho) - \frac{\sqrt{2}}{3} R_1^{IN}(\varrho) \right),$$

$$\left.\begin{aligned} &\frac{d^2 R_3^{IN}(\varrho)}{d\varrho^2} + R_3^{IN}(\varrho) = 0\,, \\ &\frac{d^2 R_1^{IN}(\varrho)}{d\varrho^2} + R_1^{IN}(\varrho) = 0\,, \end{aligned}\right\} \quad \varrho \geqslant \varrho_0 \qquad (71)$$

Comparing equation (71) with equations (5) and (62) we see that for $\varrho \geqslant \varrho_0$ R_3^{IN} and R_1^{IN} obey the same equation as R_3^N and R_1^N. Hence, provided δ_3 and δ_1 are not equal, $R_{3\alpha}^N$ and $R_{1\beta}^N$ provide two independent solutions from which R_3^{IN} and R_1^{IN} can be constructed. Thus we write

$$\left.\begin{aligned} R_3^{IN}(\varrho) &= \alpha_3\, R_{3\alpha}^N(\varrho) + \beta_3\, R_{1\beta}^N(\varrho)\,, \\ R_1^{IN}(\varrho) &= \alpha_1\, R_{3\alpha}^N(\varrho) + \beta_1\, R_{1\beta}^N(\varrho)\,, \end{aligned}\right\} \quad \varrho \geqslant \varrho_0 \qquad (72)$$

where α_3, β_3, α_1 and β_1 are arbitrary constants. We first derive implicit relations between these constants. Using equations (4) and (70) we obtain

$$\frac{d}{d\varrho} \left[R_3^N(\varrho) \frac{d}{d\varrho} R_3^{IN}(\varrho) - R_3^{IN}(\varrho) \frac{d}{d\varrho} R_3^N(\varrho) \right]$$

$$= - \tilde{V}_c^{IN}(\varrho)\, R_3^N(\varrho) \left[\frac{1}{3} R_3^{IN}(\varrho) - \frac{\sqrt{2}}{3} R_1^{IN}(\varrho) \right]$$

which on using the regularity condition at $\varrho = 0$ gives

$$\left[R_3^N(\varrho) \frac{d}{d\varrho} R_3^{IN}(\varrho) - R_3^{IN}(\varrho) \frac{d}{d\varrho} R_3^N(\varrho) \right]_{\varrho = \varrho_0}$$

$$= - \int_0^{\varrho_0} \tilde{V}_c^{IN}(\varrho)\, R_3^N(\varrho) \left[\frac{1}{3} R_3^{IN}(\varrho) - \frac{\sqrt{2}}{3} R_1^{IN}(\varrho) \right] d\varrho\,.$$

Finally, using equations (6), (63) and (72), this gives

$$\beta_3 \operatorname{Sin}(\delta_3 - \delta_1) = - \int_0^{\varrho_0} \tilde{V}_c^{IN}(\varrho)\, R_{3\alpha}^N(\varrho) \left[\frac{1}{3} R_3^{IN}(\varrho) - \frac{\sqrt{2}}{3} R_1^{IN}(\varrho) \right] d\varrho \equiv \chi_3\,. \qquad (73)$$

In a similar way we obtain

$$\alpha_1 \operatorname{Sin}(\delta_3 - \delta_1) = -\sqrt{2} \int_0^{\varrho_0} \tilde{V}_c^{IN}(\varrho)\, R_{1\beta}^N(\varrho) \left[\frac{1}{3} R_3^{IN}(\varrho) - \frac{\sqrt{2}}{3} R_1^{IN}(\varrho)\right] d\varrho \equiv \chi_1. \quad (74)$$

Equations (73) and (74) are only used in the perturbation treatment of Section 5.

Returning now to equation (72), we can choose α_3 and β_1 so as to construct two independent sets of regular solutions of equations (70) and (71). In the same spirit as in the purely nuclear case we choose the α solution to correspond to $\alpha_3 = 1$, $\beta_1 = 0$ giving

$$\left.\begin{aligned} R_{3\alpha}^{IN} &= R_{3\alpha}^N + \frac{\chi_{3\alpha}}{\operatorname{Sin}(\delta_3 - \delta_1)} R_{1\beta}^N, \\ R_{1\alpha}^{IN} &= \frac{\chi_{1\alpha}}{\operatorname{Sin}(\delta_3 - \delta_1)} R_{3\alpha}^N \end{aligned}\right\} \quad \varrho \geqslant \varrho_0 \quad (75)$$

where $\chi_{3\alpha}$ and $\chi_{1\alpha}$ are given by the implicit relations (73) and (74) with $R_{3\alpha}^{IN}$ and $R_{1\alpha}^{IN}$ substituted for R_3^{IN} and R_1^{IN}. Similarly, we choose the β solution to correspond to $\alpha_3 = 0$, $\beta_1 = 1$ giving

$$\left.\begin{aligned} R_{3\beta}^{IN} &= \frac{\chi_{3\beta}}{\operatorname{Sin}(\delta_3 - \delta_1)} R_{1\beta}^N, \\ R_{1\beta}^{IN} &= R_{1\beta}^N + \frac{\chi_{1\beta}}{\operatorname{Sin}(\delta_3 - \delta_1)} R_{3\alpha}^N. \end{aligned}\right\} \quad \varrho \geqslant \varrho_0 \quad (76)$$

The most general regular solution of equations (70) and (71) can now be written in the form

$$\begin{Bmatrix} R_3^{IN}(\varrho) \\ R_1^{IN}(\varrho) \end{Bmatrix} = A_\alpha \begin{Bmatrix} R_{3\alpha}^{IN}(\varrho) \\ R_{1\alpha}^{IN}(\varrho) \end{Bmatrix} + A_\beta \begin{Bmatrix} R_{3\beta}^{IN}(\varrho) \\ R_{1\beta}^{IN}(\varrho) \end{Bmatrix}$$

where A_α and A_β are arbitrary constants. Proceeding as before, using equations (75) and (76) together with the known asymptotic forms of $R_{3\alpha}^N$ and $R_{1\beta}^N$, we can obtain in the 3/2, 1/2 basis the s-wave S-matrix elements in the presence of the nuclear and inner Coulomb potential. The values are

$$S_{IN}^{3/2\,3/2} = \left[e^{2i\delta_3} + \frac{\chi_{3\alpha} + \chi_{1\beta}}{\operatorname{Sin}(\delta_3 - \delta_1)} e^{i(\delta_3+\delta_1)} + \frac{\chi_{3\alpha}\chi_{1\beta} - \chi_{3\beta}\chi_{1\alpha}}{\operatorname{Sin}^2(\delta_3 - \delta_1)} e^{2i\delta_1}\right] D_{IN}^{-1}$$

$$S_{IN}^{3/2\,1/2} = -2i\,\chi_{3\beta}\, e^{i(\delta_3+\delta_1)} \cdot D_{IN}^{-1}, \qquad S_{IN}^{1/2\,3/2} = 2i\,\chi_{1\alpha}\, e^{i(\delta_3+\delta_1)} \cdot D_{IN}^{-1}$$

$$S_{IN}^{1/2\,1/2} = \left[e^{2i\delta_1} + \frac{\chi_{3\alpha} + \chi_{1\beta}}{\operatorname{Sin}(\delta_3 - \delta_1)} e^{i(\delta_3+\delta_1)} + \frac{\chi_{3\alpha}\chi_{1\beta} - \chi_{3\beta}\chi_{1\alpha}}{\operatorname{Sin}^2(\delta_3 - \delta_1)} e^{2i\delta_3}\right] D_{IN}^{-1} \quad (77)$$

where

$$D_{IN} = 1 + \frac{\chi_{3\alpha}\, e^{i(\delta_3-\delta_1)} + \chi_{1\beta}\, e^{-i(\delta_3-\delta_1)}}{\operatorname{Sin}(\delta_3 - \delta_1)} + \frac{\chi_{3\alpha}\chi_{1\beta} - \chi_{3\beta}\chi_{1\alpha}}{\operatorname{Sin}^2(\delta_3 - \delta_1)}. \quad (78)$$

Thus, these S-matrix elements are specified completely in terms of the two nuclear phases δ_3 and δ_1 and the four constants $\chi_{3\alpha}$, $\chi_{1\alpha}$, $\chi_{3\beta}$ and $\chi_{1\beta}$. These constants can be obtained by solving the coupled equations (70) subject to the boundary conditions (75) or (76), adjusting $\chi_{3\alpha}$ and $\chi_{1\alpha}$ or $\chi_{3\beta}$ and $\chi_{1\beta}$ until the regularity condition is satisfied. Alternatively, equations (73) and (74) can be used to give perturbation expressions for these constants.

The form of these S-matrix elements is unchanged in the case of general l and again perturbation expressions for the χ's can be obtained from the general l equivalents of equations (73) and (74). These have the same form except that the general l radial wave functions must be substituted for the s-wave radial wave functions.

c) *Full Coulomb Corrections*

Finally, we consider the case when, in addition to the nuclear potential, the full Coulomb potential is present. We now work in the $-$, 0 basis and we see from equation (44) that the s-wave radial wave functions obey the equations

$$\frac{d^2R_-^{TOT}(\varrho)}{d\varrho^2} + \left(1 - \frac{1}{3}\tilde{U}_3 - \frac{2}{3}\tilde{U}_1 + \tilde{V}_c^{IN}\right) R_-^{TOT}(\varrho) = \frac{\sqrt{2}}{3}(\tilde{U}_3 - \tilde{U}_1)\, R_0^{TOT}(\varrho)\ ,$$

$$\varrho \leqslant \varrho_0 \qquad (79)$$

$$\frac{d^2R_0^{TOT}(\varrho)}{d\varrho^2} + \left(1 - \frac{2}{3}\tilde{U}_3 - \frac{1}{3}\tilde{U}_1\right) R_0^{TOT}(\varrho) = \frac{\sqrt{2}}{3}(\tilde{U}_3 - \tilde{U}_1)\, R_-^{TOT}(\varrho)\ ,$$

$$\frac{d^2R_-^{TOT}(\varrho)}{d\varrho^2} + (1 + \tilde{V}_{c_p})\, R_-^{TOT}(\varrho) = 0\ ,$$

$$\varrho \geqslant \varrho_0 \qquad (80)$$

$$\frac{d^2R_0^{TOT}(\varrho)}{d\varrho^2} + R_0^{TOT}(\varrho) = 0\ .$$

Transforming equation (70) into the corresponding equations for R_-^{IN} and R_0^{IN} and comparing with equation (79), we see that, for $\varrho \leqslant \varrho_0$, R_-^{TOT} and R_0^{TOT} are solutions of the same set of coupled equations as R_-^{IN} and R_0^{IN}. Thus, transforming $R_{3\alpha}^{IN}, R_{3\beta}^{IN}$, $R_{1\alpha}^{IN}$, $R_{1\beta}^{IN}$, to the charge basis we can write

$$\left\{\begin{matrix} R_0^{TOT}(\varrho) \\ R_0^{TOT}(\varrho) \end{matrix}\right\} = A_\alpha \left\{\begin{matrix} R_{-\alpha}^{IN}(\varrho) \\ R_{0\alpha}^{IN}(\varrho) \end{matrix}\right\} + A_\beta \left\{\begin{matrix} R_{-\beta}^{IN}(\varrho) \\ R_{0\beta}^{IN}(\varrho) \end{matrix}\right\} \qquad \varrho \leqslant \varrho_0 \qquad (81)$$

where A_α and A_β are arbitrary constants.

If we now transform equation (71) into the corresponding equations for R_-^{IN}, R_0^{IN} and compare with equation (80), we see that R_0^{TOT} also satisfies the same equation as R_0^{IN} for $\varrho \geqslant \varrho_0$.

Thus, we can write

$$R_0^{TOT}(\varrho) = A'_\alpha\, R_{0\alpha}^{IN}(\varrho) + A'_\beta\, R_{0\beta}^{IN}(\varrho)\ . \qquad \varrho \geqslant \varrho_0 \qquad (82)$$

Matching value and derivative of R_0^{TOT} at ϱ_0, using equations (81) and (82) we then obtain

$$A'_\alpha = A_\alpha, \qquad A'_\beta = A_\beta\ .$$

Finally, we consider the Coulomb equation (with Coulomb parameter $-\eta$)

$$\frac{d^2u(\varrho)}{d\varrho^2} + (1 + \tilde{V}_{c_p})\, u(\varrho) = 0 \tag{83}$$

and define two independent solutions u_α and u_β such that

$$\left.\frac{u'_\alpha(\varrho)}{u_\alpha(\varrho)}\right|_{\varrho=\varrho_0} = \left.\frac{R^{IN\prime}_{-\alpha}(\varrho)}{R^{IN}_{-\alpha}(\varrho)}\right|_{\varrho=\varrho_0}$$

$$\left.\frac{u'_\beta(\varrho)}{u_\beta(\varrho)}\right|_{\varrho=\varrho_0} = \left.\frac{R^{IN\prime}_{-\beta}(\varrho)}{R^{IN}_{-\beta}(\varrho)}\right|_{\varrho=\varrho_0} \tag{84}$$

and with asymptotic behaviour

$$u_\alpha(\varrho) \underset{\varrho\to\infty}{\sim} \operatorname{Sin}(\varrho - \sigma_0 + \eta \ln 2\,\varrho + \tau_\alpha)\,, \tag{85}$$

$$u_\beta(\varrho) \underset{\varrho\to\infty}{\sim} \operatorname{Sin}(\varrho - \sigma_0 + \eta \ln 2\,\varrho + \tau_\beta)\,.$$

Comparing equations (80) and (83), we see that R^{TOT}_- satisfies the same equation as u for $\varrho \geqslant \varrho_0$ and so we can write

$$R^{TOT}_-(\varrho) = A''_\alpha\, u_\alpha(\varrho) + A''_\beta\, u_\beta(\varrho) \qquad \varrho \geqslant \varrho_0 \tag{86}$$

Defining constants X_α and X_β by

$$u_\alpha(\varrho_0) = X_\alpha^{-1}\, R^{IN}_{-\alpha}(\varrho_0)\,, \qquad u_\beta(\varrho_0) = X_\beta^{-1}\, R^{IN}_{-\beta}(\varrho_0)\,, \tag{87}$$

we can match value and derivative of R^{TOT}_- at ϱ_0. Using equations (81), (84), (86) and (87) we obtain

$$A''_\alpha = X_\alpha\, A_\alpha\,, \qquad A''_\beta = X_\beta\, A_\beta\,.$$

In this way, we have obtained the general regular solution of the coupled equations (79) and (80) in the form

$$\begin{Bmatrix} R^{TOT}_-(\varrho) \\ R^{TOT}_0(\varrho) \end{Bmatrix} = A_\alpha \begin{Bmatrix} X_\alpha\, u_\alpha(\varrho) \\ R^{IN}_{0\alpha}(\varrho) \end{Bmatrix} + A_\beta \begin{Bmatrix} X_\beta\, u_\beta(\varrho) \\ R^{IN}_{0\beta}(\varrho) \end{Bmatrix} \qquad \varrho \geqslant \varrho_0 \tag{88}$$

Using the known asymptotic forms of u_α, $R^{IN}_{0\alpha}$, u_β and $R^{IN}_{0\beta}$ we can then obtain the amplitudes of the incoming and outgoing radial Coulomb and free waves. However, these are related via equation (47) and so we can obtain the s-wave S-matrix elements in the $-$, 0 basis. The values are

$$S^{--}_{TOT} = \Bigg[X_\alpha\, e^{i(\tau_\alpha+\delta_3)} \left\{\sqrt{\frac{1}{3}} + \frac{\sqrt{2/3}\,\chi_{3\beta} + \sqrt{1/3}\,\chi_{1\beta}\, e^{-i(\delta_3-\delta_1)}}{\operatorname{Sin}(\delta_3-\delta_1)}\right\}$$

$$- X_\beta\, e^{i(\tau_\beta+\delta_1)} \left\{\sqrt{\frac{2}{3}} + \frac{\sqrt{1/3}\,\chi_{1\alpha} + \sqrt{2/3}\,\chi_{3\alpha}\, e^{i(\delta_3-\delta_1)}}{\operatorname{Sin}(\delta_3-\delta_1)}\right\}\Bigg] D^{-1}_{TOT}\,,$$

$$S^{-0}_{TOT} = -\,2\,i\,X_\alpha\,X_\beta\,\operatorname{Sin}(\tau_\alpha-\tau_\beta)\,e^{i(\delta_3+\delta_1)}\,D^{-1}_{TOT} \tag{89}$$

$$S^{0-}_{TOT} = \Bigg[\left\{\sqrt{\frac{1}{3}} + \frac{\sqrt{2/3}\,\chi_{3\beta} + \sqrt{1/3}\,\chi_{1\beta}\, e^{-i(\delta_3-\delta_1)}}{\operatorname{Sin}(\delta_3-\delta_1)}\right\} \left\{\sqrt{\frac{2}{3}} + \frac{\sqrt{1/3}\,\chi_{1\alpha} + \sqrt{2/3}\,\chi_{3\alpha}\, e^{-i(\delta_3-\delta_1)}}{\operatorname{Sin}(\delta_3-\delta_1)}\right\} e^{2i\delta_3}$$
$$- \left\{\sqrt{\frac{2}{3}} + \frac{\sqrt{1/3}\,\chi_{1\alpha} + \sqrt{2/3}\,\chi_{3\alpha}\, e^{i(\delta_3-\delta_1)}}{\operatorname{Sin}(\delta_3-\delta_1)}\right\} \left\{\sqrt{\frac{1}{3}} + \frac{\sqrt{2/3}\,\chi_{3\beta} + \sqrt{1/3}\,\chi_{1\beta}\, e^{i(\delta_3-\delta_1)}}{\operatorname{Sin}(\delta_3-\delta_1)}\right\} e^{2i\delta_1}\Bigg] D^{-1}_{TOT}$$

$$S^{00}_{TOT} = \Bigg[X_\alpha\, e^{-i(\tau_\alpha-\delta_3)} \left\{\sqrt{\frac{1}{3}} + \frac{\sqrt{2/3}\,\chi_{3\beta} + \sqrt{1/3}\,\chi_{1\beta}\, e^{i(\delta_3-\delta_1)}}{\operatorname{Sin}(\delta_3-\delta_1)}\right\} e^{2i\delta_1}$$
$$- X_\beta\, e^{-i(\tau_\beta-\delta_1)} \left\{\sqrt{\frac{2}{3}} + \frac{\sqrt{1/3}\,\chi_{1\alpha} + \sqrt{2/3}\,\chi_{3\alpha}\, e^{-i(\delta_3-\delta_1)}}{\operatorname{Sin}(\delta_3-\delta_1)}\right\} e^{2i\delta_3}\Bigg] D^{-1}_{TOT}$$

where

$$D_{TOT} = X_\alpha\, e^{-i(\tau_\alpha-\delta_3)} \left\{\sqrt{\frac{1}{3}} + \frac{\sqrt{2/3}\,\chi_{3\beta} + \sqrt{1/3}\,\chi_{1\beta}\, e^{-i(\delta_3-\delta_1)}}{\operatorname{Sin}(\delta_3-\delta_1)}\right\}$$
$$- X_\beta\, e^{-i(\tau_\beta-\delta_1)} \left\{\sqrt{\frac{2}{3}} + \frac{\sqrt{1/3}\,\chi_{1\alpha} + \sqrt{2/3}\,\chi_{3\alpha}\, e^{i(\delta_3-\delta_1)}}{\operatorname{Sin}(\delta_3-\delta_1)}\right\}. \tag{90}$$

Thus, the s-wave S-matrix elements in the simultaneous presence of the nuclear and full Coulomb potential are specified by the purely nuclear phases δ_3 and δ_1, the four χ's related to the inner part of the Coulomb potential and the constants X_α, X_β and phases τ_α and τ_β defined by equations (84), (85) and (87). Similar results are obtained in the general l case, the S-matrix elements simply being obtained by substituting the general l equivalents of the χ's, X's and τ's.

We show in Appendix I how these results go over into the results of Van Hove [4] in the limit of vanishing inner Coulomb potential and in Appendix II we demonstrate explicitly that the matrix elements given by equations (89) and (90) correspond to a symmetric unitary matrix.

5. Perturbation Expressions for the Multichannel Coulomb Corrections

In equations (89) and (90) we have obtained expressions for the s-wave S-matrix elements when both the nuclear and Coulomb potentials are present. To evaluate these matrix elements we need values for the χ's, X's and τ's as well as the two purely nuclear phases δ_3 and δ_1. As described earlier, the χ's can be obtained by solving numerically equation (70). Alternatively perturbation expressions can be obtained by substituting the charge independent limiting values of R_3^{IN} and R_1^{IN} into equations (73) and (74). In this case, to first order in the Coulomb parameter, we obtain

$$\chi_{3\alpha} = -\frac{1}{3}\int_0^{\varrho_0} \tilde{V}_c^{IN}(\varrho)\,(R_{3\alpha}^N(\varrho))^2\,d\varrho + 0(\eta^2)\,,$$

$$\chi_{3\beta} = -\chi_{1\alpha} = \frac{\sqrt{2}}{3}\int_0^{\varrho_0} \tilde{V}_c^{IN}(\varrho)\,R_{3\alpha}^N(\varrho)\,R_{1\beta}^N(\varrho)\,d\varrho + 0(\eta^2)\,,$$

$$\chi_{1\beta} = \frac{2}{3}\int_0^{\varrho_0} \tilde{V}_c^{IN}(\varrho)\,(R_{1\beta}^N(\varrho))^2\,d\varrho + 0(\eta^2)\,. \tag{91}$$

Using these values together with equations (75) and (76) we can calculate $R_{-\alpha}^{IN}$ and $R_{-\beta}^{IN}$ from

$$\left.\begin{aligned} R_{-\alpha}^{IN} &= \sqrt{\frac{1}{3}}\;R_{3\alpha}^N + \frac{\sqrt{1/3}\,\chi_{3\alpha}\,R_{1\beta}^N - \sqrt{2/3}\,\chi_{1\alpha}\,R_{3\alpha}^N}{\mathrm{Sin}\,(\delta_3-\delta_1)} \\ R_{-\beta}^{IN} &= -\sqrt{\frac{2}{3}}\;R_{1\beta}^N + \frac{\sqrt{1/3}\,\chi_{3\beta}\,R_{1\beta}^N - \sqrt{2/3}\,\chi_{1\beta}\,R_{3\alpha}^N}{\mathrm{Sin}\,(\delta_3-\delta_1)}\,. \end{aligned}\right\}\quad \varrho \geq \varrho_0 \tag{92}$$

Now equations (84), (85) and (87) imply that

$$R_{-\alpha}^{IN}(\varrho_0) = X_\alpha\,\{\mathrm{Cos}\,\tau_\alpha\,F_0(-\eta,\varrho_0) + \mathrm{Sin}\,\tau_\alpha\,G_0(-\eta,\varrho_0)\}\,,$$

$$R_{-\alpha}^{IN\prime}(\varrho_0) = X_\alpha\,\{\mathrm{Cos}\,\tau_\alpha\,F_0'(-\eta,\varrho_0) + \mathrm{Sin}\,\tau_\alpha\,G_0'(-\eta,\varrho_0)\} \tag{93}$$

where F_0 and G_0 are the regular and irregular $l = 0$ Coulomb wave functions (note that the Coulomb parameter is $-\eta$). Thus using equations (92) and (93) and matching value and derivative of $R_{-\alpha}^{IN}$ at ϱ_0 we obtain

$$\mathrm{Tan}\,\tau_\alpha = \frac{F_0(-\eta,\varrho_0)\,R_{-\alpha}^{IN\prime}(\varrho_0) - F_0'(-\eta,\varrho_0)\,R_{-\alpha}^{IN}(\varrho_0)}{G_0'(-\eta,\varrho_0)\,R_{-\alpha}^{IN}(\varrho_0) - G_0(-\eta,\varrho_0)\,R_{-\alpha}^{IN\prime}(\varrho_0)} \tag{94}$$

$$X_\alpha = \frac{R_{-\alpha}^{IN}(\varrho_0)}{\mathrm{Cos}\,\tau_\alpha\,F_0(-\eta,\varrho_0) + \mathrm{Sin}\,\tau_\alpha\,G_0(-\eta,\varrho_0)} \tag{95}$$

Values for τ_β and X_β can be obtained from similar relations involving $R_{-\beta}^{IN}$ in place of $R_{-\alpha}^{IN}$. The corresponding expressions for general l are obtained by substituting the appropriate Coulomb and radial wave functions in place of F_0, G_0 and R_-^{IN}. Perturbation values for the τ's and the X's can be obtained with calculating the Coulomb wave functions by use of the results of Ref. [11]

$$F_l(-\eta,\varrho) = \varrho\,j_l(\varrho) - \eta\,f_l(\varrho) + 0(\eta^2)\,,$$

$$G_l(-\eta,\varrho) = -\varrho\,n_l(\varrho) - \eta\,g_l(\varrho) + 0(\eta^2)$$

where

$$f_l(\varrho) = 2\,\varrho\, j_l(\varrho) \int_{\varrho}^{\infty} \varrho'\, j_l(\varrho')\, n_l(\varrho')\, d\varrho'$$
$$+ 2\,\varrho\, n_l(\varrho) \left[\frac{1}{2} \ln 2\,\varrho - \frac{1}{2\,\eta}\,\sigma_l - \int_{\varrho}^{\infty} \left(\varrho'\, j_l^2(\varrho') - \frac{1}{2\,\varrho'}\right) d\varrho'\right],$$

$$g_l(\varrho) = 2\,\varrho\, n_l(\varrho) \int_{\varrho}^{\infty} \varrho'\, j_l(\varrho')\, n_l(\varrho')\, d\varrho'$$
$$+ 2\,\varrho\, j_l(\varrho) \left[\frac{1}{2} \ln 2\,\varrho - \frac{1}{2\,\eta}\,\sigma_l - \int_{\varrho}^{\infty} \left(\varrho'\, n_l^2(\varrho') - \frac{1}{2\,\varrho'}\right) d\varrho'\right].$$

The values thus obtained for the χ's, X's and τ's when substituted into equations (89) and (90) then give the Coulomb corrected S-matrix elements. We have seen in Section 3 that these S-matrix elements can also be expressed in terms of two real eigenphases and a mixing parameter, and it is of interest to obtain perturbation expressions for these three quantities.

We saw in equation (50) that the mixing parameter ε had a finite limit in the case of vanishing Coulomb interaction. For calculating perturbation expressions, it is more convenient to define a new quantity ω such that

$$\mathrm{Sin}\,\varepsilon = \frac{\sqrt{2/3} - \sqrt{1/3}\,\mathrm{Tan}\,\omega}{(1 + \mathrm{Tan}^2\omega)^{1/2}} \qquad \mathrm{Cos}\,\varepsilon = \frac{\sqrt{1/3} + \sqrt{2/3}\,\mathrm{Tan}\,\omega}{(1 + \mathrm{Tan}^2\,\omega)^{1/2}} \tag{96}$$

The limiting value of Tan ω is

$$\underset{\eta \to 0}{L\,t}\ \mathrm{Tan}\,\omega = 0\,. \tag{97}$$

In terms of the eigenphases, $\tau_3 \equiv \tau_{0+}^{(3/2)}$ and $\tau_1 \equiv \tau_{0+}^{(3/2)}$, and Tan$\omega$ the S-matrix elements are given by

$$S_{TOT}^{--} = \frac{(\sqrt{1/3} + \sqrt{2/3}\,\mathrm{Tan}\omega)^2\, e^{2i\tau_3} + (\sqrt{2/3} - \sqrt{1/3}\,\mathrm{Tan}\,\omega)^2\, e^{2i\tau_1}}{1 + \mathrm{Tan}^2\,\omega}$$

$$S_{TOT}^{-0} = S_{TOT}^{0-} = \frac{(\sqrt{2/3} - \sqrt{1/3}\,\mathrm{Tan}\,\omega)\,(\sqrt{1/3} + \sqrt{2/3}\,\mathrm{Tan}\,\omega)\,(e^{2i\tau_3} - e^{2i\tau_1})}{1 + \mathrm{Tan}^2\omega}$$

$$S_{TOT}^{00} = \frac{(\sqrt{2/3} - \sqrt{1/3}\,\mathrm{Tan}\,\omega)^2\, e^{2i\tau_3} + (\sqrt{1/3} + \sqrt{2/3}\,\mathrm{Tan}\,\omega)^2\, e^{2i\tau_1}}{1 + \mathrm{Tan}^2\,\omega} \tag{98}$$

Making use of the limiting values of equations (49) and (97), equation (98) gives

$$S_{TOT}^{--} + S_{TOT}^{00} = e^{2i\tau_3} + e^{2i\tau_1}$$

$$S_{TOT}^{-0} = S_{TOT}^{0-} = \frac{\sqrt{2}}{3}\,(e^{2i\tau_3} - e^{2i\tau_1}) + \frac{\mathrm{Tan}\,\omega}{3}\,(e^{2i\delta_3} - e^{2i\delta_1}) + 0(\eta^2)\,. \tag{99}$$

We now define Δ's and ε's such that

$$\tau_\alpha = \delta_3 + \Delta_3 \qquad \tau_\beta = \delta_1 + \Delta_1$$

$$X_\alpha = \sqrt{\frac{1}{3}}\,(1 + \varepsilon_3) \qquad X_\beta = -\sqrt{\frac{2}{3}}\,(1 + \varepsilon_1) \tag{100}$$

and note that the Δ's and ε's vanish in the limit of no Coulomb interaction. Substituting equation (100) into equations (89) and (90) we then obtain the first order results

$$S^{--}_{TOT} + S^{00}_{TOT} = e^{2i\delta_3}\left(1 + 2i\,\frac{\Delta_3 - 2\chi_{3\alpha}}{3}\right) + e^{2i\delta_1}\left(1 + 2i\,\frac{2\Delta_1 + \chi_{1\beta}}{3}\right) + 0(\eta^2)$$

$$S^{-0}_{TOT} = \frac{\sqrt{2}}{3}\,(e^{2i\delta_3} - e^{2i\delta_1})\left(1 + \frac{\varepsilon_1 + 2\varepsilon_3}{3} + \frac{\Delta_3 - \Delta_1}{\operatorname{Tan}(\delta_3 - \delta_1)} + i\,\frac{2\Delta_1 + \Delta_3}{3}\right.$$
$$\left. - \frac{2\chi_{3\alpha}\,e^{i(\delta_3-\delta_1)} + \chi_{1\beta}\,e^{-i(\delta_3-\delta_1)}}{3\operatorname{Sin}(\delta_3 - \delta_1)}\right) + 0(\eta^2)\,. \tag{101}$$

In obtaining these results, we have also used the fact that the χ's vanish in the limit of no Coulomb interaction and that

$$\chi_{3\beta} + \chi_{1\alpha} = 0$$

(see Appendix II, equation (A2.4)).

Comparing equations (99) and (101), we then obtain the first order results

$$\tau_3 = \delta_3 + \frac{\Delta_3 - 2\chi_{3\alpha}}{3} + 0(\eta^2)\,, \qquad \tau_1 = \delta_1 + \frac{2\Delta_1 + \chi_{1\beta}}{3} + 0(\eta^2)\,,$$

$$\operatorname{Tan}\omega = \sqrt{2}\left\{\frac{2\Delta_3 - \Delta_1}{3\operatorname{Tan}(\delta_3 - \delta_1)} + \frac{2\varepsilon_3 + \varepsilon_1}{3}\right\} + 0(\eta^2)\,.$$

Similar results can, of course, be obtained in the general l case.

APPENDIX I

Outer Coulomb Corrections Only

In the case of $\pi^+ p$ scattering, the vanishing of the inner part of the Coulomb potential implies from equation (19) that

$$\tilde{\tau}^{(+)}_{l\pm} \equiv \delta^{(3/2)}_{l\pm}$$

Then, using equation (29), the outer Coulomb corrections are obtained by calculating $\tau^{(+)}_{l\pm}$ from

$$\left.\frac{\varrho\, j_l(\varrho) - \operatorname{Tan}\delta^{(3/2)}_{l\pm}\,\varrho\, n_l(\varrho)}{(\varrho\, j_l(\varrho))' - \operatorname{Tan}\delta^{(3/2)}_{l\pm}(\varrho\, n_l(\varrho))'}\right|_{\varrho=\varrho_0} = \left.\frac{F_l(\varrho) + \operatorname{Tan}\tau^{(+)}_{l\pm}\,G_l(\varrho)}{F'_l(\varrho) + \operatorname{Tan}\tau^{(+)}_{l\pm}\,G'_l(\varrho)}\right|_{\varrho=\varrho_0}$$

this result agreeing with the work of Van Hove.

In the case of the processes $\pi^- p \to \pi^- p$ and $\pi^- p \to \pi^0 n$, the vanishing of the inner part of the Coulomb potential implies from equations (73) and (74) that the χ's vanish. Thus, using equations (75) and (76) and forming the α and β solutions for R_-^{IN} and R_0^{IN}, we have

$$R^{IN}_{-\alpha} = \sqrt{\frac{1}{3}}\; R^N_{3\alpha} \qquad R^{IN}_{-\beta} = -\sqrt{\frac{2}{3}}\; R^N_{1\beta}$$

$$R^{IN}_{0\alpha} = \sqrt{\frac{2}{3}}\; R^N_{3\alpha} \qquad R^{IN}_{0\beta} = \sqrt{\frac{1}{3}}\; R^N_{1\beta} \tag{A1.1}$$

Now in the case of only outer Coulomb corrections, Van Hove defines two independent solutions of equation (83), u_3 and u_1, such that

$$u_3(\varrho) \underset{\varrho \to \varrho_0}{\sim} C_3^{-1}\, R^N_{3\alpha}(\varrho)\,, \qquad u_1(\varrho) \underset{\varrho \to \varrho_0}{\sim} C_1^{-1}\, R^N_{1\beta}(\varrho) \tag{A1.2}$$

and with asymptotic behaviour

$$u_3(\varrho) \underset{\varrho \to \infty}{\sim} \operatorname{Sin}(\varrho - \sigma_0 + \eta \ln 2\,\varrho + \tau_3)\,,$$

$$u_1(\varrho) \underset{\varrho \to \infty}{\sim} \operatorname{Sin}(\varrho - \sigma_0 + \eta \ln 2\,\varrho + \tau_1)\,. \tag{A1.3}$$

Comparing equations (A1.2) and (A1.3) with equations (85) and (87) using equation (A1.1), we see that in the case of no inner Coulomb corections

$$\tau_\alpha = \tau_3 \qquad \tau_\beta = \tau_1\,,$$

$$X_\alpha = \sqrt{\frac{1}{3}}\; C_3 \qquad X_\beta = -\sqrt{\frac{2}{3}}\; C_1\,. \tag{A1.4}$$

Substituting these results into equations (89) and (90) and setting the χ's equal to zero we obtain the s-wave S-matrix elements for the case of only outer Coulomb corrections

$$S^{--}_{out} = \left[\frac{1}{3}\, C_3\, e^{i(\tau_3+\delta_3)} + \frac{2}{3}\, C_1\, e^{i(\tau_1+\delta_1)}\right] . \; D^{-1}_{out}\,,$$

$$S^{-0}_{out} = \left[\frac{\sqrt{2}}{3}\, C_1\, C_3\, e^{i(\delta_1+\delta_3)}\, 2\, i\, \operatorname{Sin}(\tau_3 - \tau_1)\right] . \; D^{-1}_{out}\,,$$

$$S^{0-}_{out} = \frac{\sqrt{2}}{3}\, (e^{2i\delta_3} - e^{2i\delta_1}) \; . \; D^{-1}_{out}\,, \tag{A1.5}$$

$$S^{00}_{out} = \left[\frac{1}{3}\, C_3\, e^{i(\delta_1-\tau_3)} + \frac{2}{3}\, C_1\, e^{i(\delta_3-\tau_1)}\right] e^{i(\delta_3+\delta_1)} \; . \; D^{-1}_{out}$$

where

$$D_{out} = \frac{1}{3}\, C_3\, e^{-i(\tau_3-\delta_3)} + \frac{2}{3}\, C_1\, e^{-i(\tau_1-\delta_1)}\,. \tag{A1.6}$$

These results when used to calculate the scattering amplitudes agree with Van Hove's outer Coulomb corrections for $\pi^- p$ elastic and charge exchange scattering.

APPENDIX II

Symmetry and Unitarity of the S-Matrix with Full Coulomb Corrections

We first prove the symmetry by showing that S_{TOT}^{-0} and S_{TOT}^{0-} given by equations (89) and (90) are equal. From these equations we have

$$D_{TOT}\,(S_{TOT}^{0-} - S_{TOT}^{-0}) = 2\,i\,e^{i(\delta_3+\delta_1)} \left[\frac{\sqrt{2}}{3}\,\mathrm{Sin}\,(\delta_3-\delta_1) + \frac{\chi_{1\alpha} + 2\,\chi_{3\beta}}{3} + \frac{\sqrt{2}\,(\chi_{1\alpha}\,\chi_{3\beta} - \chi_{3\alpha}\,\chi_{1\beta})}{3\,\mathrm{Sin}\,(\delta_3-\delta_1)} + X_\alpha\,X_\beta\,\mathrm{Sin}\,(\tau_\alpha-\tau_\beta)\right]. \tag{A2.1}$$

Now, by construction, u_α and u_β are two independent solutions of equation (83) and so their Wronskian must be independent of ϱ. Evaluating this Wronskian using the asymptotic forms given by equation (85), we have

$$W[u_\alpha, u_\beta] = \mathrm{Sin}\,(\tau_\alpha - \tau_\beta)\,.$$

However, from equation (87) we have

$$W[u_\alpha, u_\beta] = (X_\alpha\,X_\beta)^{-1}\,W[R_{-\alpha}^{IN}, R_{-\beta}^{IN}]_{\varrho=\varrho_0}\,.$$

Using equation (92) and equating the two expressions for the Wronskians we then obtain

$$X_\alpha\,X_\beta\,\mathrm{Sin}\,(\tau_\alpha-\tau_\beta) = -\frac{\sqrt{2}}{3}\,\mathrm{Sin}\,(\delta_3-\delta_1) + \frac{2\,\chi_{1\alpha} + \chi_{3\beta}}{3} - \frac{\sqrt{2}\,(\chi_{1\alpha}\,\chi_{3\beta} - \chi_{3\alpha}\,\chi_{1\beta})}{3\,\mathrm{Sin}\,(\delta_3-\delta_1)}\,. \tag{A2.2}$$

Substituting equation (A2.2) into (A2.1), we have

$$D_{TOT}\cdot(S_{TOT}^{0-} - S_{TOT}^{-0}) = 2\,i\,e^{i(\delta_3+\delta_1)}\,(\chi_{1\alpha} + \chi_{3\beta})\,. \tag{A2.3}$$

We now make use of the fact that the sets $R_{3\alpha}^{IN}$, $R_{1\alpha}^{IN}$ and $R_{3\beta}^{IN}$, $R_{1\beta}^{IN}$ are two independent regular solutions of the coupled equations (70). This means that

$$W[R_{3\alpha}^{IN}, R_{3\beta}^{IN}] + W[R_{1\alpha}^{IN}, R_{1\beta}^{IN}] = 0$$

and when we evaluate these Wronskians at ϱ_0 we obtain

$$\chi_{1\alpha} + \chi_{3\beta} = 0\,. \tag{A2.4}$$

Finally, substituting equation (A2.4) into (A2.3), we obtain the desired result

$$S_{TOT}^{-0} = S_{TOT}^{0-} \tag{A2.5}$$

We now wish to prove that the S-matrix is unitary. For this, it is sufficient to prove the three relations

$$|S_{TOT}^{--}|^2 + |S_{TOT}^{-0}|^2 = 1\,, \tag{A2.6}$$

$$|S_{TOT}^{00}|^2 + |S_{TOT}^{0-}|^2 = 1\,, \tag{A2.7}$$

$$|S_{TOT}^{--}\, S_{TOT}^{0-}{}^* + S_{TOT}^{00}{}^*\, S_{TOT}^{-0}|^2 = 0\,. \tag{A2.8}$$

First, we rewrite equation (A2.6) in the form

$$|D_{TOT} \cdot S_{TOT}^{-0}|^2 = |D_{TOT}|^2 - |D_{TOT} \cdot S_{TOT}^{--}|^2\,. \tag{A2.9}$$

Using equations (89) and (90), we have

$$|D_{TOT}|^2 - |D_{TOT} \cdot S_{TOT}^{--}|^2 = -4\, X_\alpha\, X_\beta\, \mathrm{Sin}\,(\tau_\alpha - \tau_\beta)$$
$$\times \left\{ \frac{\sqrt{2}}{3}\, \mathrm{Sin}\,(\delta_3 - \delta_1) + \frac{\chi_{1\alpha} + 2\,\chi_{3\beta}}{3} + \frac{\sqrt{2}\,(\chi_{1\alpha}\,\chi_{3\beta} - \chi_{3\alpha}\,\chi_{1\beta})}{\mathrm{Sin}\,(\vartheta_3 - \delta_1)} \right\}$$

which, on using equations (A2.2) and (A2.4), becomes

$$|D_{TOT}|^2 - |D_{TOT} \cdot S_{TOT}^{--}|^2 = 4\, X_\alpha^2\, X_\beta^2\, \mathrm{Sin}^2\,(\tau_\alpha - \tau_\beta)\,.$$

However, equations (89) and (90) give

$$|D_{TOT} \cdot S_{TOT}^{-0}|^2 = 4\, X_\alpha^2\, X_\beta^2\, \mathrm{Sin}^2\,(\tau_\alpha - \tau_\beta)$$

and so

$$|D_{TOT} \cdot S_{TOT}^{-0}|^2 = |D_{TOT}|^2 - |D_{TOT} \cdot S_{TOT}^{--}|^2\,. \qquad \text{Q.E.D.}$$

In order to prove equation (A2.7), we note from equations (89) and (90) that

$$(D_{TOT} \cdot S_{TOT}^{00})^*\, e^{2i(\delta_3 + \delta_1)} = (D_{TOT} \cdot S_{TOT}^{--}) \tag{A2.10}$$

and so, using equation (A2.5), we see that equation (A2.7) is equivalent to equation (A2.6) which we have just proved to be true.

Finally to prove equation (A2.8) we note that

$$D_{TOT} \cdot S_{TOT}^{-0} = -2\, i\, X_\alpha\, X_\beta\, \mathrm{Sin}\,(\tau_\alpha - \tau_\beta)\, e^{i(\delta_1 + \delta_3)}\,. \tag{A2.11}$$

But, using equations (A2.5) and (A2.10), equation (A2.8) is equivalent to

$$|D_{TOT} \cdot S_{TOT}^{--}\, (D_{TOT} \cdot S_{TOT}^{-0})^* + (D_{TOT} \cdot S_{TOT}^{-0})\, e^{-2i(\delta_1 + \delta_3)})|^2 = 0\,. \tag{A2.12}$$

Since X_α, X_β, τ_α, τ_β, δ_1 and δ_3 are all real, we see at once from equation (A2.11) that (A2.12) is true.

Thus, we have checked that our explicit form for the S-matrix elements in the simultaneous presence of the nuclear and Coulomb potentials corresponds to a symmetric and unitary S-matrix. Similar results follow at once in the case when only outer Coulomb corrections are included. In the case when only inner Coulomb corrections are taken into account, the S-matrix elements given by equations (77) and (78) can also be shown to correspond to a symmetric and unitary S-matrix. The symmetry follows at once by comparing the expressions for $S_{IN}^{3/2\ 1/2}$ and $S_{IN}^{1/2\ 2/3}$ using equation (A2.4). The unitarity can be established by proving relations analogous to equations (A2.6), (A2.7) and (A2.8).

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