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On the Logarithmic Power of Kernel Integrals

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Abstract. Sufficient conditions are given under which the asymptotic behaviour of integrals is described by pure power counting, excluding therefore logarithmic powers.

1. Introduction

Weinberg's theorem on the asymptotic behaviour of complicated integrals [3] has been used lately quite often in estimates on kernels of biquantized operators, or bilinear forms (for references, see [2]). The original theorem gives only results for the asymptotic behaviour in *powers* of the variables, leaving open a possible occurence of powers of *logarithmic terms*. Fink [1] has discussed and solved this problem in Weinberg's terminology and he applied his results to self-energy graphs.

In this note we use Fink's argument for a wide class of functions occurring in the kernel estimates mentioned above. It is hoped that this should clarify the question of when logarithmic terms do occur and when they can be excluded with certainty.

In section 2 we state the results in a simple and nonspecialized language which should allow a very quick decision in any particular case, as to whether logarithmic terms will appear or not. In section 3 we prove these results in the form of a more general theorem. Here we use the terminology of [1] and [3]. We have omitted the definitions in order to avoid unnecessary repetitions, and the reader is asked to look up the definitions in the papers of Weinberg and Fink. As a corollary of the theorem we give an upper bound for the logarithmic power for the case where it is not zero.

2. Formulation of the Theorem

We will describe the asymptotic behaviour of the functions by means of the following definition:

Definition: We shall say that the real function $f(k)$ behaves like k^α if f satisfies: For some $b > 0$ there exists a $M = M(b, f) < \infty$ such that

$$|f(k)| \leqslant M |k|^\alpha \text{ for } |k| \geqslant b .$$

If $f(k)$ behaves like k^α we write $f(k) \sim k^\alpha$ [2]).

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[2]) $f(k) \sim k^\alpha$ implies, in the terminology of Weinberg, that f is of class A_1, with asymptotic coefficient $\alpha(R^1) = \alpha$.

Theorem 1: *Let $k_1, \dots k_n, p \in R^\nu$. Let*

$$g(k_1, \dots k_n, p) = \prod_{j=1}^{m} f_j\,(\sum_{i=1}^{n} a_{j\,i}\, k_i + a_{j\,0}\, p) \;\cdot$$
$$\prod_{j=m+1}^{m'} f_j\,(\sum_{i=1}^{n} a_{j\,i}\, |k_i| + a_{j\,0}\, |p| + c_j)$$

where $f_j(k) = f_j(|k|) \sim k^{\alpha_j}$, $j = 1, \dots m'$; $a_{j\,i} \geqslant 0$ for $j = m+1, \dots m'$, $i = 0, \dots n$ and $c_j \geqslant 0$, $j = m+1, \dots m'$.

If $I = \{i_1, \dots i_q\}$ is a subset of $\{m+1, \dots m'\}$, $0 \leqslant q \leqslant m' - m$, denote by P_I the set of variables occuring in $f_{i_1}, \dots f_{i_q}$ which have a coefficient $a_{j\,i}$ different from zero. Let N_I be the number of such variables. For any subset $J = \{j_1, \dots j_r\}$ of $\{1, \dots m\}$, $0 \leqslant r \leqslant m$, let A_J be the following matrix

$$A_J = \begin{pmatrix} a_{j_1 0} & \dots & a_{j_1 n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{j_r 0} & \dots & a_{j_r n} \end{pmatrix}, \text{ and define}$$

$B_{I,J}$ as the matrix obtained from A_J by striking out the columns corresponding to the variables in P_+. Suppose that the following inequality holds for all pairs of subsets I, J satisfying $r + q > 0$ and $N_I + \operatorname{rank} B_{I,J} \leqslant n$:

$$\sum_{k=1}^{r} \alpha_{j_k} + \sum_{k=1}^{q} \alpha_{i_k} + \nu\, N_I + \nu \operatorname{rank} B_{I,J} > 0\,. \tag{1}$$

Then, if $G(p) = G(|p|) \equiv \int d^\nu k_1 \dots d^\nu k_n\, g(k_1, \dots k_n, p)$ exists, one has

$$G(|p|) \sim p^{\sum_{i=1}^{m'} \alpha_i + n\,\nu}$$

and there are no logarithmic terms.

Remark: The relation (1) measures the connection between the coefficients α_i and the linear dependence of the arguments of the function. We will state this in a more compact form in Theorem 2.

Examples: Let $\mu(k) = (k^2 + m^2)^{1/2}$, $m > 0$, $k \in R^\nu$.

1. Let $g(k_1, k_2, p) = \mu\,(k_1 - p)^\alpha\, \mu\,(k_2 - p)^\beta\, \mu\,(k_1 - k_2)^\gamma$. Then (1) amounts to $\alpha, \beta, \gamma > -\nu$.

2. Let $g(k_1, k_2, p) = \mu\,(k_1 - p)^\alpha\, \mu\,(k_2 - p)^\beta \mu\,(k_1 - k_2)^\gamma\, \mu\,(k_1 + k_2)^\delta\, \mu\,(k_1 + 2\,k_2)^\varepsilon$. Then (1) amounts to $\alpha, \beta, \gamma, \delta, \varepsilon > -\nu$, $\gamma + \delta + \varepsilon > -2\,\nu$.

3. Let $g(k_1, k_2, p) = \mu\,(k_1 - p)^\alpha\, \mu\,(k_2 - p)^\beta\, \mu\,(k_1 - k_2)^\gamma\, \mu\,(k_1 - 2\,k_2)^\delta\, (\mu(k_1) + \mu(k_2))^\varepsilon$. Then g can be majorized by a function of the class described by the theorem, $(\mu(k_1) + \mu(k_2))^\varepsilon < (|k_1| + |k_2| + m)^\varepsilon \cdot \text{const}$ and (1) amounts to $\alpha, \beta, \gamma, \delta, \varepsilon > -\nu$, $\gamma + \delta + \varepsilon > -2\,\nu$.

3. General Formulation and Proofs

We shall generalize now the class of functions considered, by introducing the following definitions. We consider functions of the variables $k_0, \dots k_n \in R^1$. Let

$$f_A(k_0, \dots k_n) = f(\sum_{i=0}^{n} a_i\, k_i)\;, \tag{2}$$

where A is the one-dimensional subspace corresponding to the vector $(a_0, \dots a_n) \in R^{n+1}$, $A = \{a_0, \dots a_n\}$. Let $f(k) \sim k^\alpha$. It will be convenient to write $U = \{k_0, \dots k_n\}$ and to say that $f_A(\lambda\, U) \sim \lambda^\alpha$ if and only if U is not orthogonal to A; $(\lambda \in R^1)$. We want to generalize this to higher-dimensional A's. Let

$$f_{\{A_1, \dots A_q\}}(U) = h\Big(\sum_{i=1}^{q} |g_{A_i}(U)|\Big) \quad , \tag{3}$$

where all g_{Ai} are obtained from one g as in (2), where $g(k) \sim k^\gamma$, $h(k) \sim k^{\gamma'}$ and $A = \{A_1, \dots A_q\}$ is the subspace spanned by $A_1, \dots A_q$. It is clear that $f_A(\lambda\, U) \sim \lambda^{\gamma\gamma'}$ if and only if U is not orthogonal to A.

Definition: Let C be the class of functions obtained by repeating the above construction (i.e. take in (3) also g_{Ai} where $\dim A_i > 1$).

Example: $(\mu(k_1) + \dots + \mu(k_r))^\alpha$, $k_i \in R^\nu$ is a function in C: Indeed $\mu(k_i) = (k_{i,1}^2 + \dots + k_{i,\nu}^2 + m^2)^{1/2}$ is in C and hence also the sum of μ's to the power α.

With these definitions we restate Theorem 1 in a more general and more compact form.

Theorem 2: *Let R^{n+1} be the space of the variables $(k_0, \dots k_n)$. Let $g(k_0, \dots k_n) = g(U) = \prod_{i=1}^{m} f_{L_i}^{(i)}(U)$ where the functions $f_{L_i}^{(i)}$ are in C; they are derived from functions $f^{(i)}$, $i = 1, \dots m$; and $f_{L_i}^{(i)}(\lambda\, U) \sim \lambda^{\alpha_i}$ whenever U is not orthogonal to the linear subspace L_i. Suppose that for any subset $\{i_1, \dots i_q\}$ of $\{1, \dots m\}$ the following is true:*

$$\textit{If } \dim\{L_{i_1}, \dots L_{i_q}\} \leqslant n \textit{ then } \sum_{j=1}^{q} \alpha_{i_j} > -\dim\{L_{i_1}, \dots L_{i_q}\}\,. \tag{4}$$

Then, if $G(k_0) = \int dk_1 \dots dk_n\, g(k_0, \dots k_n)$ exists, one has $G(k_0) \sim k_0^{\sum_{i=1}^{m} \alpha_i + n}$. There are no logarithmic terms.

Remark: The case where (4) is not always fulfilled is discussed at the end of the paper.

Proof: We proceed by several lemmata, using the terminology of Weinberg and Fink, without restating the precise definitions. It is always to be understood that the asymptotic powers etc. are taken with respect to the function g, *satisfying the hypotheses of the theorem.*

Definition: Let $A = \{L_{i_1}, \dots L_{i_q}\}$. Then S_A is defined to be the orthogonal complement of A in R^{n+1}.

In the following lemmata we always assume that the L_i have been renumbered in such a way that $A = \{L_1, \dots L_q\}$. Let S be a linear subspace of R^{n+1} and define, as in [3], $\alpha(S)$ to be the asymptotic power corresponding to S.

Lemma 1: *Let $A = \{L_1, \dots L_q\}$ and let $L_i \notin A$, $i = q+1, \dots m$.* Let $S \subset R^{n+1}$, *S orthogonal to A and let L_i be not orthogonal to S, $i = q+1, \dots m$. Then*

$$\alpha(S) + \dim S < \alpha(S_A) + \dim S_A\,.$$

Proof: It is easily seen that for any $T \subset R^{n+1}$, $\alpha(T) = \sum^* \alpha_i$, where $\sum^*$ extends over all i, $1 \leqslant i \leqslant m$, such that L_i is not orthogonal to T. Therefore $\alpha(S) = \alpha(S_A) = \alpha_{q+1} + \dots + \alpha_m$. It is clear that $\dim S + \dim A \leqslant n+1$, and as $\dim S_A + \dim A = n+1$, the lemma is proved.

We come now to the central lemma, in which we use the condition (4) on g.

Lemma 2: *Let S be a linear subspace of* R^{n+1}, $0 < \dim S < n+1$. *Then*

$$\alpha(S_\emptyset) + \dim S_\emptyset > \alpha(S) + \dim S \, .$$

Proof: Note that $S_\emptyset = R^{n+1}$. Let first $S = S_A$ with $A = \{L_1, \dots L_q\}$, $L_i \notin A$, $i = q+1, \dots m$. Then

$$\begin{aligned} &\alpha(S_\emptyset) + \dim S_\emptyset - \alpha(S_A) - \dim S_A = \\ &= \alpha_1 + \dots + \alpha_m + (n+1) - \alpha_{q+1} - \dots - \alpha_m - (n+1) + \dim A = \\ &= \alpha_1 + \dots + \alpha_q + \dim\{L_1, \dots L_q\} > 0 \, , \text{ by (4).} \end{aligned}$$

(Note that automatically $\dim A \leqslant n$.) The case of a general S is now an immediate consequence of Lemma 1, and hence Lemma 2 is proved.

Let I be a linear subspace of R^{n+1} and let, as in [3],

$$\alpha_I(S) = \max_{\substack{S' \\ A(I)S' = S}} \alpha(S') + \dim S' - \dim S$$

be the asymptotic power of g with respect to S after integration over I. $A(I)S'$ is the projection of S' along I onto a complement E of I. Let P_I be the projection onto the orthogonal complement of I.

Lemma 3: *Let I_j denote the space* $\{k_1\} \oplus \dots \oplus \{k_j\}$, $j = 1, \dots n$[3]). *Then*

$$\alpha_{I_j}(P_{I_j} S_\emptyset) = \alpha(S_\emptyset) + \dim S_\emptyset - \dim P_{I_j} S_\emptyset \, .$$

Proof: Write $S_j = P_{I_j} S_\emptyset$. By definition

$$\alpha_{I_j}(S_j) = \max_{A(I_j)S = S_j} \alpha(S) + \dim S - \dim S_j \, .$$

By [3], Appendix (B), as I_j is disjoint from S_j, $S = S_\emptyset = S_j + I_j$ is a possible choice for S. By Lemma 2, if $S \subset S_\emptyset$ and $S \neq S_\emptyset$, then $\alpha(S) + \dim S < \alpha(S_\emptyset) + \dim S_\emptyset$. Therefore the maximum is attained for the choice $S = S_\emptyset$. This proves the lemma.

Before we can look at the question of logarithmic powers, we need two further technical lemmata.

Lemma 4: *Let $S \subset S_j$ and* $\dim S < \dim S_j$. *Then*

$$\alpha_{I_j}(S_j) + \dim S_j > \alpha_{I_j}(S) + \dim S, \; j = 1, \dots n \, .$$

Proof: By definition,

$$\alpha_{I_j}(S) + \dim S = \max_{A(I_j)S' = S} \alpha(S') + \dim S' \, .$$

By Lemma 3,

$$\alpha_{I_j}(S_j) + \dim S_j = \alpha(S_\emptyset) + \dim S_\emptyset \, .$$

Note that $\dim S' \leqslant \dim I_j + \dim S < \dim I_j + \dim S_j = n+1$, by [3], Appendix (B) and by hypothesis. Therefore, by Lemma 2,

$$\alpha(S') + \dim S' < \alpha_{I_j}(S_j) + \dim S_j$$

for all S' with $A(I_j)\, S' = S$. QED.

We attack now the question of possible logarithmic powers, for which we have to determine the number of 'maximizing subspaces' ([1], sect. F).

[3]) $\{k_j\} = \{(0, \dots 1, \dots 0)\}$ with the 1 at the $j+1-st$ place.

Lemma 5: *For all $j = 1, \dots n$, the maximum*

$$Q = \max_{\Lambda(\{k_j\})\, S = S_j} \alpha_{I_{j-1}}(S) + \dim S$$

is attained for $S = S_{j-1}$, and for no other S.

Proof: By equation IV.10 in [3], Q is equal to $\alpha_{I_j}(S_j)$. By [3], Appendix (B), $S \subset S_j + \{k_j\} = S_{j-1}$, so $S = S_{j-1}$ is indeed a possible choice for S, and necessarily all the other choices of S with $\Lambda(\{k_j\})\ S = S_j$ fulfil $S \subset S_{j-1}$. Now by Lemma 4, if $S \subset S_{j-1}$ and $S \neq S_{j-1}$ then $\alpha_{I_{j-1}}(S) + \dim S < \alpha_{I_{j-1}}(S_{j-1}) + \dim S_{j-1}$. The maximum in (5) is therefore attained only for $S = S_{j-1}$. QED.

We prove now Theorem 2 by using Theorem 4 of Fink. Indeed, our g is in his class B_{n+1}. By definition,

$$\alpha_{I_n}(\{k_0\}) = \max_{\Lambda(I_n)\, S = \{k_0\}} \alpha(S) + \dim S - 1 .$$

The maximum is attained for $S = S_\varnothing = R^{n+1}$, and

$$\alpha_{I_n}(\{k_0\}) = \sum_{i=1}^{m} \alpha_i + n .$$

Consider now the integrations in the order $k_n, \dots k_1$. By Lemma 5 the maximizing subspace for the k_n integration is S_{n-1}, and p_n, the dimension number relative to $S_n = \{k_0\}$ after integration over I_{n-1}[4]), equals one. Now proceed by induction: The basic observation is that each p_j has to be taken with respect to S_j which was the maximizing subspace for the preceding integration (over k_{j+1}). Using now Lemma 5 again, we see that p_j, the dimension number relative to S_j after integration over I_{j-1}, equals one for $j = n, n-1, \dots 1$. Hence all $p_j = 1$, $j = 1, \dots n$ and inserting in Theorem 4 of Fink [1], we get Theorem 2.

By following the steps of the proof of Theorem 2, we arrive at the

Corollary: *If s is the number of dimensions d for which*

$$\sum_{j=1}^{q} \alpha_{i_j} = -\dim\{L_{i_1}, \dots L_{i_q}\} = -d \ (\geqslant -n)$$

for at least one choice of $i_1, \dots i_q$, then

$$G(k_0) \sim k_0^{\sum_{i=1}^{m} \alpha_i + n} (\log |k_0|)^s , \text{ if it exists.}$$

Proof of Theorem 1: As $f_j(k) = f_j(|k|)$, it is of course in the class $\mathcal{C}$. Note that

$$\nu\,(N_I + \operatorname{rank} B_{I,J}) = \dim\{L_{i_1}, \dots L_{i_q}, L_{j_1}, \dots L_{j_r}\} .$$

p has to be considered as a one-dimensional variable $|p| = k_0$. The rest is an easy consequence of Theorem 2.

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[4]) See [1], sect. F.