Zeitschrift: Helvetica Physica Acta
Band: 44 (1971)
Heft: 1

Artikel: Explicit solution for quadratic interactions

Autor: Zabey, Ph.Ch. / Ducommun, M.

DOI: https://doi.org/10.5169/seals-114270

Explicit Solution for Quadratic Interactions

by **Ph. Ch. Zabey** and **M. Ducommun**
Institute for Theoretical Physics, Geneva, Switzerland

(31. VII. 70)

Abstract. Explicit operator solutions are given for quadratic interactions, both for scalar and spinor fields. Their existence is rigorously established.

1. Introduction

We intend to construct in this paper an entirely soluble model for mass and field intensity renormalizations, both for scalar and spinor fields.

The difficulty, with such a program, is due essentially to Haag's theorem [1], which asserts that there does not exist unitary operators representing the time evolution of a system with interactions. Indeed, we will meet this difficulty in the fact that the interaction terms we introduce for the renormalizations are not defined on the Fock space of the free field.

To bypass Haag's theorem, the main idea, due to Guenin [2, 3], is to consider the time evolution of the system as an automorphism of the algebra of field operators. This automorphism is not unitarily implementable, but we will show that it can be reached by some limiting procedure performed on unitarily implementable automorphisms.

To work out this procedure, we must first introduce cut-offs in order to make of the interaction terms well-defined operators on the bare Fock space, and then we prove the existence of the solution and its convergence when cut-offs are removed. Finally, for our model to be complete, we must evaluate explicitely the physical vacuum, allowing therefore the Wightman's construction. Usual methods for obtaining the physical vacuum do not work in our case, but we will show that the right result can be obtained if we use a dressing transformation.

We will essentially present the calculations for the spinor field, and give the main results for the scalar field. Proofs are quite similar in both cases; for the scalar field, they have been partly given in [2], and the explicit form of the dressing transformation has been proposed by Eckmann and Guenin [4].

2. The Case of the Spinor Field

We start with a free spinor field, usually written as:

$$\Psi(x)=\left(\frac{1}{2\pi}\right)^{s/2}\int d^s p\left(\frac{m}{w_p}\right)^{1/2} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\sum_r\{b_r(\boldsymbol{p})\,u_r(\boldsymbol{p})\,e^{-iw_p t}+d_r^*(-\boldsymbol{p})\,v_r(-\boldsymbol{p})\,e^{iw_p t}\}.$$

Here, x stands for $(t, \boldsymbol{x})$, ω_p for $(p^2 + m^2)^{1/2}$ and s for the number of space dimension; the symbol $\sim$ will denote the transition to adjoints spinors. The following rules apply:

$$\tilde{u}_r(\boldsymbol{p})\, u_{r'}(\boldsymbol{p}) = -\tilde{v}_r(\boldsymbol{p})\, v_{r'}(\boldsymbol{p}) = \delta_{rr'}$$

$$[b_r(\boldsymbol{p}), b^*_{r'}(\boldsymbol{p}')]_+ = [d_r(\boldsymbol{p}), d^*_{r'}(\boldsymbol{p}')]_+ = \delta_{rr'}\, \delta(\boldsymbol{p} - \boldsymbol{p}')$$

all other anticommutators being zero. The free hamiltonian H_0 is:

$$H_0 = \int d^s p\, \omega_p \sum_r \{b^*_r(\boldsymbol{p})\, b_r(\boldsymbol{p}) + d^*_r(-\boldsymbol{p})\, d_r(-\boldsymbol{p})\} \quad .$$

We now introduce mass and field intensity renormalizations by means of the following formal interaction terms:

$$V = \frac{\delta m^2}{4\,m} \int_{t=o} d^s x\, :\tilde{\Psi}\Psi:(x) + \frac{\mathfrak{G}}{4\mathrm{m}} \int_{t=o} d^s x\, :\Box\, \tilde{\Psi}\Psi:(x)$$

which includes a factor $1/2m$ for coherence in units; $\Box$ is equal to $\partial_t^2 - \Delta$. As in [2], V is not defined on the bare Fock space, even if we substitute to it a once cut-off interaction V_f of the form:

$$V_f = \frac{\delta \mathrm{m}^2}{4\mathrm{m}} \int_{t=o} d^s x\, f(\boldsymbol{x})\, :\tilde{\Psi}\Psi:(x) + \frac{\mathfrak{G}}{4\,m} \int_{t=o} d^s x\, f(\boldsymbol{x})\, :\Box\, \tilde{\Psi}\Psi:(x)$$

Let us therefore consider a twice cut-off interaction V_α defined by:

$$V_\alpha = \int_{t=o} d^s x \int_{t'=o} d^s x'\, f_{m\alpha}(\boldsymbol{x})\, \varphi_\alpha(\boldsymbol{x}')\, :\tilde{\Psi}(x + x')\, \Psi(x - x'):$$

$$+ \int_{t=0} d^s x \int_{t'=0} d^s x'\, f_{\mathfrak{G}\alpha}(\boldsymbol{x})\, \varphi_\alpha(\boldsymbol{x}')\, :\Box_x \tilde{\Psi}(x + x')\, \Psi(x - x'):$$

with:

$$f_{m\alpha}(\boldsymbol{x}) = \frac{\delta m^2}{4\,m} \exp\left(-\frac{\alpha\, \|\boldsymbol{x}\|^2}{4}\right) \xrightarrow{in\ S'} \frac{\delta m^2}{4\,m} \text{ when } \alpha \to 0\ ,$$

$$f_{\mathfrak{G}\alpha}(\boldsymbol{x}) = \frac{\mathfrak{G}}{4\,m} \exp\left(-\frac{\alpha\, \|\boldsymbol{x}\|^2}{4}\right) \xrightarrow{in\ S'} \frac{\mathfrak{G}}{4\,m} \text{ when } \alpha \to 0\ ,$$

$$\varphi_\alpha(\boldsymbol{x}) = \left(\frac{1}{\pi\,\alpha}\right)^{s/2} \exp\left(-\frac{\|\boldsymbol{x}\|^2}{\alpha}\right) \xrightarrow{in\ S'} \delta(\boldsymbol{x}) \text{ when } \alpha \to 0\ .$$

With these cut-offs, parametrized by α, $H_\alpha = H_0 + V_\alpha$ is, after Kato's well-known theorem, a well defined essentially self adjoint operator on the manifold generated by the states with a finite number of particles; we take $|\mathfrak{G}| \leqslant 1/2$ and $\delta m^2 < m^2$.

Our aim is now to look for some limit $\breve{\Psi}(x)$ of the field $\breve{\Psi}_\alpha(x)$ defined by $\breve{\Psi}_\alpha(x) = \exp(i\, H_\alpha\, t)\, \Psi(0, \boldsymbol{x}) \exp(-i\, H_\alpha\, t)$ when α goes to zero. But, by Haag's theorem, we know that we can not compute any limit on the exponential $\exp(-i\, H_\alpha\, t)$. Our method is then to try an ansatz:

$$\sum_r b_r(\boldsymbol{p}, t)\, u_r(\boldsymbol{p}, t) = \sum_r \int d^s k\, A_r(\boldsymbol{p}, \boldsymbol{k}, t)\, b_r(\boldsymbol{k})\, u_r(\boldsymbol{k})$$

$$+ \sum_r \int d^s k\, B_r(\boldsymbol{p}, \boldsymbol{k}, t)\, d^*_r(-\boldsymbol{k})\, v_r(-\boldsymbol{k}) \tag{2.1}$$

and to write:

$$\breve{\Psi}_\alpha(x) = \left(\frac{1}{2\pi}\right)^{s/2} \int d^s p \left(\frac{m}{w_p}\right)^{1/2} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \sum_r \left\{ b_r(\boldsymbol{p}, t)\, u_r(\boldsymbol{p}, t) + d_r^*(-\boldsymbol{p}, t)\, v_r(-\boldsymbol{p}, t) \right\} .$$

$A_r(\boldsymbol{p}, \boldsymbol{k}, t)$ and $B_r(\boldsymbol{p}, \boldsymbol{k}, t)$ are unknown functions we have to determine by solving Heisenberg's equation:

$$i\, \partial_t\, \breve{\Psi}_\alpha(x) = [\breve{\Psi}_\alpha(x), H_\alpha]$$

with the initial condition $\breve{\Psi}_\alpha(0, \boldsymbol{x}) = \Psi(0, \boldsymbol{x})$.

After some calculations, we obtain the following system:

$$i\, \partial_t\, A_r(\boldsymbol{p}, \boldsymbol{k}, t) = \omega_k\, A_r(\boldsymbol{p}, \boldsymbol{k}, t) + \int d^s q \left(\frac{m}{w_k}\right)^{1/2} \left(\frac{m}{w_p}\right)^{1/2} \sum_{r'} \left\{ A_{r'}(\boldsymbol{p}, \boldsymbol{q}, t)\, D_\alpha^-(\boldsymbol{k}, -\boldsymbol{q}) - B_{r'}(\boldsymbol{p}, \boldsymbol{q}, t)\, D_\alpha^+(\boldsymbol{k}, -\boldsymbol{q}) \right\} , \tag{2.2a}$$

$$i\, \partial_t\, B_r(\boldsymbol{p}, \boldsymbol{k}, t) = -\omega_k\, B_r(\boldsymbol{p}, \boldsymbol{k}, t) + \int d^s q \left(\frac{m}{w_k}\right)^{1/2} \left(\frac{m}{w_p}\right)^{1/2} \sum_{r'} \left\{ A_{r'}(\boldsymbol{p}, \boldsymbol{q}, t)\, D_\alpha^+(\boldsymbol{k}, -\boldsymbol{q}) - B_{r'}(\boldsymbol{p}, \boldsymbol{q}, t)\, D_\alpha^-(\boldsymbol{k}, -\boldsymbol{q}) \right\} \tag{2.2b}$$

where:

$$D_\alpha^\pm(\boldsymbol{p}, \boldsymbol{p}') = \hat{\varphi}\, (\boldsymbol{p} - \boldsymbol{p}')\, \{ \hat{f}_{m\alpha}\, (\boldsymbol{p} + \boldsymbol{p}') + \hat{f}_{\mathfrak{G}\alpha}(\boldsymbol{p}+\boldsymbol{p}')\, ((\boldsymbol{p} + \boldsymbol{p}')^2 - (\omega_p \pm \omega_{p'})^2) \}$$

and whose initial conditions are:

$$A_r(\boldsymbol{p}, \boldsymbol{k}, 0) = \delta\, (\boldsymbol{p} - \boldsymbol{k})\ , \quad B_r(\boldsymbol{p}, \boldsymbol{k}, 0) = 0 \tag{2.2c}$$

the symbol ^ denoting Fourier transformation. Our next task is to find a solution of (2.2) and to show that it admits a limit when we remove the cut-offs, i.e. when $\alpha \to 0$. More precisely, we want to establish that:

(i) A_r and B_r are distributions in the $\boldsymbol{p}$ variable,

(ii) A_r and B_r belong to $S(R^s)$ in the $\boldsymbol{k}$ variable,

(iii) A_r and B_r converge to some limits in the $S(R^s)$ topology when $\alpha \to 0$.

For this purpose, we write the system (2.2) in a more compact way: let $\psi(\boldsymbol{p}) \in S(R^s)$, and define successively:

$$U(\boldsymbol{k}, t) = \int d^s p\, \psi(\boldsymbol{p}) \begin{pmatrix} \exp(i\,\omega_k\, t)\, A_1(\boldsymbol{p}, \boldsymbol{k}, t) \\ \exp(i\,\omega_k\, t)\, A_2(\boldsymbol{p}, \boldsymbol{k}, t) \\ \exp(-i\,\omega_k\, t)\, B_1(\boldsymbol{p}, \boldsymbol{k}, t) \\ \exp(-i\,\omega_k\, t)\, B_2(\boldsymbol{p}, \boldsymbol{k}, t) \end{pmatrix},$$

$$C_\alpha^\pm(\boldsymbol{p}, \boldsymbol{p}') = \frac{\delta m^2}{4\, m} + \frac{\mathfrak{G}}{4\, m}\, ((\boldsymbol{p} - \boldsymbol{p}')^2 - (\omega_p \pm \omega_{p'})^2)\ ,$$

$$f_\alpha(\boldsymbol{x}) = \exp\left(-\frac{\alpha\, \|\boldsymbol{x}\|^2}{4}\right)$$

$$M(\boldsymbol{k}, \boldsymbol{q}, t) = -i \left(\frac{m}{w_k}\right)^{1/2} \left(\frac{m}{w_q}\right)^{1/2} \hat{\varphi}_\alpha(\boldsymbol{k}+\boldsymbol{q}) \left(\begin{array}{c|c} C_\alpha^-(\boldsymbol{k}, -\boldsymbol{q})\, e^{i(w_k - w_q)t} & -C_\alpha^+(\boldsymbol{k}, -\boldsymbol{q})\, e^{i(w_k + w_q)t} \\ \hline C_\alpha^+(\boldsymbol{k}, -\boldsymbol{q})\, e^{i(-w_k - w_q)t} & -C_\alpha^-(\boldsymbol{k}, -\boldsymbol{q})\, e^{i(-w_k + w_q)t} \end{array} \right)$$

Then we may write:

$$\partial_t U(\boldsymbol{k}, t) = \int d^s q \hat{f}_\alpha (\boldsymbol{k} - \boldsymbol{q}) M(\boldsymbol{k}, \boldsymbol{q}, t) U(\boldsymbol{q}, t)$$

$$U(\boldsymbol{k}, 0) = \psi(\boldsymbol{k}) \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

which is an integral equation giving the following successive approximations:

$$U_n(\boldsymbol{k}, t) = \int^t d\varrho_1 \int d^s q_1 \ldots \int^{\varrho_{n-1}} d\varrho_n \int d^s q_n \hat{f}_\alpha(\boldsymbol{k} - \boldsymbol{q}_1) \ldots f_\alpha(\boldsymbol{q}_{n-1} - \boldsymbol{q}_n) \times$$
$$M(\boldsymbol{k}, \boldsymbol{q}_1, \varrho_1) \ldots M(\boldsymbol{q}_{n-1}, \boldsymbol{q}_n, \varrho_n) U_0(\boldsymbol{q}_n, \varrho_n) .$$

Now, by properties of $\hat{f}_\alpha$ and of $\hat{\varphi}_\alpha$, we can see that:

$$\| M(\boldsymbol{k}, \boldsymbol{q}, t) \| \leqslant \beta \operatorname{Max}\{ | C_\alpha^\pm(\boldsymbol{k}, -\boldsymbol{q}) | \} ,$$

$$\| \int d^s q \, C_\alpha^\pm(\boldsymbol{k}, -\boldsymbol{q}) \hat{f}_\alpha (\boldsymbol{k} - \boldsymbol{q}) \| < \gamma \, \omega_k$$

and consequently that:

$$\| U_n(\boldsymbol{k}, t) \| \leqslant \frac{\beta^n \gamma^n t^n w_k^n}{n!} \| U_0(\boldsymbol{k}, 0) \|$$

which converges uniformely in α. This establishes (i) and (iii). We obtain (ii) by similar techniques on $\prod_i (1 + k_i)^p \| U_n(\boldsymbol{k}, t) \|$ and on its derivatives for each p.

These results enable us to solve (2.2) after having removed the cut-offs, that is to solve the following system:

$$i \, \partial_t A_r(\boldsymbol{p}, \boldsymbol{k}, t) = \omega_k A_r(\boldsymbol{p}, \boldsymbol{k}, t) + \frac{\delta m^2}{4 m} \sum_{r'} A_{r'}(\boldsymbol{p}, \boldsymbol{k}, t)$$
$$- \frac{\delta m^2 - 4 \mathfrak{G} w_k^2}{4 m} \sum_{r'} B_{r'}(\boldsymbol{p}, \boldsymbol{k}, t) ,$$

$$i \, \partial_t B_r(\boldsymbol{p}, \boldsymbol{k}, t) = -\omega_k B_r(\boldsymbol{p}, \boldsymbol{k}, t) - \frac{\delta m^2}{4 m} \sum_{r'} B_{r'}(\boldsymbol{p}, \boldsymbol{k}, t)$$
$$+ \frac{\delta m^2 - 4 \mathfrak{G} w_k^2}{4 m} \sum_{r'} A_{r'}(\boldsymbol{p}, \boldsymbol{k}, t) .$$

The solution is then:

$$A_r(\boldsymbol{p}, \boldsymbol{k}, t) = \delta(\boldsymbol{k} - \boldsymbol{p}) \left\{ \left(\frac{1}{2} - \frac{2 \omega_k^2 + \delta m^2}{4 \omega_k \varepsilon_k} \right) e^{i \varepsilon_k t} + \left(\frac{1}{2} + \frac{2 \omega_k + \delta m^2}{4 \omega_k \varepsilon_k} \right) e^{-i \varepsilon_k t} \right\} ,$$

$$B_r(\boldsymbol{p}, \boldsymbol{k}, t) = \delta(\boldsymbol{k} - \boldsymbol{p}) \frac{4 \mathfrak{G} \omega_k^2 - \delta m^2}{4 \omega_k \varepsilon_k} \left\{ e^{i \varepsilon_k t} - e^{-i \varepsilon_k t} \right\}$$

for each r, where $\varepsilon_k^2 = \delta m^2 (1 + 2 \mathfrak{G}) + \omega_k^2 (1 - 4 \mathfrak{G}^2)$. Going back to our ansatz (2.1), we obtain the detailed expression for $\sum_r b_r(\boldsymbol{p}, t) \, u_r(\boldsymbol{p}, t)$ as well as for $\sum_r d_r(\boldsymbol{p}, t) \, v_r(\boldsymbol{p}, t)$, and finally we can write:

$$\breve{\Psi}(x) = \left(\frac{1}{2\pi} \right)^{s/2} \int d^s k \left(\frac{m}{\varepsilon_k} \right)^{1/2} e^{i \boldsymbol{k} \cdot \boldsymbol{x}} \sum_r \left\{ \breve{b}_r(\boldsymbol{k}) \, \breve{u}_r(\boldsymbol{k}) \, e^{-i \varepsilon_k t} + \breve{d}_r^*(-\boldsymbol{k}) \, \breve{v}_r(-\boldsymbol{k}) \, e^{i \varepsilon_k t} \right\}$$

where

$$\breve{b}_r(\boldsymbol{k})\ \breve{u}_r(\boldsymbol{k}) = \frac{1}{2}\left(\frac{\varepsilon_k}{\omega_k}\right)^{1/2}\left\{(1+\alpha_k)\ b_r(\boldsymbol{k})\ u_r(\boldsymbol{k}) + (1-\alpha_k)\ d_r^*(-\boldsymbol{k})\ v_r(-\boldsymbol{k})\right\},$$

$$\breve{d}_r(\boldsymbol{k})\ \breve{v}_r(\boldsymbol{k}) = \frac{1}{2}\left(\frac{\varepsilon_k}{\omega_k}\right)^{1/2}\left\{(1+\alpha_k)\ d_r(\boldsymbol{k})\ v_r(\boldsymbol{k}) + (1-\alpha_k)\ b_r^*(-\boldsymbol{k})\ u_r(-\boldsymbol{k})\right\}$$

with $\alpha_k = \omega_k\,(1 + 2\,\mathfrak{G})/\varepsilon_k$. One can easily verify that these renormalized creation and annihilation operators satisfy quasi-canonical anticommutation rules:

$$[\breve{b}_r(\boldsymbol{k}),\ \breve{b}_{r'}^*(\boldsymbol{k}')]_+ = [\breve{d}_r(\boldsymbol{k}),\ \breve{d}_{r'}^*(\boldsymbol{k}')]_+ = (1 + 2\,\mathfrak{G})\ \delta_{rr'}\ \delta\,(\boldsymbol{k}-\boldsymbol{k}')$$

all other anticommutators being zero.

We have finally to construct the "physical", i.e. the renormalized vacuum $|\breve{0}\rangle$. We first notice that $|\breve{0}\rangle$ does no belong to the bare Fock space, which describes particles with mass m; indeed, there is no state $|\varphi\rangle$ of mass m such that $\breve{b}_r(\boldsymbol{k})\,|\varphi\rangle = 0$ for all $\boldsymbol{k}$ and r.

The first attempt is to obtain $|\breve{0}\rangle$ from the time average of a two-point functional taken in the bare vacuum $|0\rangle$; but this procedure leads to:

$$\lim_{T=\infty}\frac{1}{T}\int_{-T}^{T} d\varrho\ \langle 0|\ \tilde{\breve{\Psi}}\,(t+\varrho,\boldsymbol{x})\ \breve{\Psi}\,(t'+\varrho,\boldsymbol{x}')\ |0\rangle =$$

$$\left(\frac{1}{2\,\pi}\right)^{s}\int d^sk\,\frac{m}{\omega_k}\left\{(1+\alpha_k)^2\,e^{-i\varepsilon_k(t-t')} + (1-\alpha_k)^2\,e^{i\varepsilon_k(t-t')}\right\}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}$$

This expression obviously does not fulfill the spectral conditions, for it contains both positive and negative frequencies.

The right idea, as mentioned in section 1, is to consider a kind of dressing transformation defined by:

$$T = \exp\left\{-\int d^sk\,f(\boldsymbol{k})\sum_r\left(b_r^*(\boldsymbol{k})\ \tilde{u}_r(\boldsymbol{k})\ d_r^*(-\boldsymbol{k})\,v_r(-\boldsymbol{k}) + b_r^*(-\boldsymbol{k})\,\tilde{u}_r(-\boldsymbol{k})\ d_r^*(\boldsymbol{k})\ v_r(\boldsymbol{k})\right)\right\}$$

and to write $|\breve{0}\rangle = T\,|0\rangle$. Actually, T is only a formal expression which is not defined on the bare vacuum. Therefore, we ought to introduce some cut-offs and to study the convergence of what we obtain; such calculations are rather combersome for the spinor field [5], and we give here only the results, all details will be presented for the scalar field.

As $|\breve{0}\rangle$ is obviously an invariant, the whole point is to write that $\breve{b}_r(\boldsymbol{k})\ |\breve{0}\rangle = 0$ for all $\boldsymbol{k}$ and r. We will verify this conditions if we take:

$$f(\boldsymbol{k}) = \frac{1}{2}\left(\frac{1-\alpha_k}{1+\alpha_k}\right)$$

and we obtain then the right two-point functional:

$$\Omega\,(x, x') = (1 + 2\,\mathfrak{G})\left(\frac{1}{2\,\pi}\right)^{s}\int d^sk\left(\frac{m}{\varepsilon_k}\right)e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\ e^{-i\varepsilon_k(t-t')}$$

3. The Case of the Neutral Scalar Field

As mentionned in section 1, the case of the neutral scalar field differs slightly from the case of the spinor field. Therefore, we will give here simply some constructive points for its solution, examinating in detail only the construction of the renormalized vacuum.

A neutral scalar field is usually given by the following formulas:

$$\Phi(x) = \left(\frac{1}{2\pi}\right)^{s/2} \int \frac{d^s k}{(2\,\omega_k)^{1/2}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left\{a(\boldsymbol{k})\, e^{-i\omega_k t} + a^*(-\boldsymbol{k})\, e^{i\omega_k t}\right\}$$

$$[a(\boldsymbol{k}), a^*(\boldsymbol{k}')] = \delta\,(\boldsymbol{k} - \boldsymbol{k}')$$

all other commutators being zero. The free hamiltonian H_0 is:

$$H_0 = \int d^s k\, \omega_k\, a^*(\boldsymbol{k})\, a(\boldsymbol{k})$$

and the interaction terms for the renormalizations are:

$$V = \frac{\delta m^2}{2} \int_{t=0} d^s x\, :\Phi^2:(\boldsymbol{x}) + \frac{\mathfrak{G}}{2} \int_{t=0} d^s x\, :\Box\, \Phi^2:(x)\ .$$

The solution is easily checked and is written:

$$\breve{\Phi}(x) = \left(\frac{1}{2\pi}\right)^{s/2} \int \frac{d^s k}{(2\,\varepsilon_k)^{1/2}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left\{\breve{a}(\boldsymbol{k})\, e^{-i\varepsilon_k t} + \breve{a}^*(-\boldsymbol{k})\, e^{i\varepsilon_k t}\right\},$$

with:

$$\varepsilon_k^2 = \omega_k^2\,(1 - 4\,\mathfrak{G}^2) + \delta m^2\,(1 + 2\,\mathfrak{G})\ ,$$

$$\breve{a}(\boldsymbol{k}) = \frac{1}{2}\left(\frac{\varepsilon_k}{\omega_k}\right)^{1/2} \left\{(1 + \alpha_k)\, a(\boldsymbol{k}) + (1 - \alpha_k)\, a^*(-\boldsymbol{k})\right\},$$

$$\alpha_k = \omega_k\,(1 + 2\,\mathfrak{G})/\varepsilon_k\ .$$

Again we consider a dressing transformation:

$$T = \exp\left\{-\int d^s k\, f(\boldsymbol{k})\, a^*(\boldsymbol{k})\, a^*(-\boldsymbol{k})\right\} \equiv \exp\left\{-B\right\}\ .$$

But we have $\langle 0 \mid B^*\, B \mid 0\rangle = \int d^s k \int d^s k'\, f(\boldsymbol{k})\, \overline{f(\boldsymbol{k}')}\, \delta^2\,(\boldsymbol{k} - \boldsymbol{k}')$, which diverges. Therefore, we have to introduce (once more) two cut-offs:

$$f_\alpha(\boldsymbol{k}) = f(\boldsymbol{k}) \exp\left(-\frac{\alpha\,\|\boldsymbol{k}\|^2}{4}\right) \xrightarrow{\text{in } S'} f(\boldsymbol{k}) \text{ when } \alpha \to 0\ ,$$

$$\varphi_\alpha(\boldsymbol{k}) = \left(\frac{1}{\pi\,\alpha}\right)^{s/2} \exp\left(-\frac{\|\boldsymbol{k}\|^2}{\alpha}\right) \xrightarrow{\text{in } S'} \delta(\boldsymbol{k}) \text{ when } \alpha \to 0\ ,$$

such that $B_\alpha = \int d^s k \int d^s k'\, f_\alpha(\boldsymbol{k})\, \varphi_\alpha(\boldsymbol{k}')\, a^*\,(\boldsymbol{k}' + \boldsymbol{k})\, a^*\,(\boldsymbol{k}' - \boldsymbol{k})$ is well defined on $|0\rangle$, as we can easily show:

$$\langle 0 |\, B_\alpha^*\, B_\alpha\, | 0 \rangle = \int d^s k\, |f_\alpha(\boldsymbol{k})|^2 \int d^s k'\, |\varphi_\alpha(\boldsymbol{k}')|^2 < \infty\ .$$

The same sort of calculations can be done for B^n, and we can then define $T_\alpha = \exp(-B_\alpha)$ on $|0\rangle$.

We write $|\breve{0}_\alpha\rangle$ for $T_\alpha\,|0\rangle$. Our aim is now to establish that:

(i) $\breve{a}(\boldsymbol{k})\,|\breve{0}_\alpha\rangle$ goes to 0 in some way when $\alpha \to 0$;

(ii) $\breve{\Phi}(x)$ is well-defined on $|\breve{0}_\alpha\rangle$;

(iii) $\lim_{\alpha=0}\langle\breve{0}_\alpha|\breve{\Phi}(x)\breve{\Phi}(x')\,|\breve{0}_\alpha\rangle/\langle\breve{0}_\alpha|\breve{0}_\alpha\rangle$ exists and is equal to

$$(1+2\mathfrak{G})\left(\frac{1}{2\pi}\right)^s\int\frac{d^sk}{2\,\varepsilon_k}\,e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}\,e^{-i\varepsilon_k(t-t')}\,.$$

Point (i) is formally achieved when we obtain, using the calculation rules published in [6]:

$$\breve{a}(\boldsymbol{k})\,|\breve{0}_\alpha\rangle = \frac{1}{2}\left(\frac{1+2\mathfrak{G}}{\alpha_k}\right)^{1/2} T_\alpha\ \times$$

$$\left\{(1-\alpha_k)\ a^*(-\boldsymbol{k}) - 2\,(1+\alpha_k)\int d^sp\ f_\alpha(\boldsymbol{p})\ \varphi_\alpha\,(\boldsymbol{k}-\boldsymbol{p})\ a^*\,(\boldsymbol{k}-2\,\boldsymbol{p})\right\}\,|0\rangle$$

which shows that we have to take $f(\boldsymbol{k}) = 1/2\,(1-\alpha_k)/(1+\alpha_k)$. For points (ii) and (iii), we must calculate $\langle\breve{0}_\alpha|\,\breve{\Phi}(\varphi)\,\breve{\Phi}(\varphi')\,|\breve{0}_\alpha\rangle$, where φ and φ' are test functions:

$$\langle\breve{0}_\alpha|\,\breve{\Phi}(\varphi)\,\breve{\Phi}(\varphi')\,|\breve{0}_\alpha\rangle$$

$$= \left(\frac{1}{2\pi}\right)^s\int\frac{d^sk}{(2\,\varepsilon_k)^{1/2}}\int\frac{d^sk'}{(2\,\varepsilon_{k'})^{1/2}}\int d^sx\int d^sx'\ e^{i\boldsymbol{k}\cdot\boldsymbol{x}}e^{i\boldsymbol{k}'\cdot\boldsymbol{x}'}\varphi(\boldsymbol{x})\ \varphi(\boldsymbol{x}')\ \times$$

$$\times\left\{\begin{array}{lll} e^{-i\varepsilon_k t}\,e^{-i\varepsilon_{k'}t'} & \langle\breve{0}_\alpha|\ \breve{a}\ (\boldsymbol{k}) & \breve{a}\ (\boldsymbol{k}')\ \ |\breve{0}_\alpha\rangle \\ +\,e^{-i\varepsilon_k t}\,e^{i\varepsilon_{k'}t'} & \langle\breve{0}_\alpha|\ \breve{a}\ (\boldsymbol{k}) & \breve{a}^*(-\boldsymbol{k}')|\breve{0}_\alpha\rangle \\ +\,e^{i\varepsilon_k t}\ \ e^{-i\varepsilon_{k'}t'} & \langle\breve{0}_\alpha|\ \breve{a}^*(-\boldsymbol{k})\,\breve{a}\ (\boldsymbol{k}') & |\breve{0}_\alpha\rangle \\ +\,e^{i\varepsilon_k t}\ \ e^{i\varepsilon_{k'}t'} & \langle\breve{0}_\alpha|\ \breve{a}^*(-\boldsymbol{k})\,\breve{a}^*(-\boldsymbol{k}') & |\breve{0}_\alpha\rangle \end{array}\right\}$$

which is a well-defined expression for $\alpha \neq 0$ (point (ii)). Its first, third and fourth terms, after point (i), vanish with α; the second one is of the form:

$$\breve{a}\,(\boldsymbol{k})\,\breve{a}^*(-\boldsymbol{k}') = \breve{a}^*(-\boldsymbol{k}')\,\breve{a}\,(\boldsymbol{k}) + (1+2\,\mathfrak{G})\ \delta\ (\boldsymbol{k}+\boldsymbol{k}')$$

the δ-part of it leading to the result (iii), the $\breve{a}^*\breve{a}$-part of it going to zero after point (i).

4. Concluding Remarks and Acknowledgements

Quadratic interactions may also be written:

$$V = \frac{\mu}{2\,m}\int_{t=0} d^sx\,:\tilde{\Psi}\,\Psi:\,(x) + \frac{d}{2\,m}\int_{t=0} d^sx\,:\partial_\sigma\,\tilde{\Psi}\,\partial^\sigma\Psi:\,(x)$$

for the spinor field, and:

$$V = \mu\int_{t=0} d^sx\,:\Phi^2:\,(x) + d\int_{t=0} d^sx\,:\partial_\sigma\Phi\,\partial^\sigma\Phi:\,(x)$$

for the scalar field. This way to do gives exactly the same results as what we have achieved, as is seen by the following correspondence rule:

$$d = \mathfrak{G} \text{ and } \mu = \frac{\delta m^2}{2} - \mathfrak{G}\, m^2$$

valid for both fields.

Now, we wish to conclude by remarking that we have accomplished our program, i.e. that we have really renormalized mass and field intensity; to see this, let us take, for instance, the renormalized free hamiltonian $\breve{H}_0$ of the scalar field:

$$\breve{H}_0 = \int d^s k\ \varepsilon_k\, \breve{a}^*(\boldsymbol{k})\, \breve{a}\, (\boldsymbol{k})\ .$$

Evidently, it describes particles of mass $(m^2 + \delta m^2)^{1/2}$, whose field intensity is $(1 + 2\,\mathfrak{G})$ times the bare's one; in fact, we have immediately $:H_0: = (1 + 2\,\mathfrak{G})\, H$, the Wick product being taken with respect to the bare vacuum.

It is a pleasure for us to thank Prof. M. Guenin for having suggested this problem and for his unvaluable help during our work.

This research was partially supported by the National Science Foundation.

REFERENCES

[1] R. Haag, Dan. Mat. Fys. Medd. *29*, 12 (1965).
[2] M. Guenin, G. Velo, Helv. Phys. Acta *41*, 362 (1968).
[3] M. Guenin, Lectures Notes, 9th Summer Institute for Theoretical Physics, Boulder (1966).
[4] J.-P. Eckmann, M. Guenin, private communication, and M. Guenin, *Les systèmes à un nombre infini de degrés de liberté*, CNRS (1969). A dressing transformation is somewhat analogue to what is called displacement operator in P. Kristensen et al., Math. Scand. *14*, 129 (1964).
[5] They have nevertheless been done!
[6] M. Guenin, Helv. Phys. Acta *41*, 75 (1968).