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Autor: Oudes, G.C. / Rasche, G.

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Mass Differences as Additional Electromagnetic Corrections in Low Energy Elastic and Charge Exchange π N Scattering

by **G. C. Oades** and **G. Rasche**
Institut für Theoretische Physik der Universität Zürich

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Summary. We present a treatment of low energy $\pi^- p \rightarrow \pi^- p$ and $\pi^- p \rightarrow \pi^0 n$ scattering which includes non-relativistically both mass difference effects and the effect of the long range Coulomb potential. Using this formalism we then show how the corrections to the usual charge independent expressions can be calculated in a first order perturbation treatment.

1. Introduction

In a recent paper [1] we have given a formalism which takes into account the Coulomb corrections to the charge independent nuclear πN interaction. These corrections are important when one wishes to isolate the purely nuclear scattering amplitude for use in dispersion relations or in tests of charge independence. It is usually assumed that these Coulomb effects constitute the bulk of the low energy electromagnetic corrections but, to be consistent, it is also necessary to include the electromagnetic mass differences between the proton and the neutron and between the $\pi^\pm$ and π^0 in the analysis of the coupled processes $\pi^- p \rightarrow \pi^- p$ and $\pi^- p \rightarrow \pi^0 n$.

The usual procedure, when analysing differential cross sections, is to take mass differences into account kinematically when extracting the scattering amplitude; the dynamic effects of these mass differences on the scattering amplitude are not treated. In the present paper we include these latter effects in our non-relativistic treatment of the Coulomb corrections. In so doing we adhere as closely as possible to the notation used in Ref. [1]. In case of any confusion or for a more detailed discussion of the problem the reader is refered to this paper which also quotes some of the earlier literature.

In Section 2 we write down the Schrödinger equation in the simultaneous presence of mass differences and Coulomb effects. In Section 3 we deal with the inner Coulomb and mass difference corrections and in Section 4 we deal with the outer Coulomb and mass difference corrections. In Section 5, for use in experimental analyses, we give perturbation expressions correct to first order in the Coulomb and mass difference effects. Finally in two appendices we give the connection between our corrected S-matrix elements and the differential cross sections and we check that our S-matrix, as constructed from the coupled two channel Schrödinger equation, is unitary and symmetric.

2. The Schrödinger Equation

Since the potential energy operator in the Hamiltonian is non-controversial, we concentrate our attention on the kinetic energy term

$$H_{kin} = \frac{1}{2\,m_{op}^{(1)}} (\underset{\sim}{p}^{(1)})^2 + \frac{1}{2\,m_{op}^{(2)}} (\underset{\sim}{p}^{(2)})^2 \quad . \tag{1}$$

Here the superscripts (1) and (2) stand for the pion and nucleon respectively. Because of mass differences the mass factors are operators in isospin space with the form

$$\begin{aligned} \frac{1}{m_{op}^{(1)}} &= -\frac{1}{\mu_-}\frac{1}{2} t_3 (1 - t_3) + \frac{1}{\mu_0} (1 + t_3)(1 - t_3) \\ \frac{1}{m_{op}^{(2)}} &= \frac{1}{m_p}\frac{1}{2} (1 + \tau_3) + \frac{1}{m_n}\frac{1}{2} (1 - \tau_3) \end{aligned} \tag{2}$$

where μ_- is the mass of the π^-, μ_0 the mass of the π^0, m_p the mass of the proton and m_n the mass of the neutron. $\underset{\sim}{t}$ and $1/2\,\underset{\sim}{\tau}$ are the pion and nucleon isospin operators. The operators in (2) are diagonal in the charge basis of isospin space, $|c\rangle$ ($c = -, 0$ denoting the $\pi^- p$ and $\pi^0 n$ states), and so the inverses of these operators exist, having the form

$$\begin{aligned} m_{op}^{(1)} &= -\mu_- \frac{1}{2} t_3 (1 - t_3) + \mu_0 (1 + t_3)(1 - t_3) \\ m_{op}^{(2)} &= m_p \frac{1}{2} (1 + \tau_3) + m_n \frac{1}{2} (1 - \tau_3) \quad . \end{aligned} \tag{3}$$

The operators $m_{op}^{(1)}$ and $m_{op}^{(2)}$ commute so one can introduce total linear momentum $\underset{\sim}{P}$ and relative linear momentum $\underset{\sim}{p}$ and transform to the c.m. frame $\underset{\sim}{P} = \underset{\sim}{0}$.

Using the same algebraic manipulations as in the single channel case we obtain

$$H_{kin} = \frac{1}{2\,m_{op}} (\underset{\sim}{p})^2 , \quad \underset{\sim}{P} = \underset{\sim}{0} \tag{4}$$

where

$$\frac{1}{m_{op}} = \frac{1}{m_{op}{}^{(1)}} + \frac{1}{m_{op}{}^{(2)}} \; .$$

In the charge basis we have in matrix notation

$$\frac{1}{m_{op}} = \begin{pmatrix} \dfrac{1}{m_p} + \dfrac{1}{\mu_-} & 0 \\ 0 & \dfrac{1}{m_n} + \dfrac{1}{\mu_0} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{m_-} & 0 \\ 0 & \dfrac{1}{m_0} \end{pmatrix} \tag{5}$$

where m_- and m_0 are the reduced masses of the $\pi^- p$ and $\pi^0 n$ systems. The inverse of (5) is obviously

$$m_{op} = \begin{pmatrix} m_- & 0 \\ 0 & m_0 \end{pmatrix} \quad . \tag{6}$$

Thus we see that taking into account mass differences in the kinetic energy for the relative motion amounts to replacing the reduced mass by a mass operator which, in the charge basis, has the form given by (6). In addition, one has to introduce a rest mass operator which takes into account the difference in the rest energies of the two channels. The form of this operator is:

$$M_{op} = m_p \frac{1}{2} (1 + \tau_3) + m_n \frac{1}{2} (1 - \tau_3) + \mu_0 (1 + t_3) (1 - t_3) - \mu_- \frac{1}{2} t_3 (1 - t_3)$$

or, in the charge basis

$$M_{op} = \begin{pmatrix} m_p + \mu_- & 0 \\ 0 & m_n + \mu_0 \end{pmatrix}$$

Using units with $\hbar = c = 1$ we arrive at the Schrödinger equation

$$\left(- \frac{1}{2\, m_{op}} \Delta + U_{op} + V_{op} + M_{op} \right) \psi = E \psi$$

where U_{op} is the strong, charge independent nuclear potential and V_{op} is the long range Coulomb potential. U_{op} is diagonal in the total isospin basis of isospin space. We assume that no other channels are open so that U_{op} is a real potential. The extension to the inelastic scattering region by the introduction of a complex U_{op} is trivial. The only essential difference being that our 2×2 S-matrix is no longer unitary. V_{op} has the form

$$V_{op} = V_c(r)\, t_3\, 1/2\, (1 + \tau_3)\,,$$

$-V_c(r)$ being the static Coulomb potential in the $\pi^- p$ state. The wave function ψ depends on the relative coordinates in the c.m. frame, on the nucleon spin variable, and on the isospin variables of the pion and of the nucleon.

We now go over to the partial waves corresponding to diagonal absolute value of the total angular momentum, 3rd component of the total angular momentum and parity,

$$\psi_{l\pm}(x, s) = [R_{l\pm}^{(-)}(r)\, |-\rangle + R_{l\pm}^{(0)}(r)\, |0\rangle] \frac{1}{r} \Omega_{l\pm,s}(\theta, \phi)$$

where $\Omega_{l\pm,s}$ is defined in Ref. [1]. The radial wave equations for the functions $R_{l\pm}$ can now be written down. For simplicity we only consider the case of $l = 0$, although the results can be immediately generalised, and we use the notation of Ref. [1]

$$R_{o+}^{(c)} \equiv R_c^{TOT}\,.$$

The superscript ^{TOT} means that the radial wave functions are exact solutions of the coupled channel system in distinction to solutions of other equations which will be introduced later.

The system of coupled equations determining R_c^{TOT} can be written in matrix form as

$$\left[\frac{1}{2} \begin{pmatrix} \frac{1}{m_-} & 0 \\ 0 & \frac{1}{m_0} \end{pmatrix} \frac{d^2}{dr^2} + \begin{pmatrix} E - m_p - \mu_- & 0 \\ 0 & E - m_n - \mu_0 \end{pmatrix} - \begin{pmatrix} \frac{1}{3} U_3 + \frac{2}{3} U_1 - V_c & \frac{\sqrt{2}}{3} (U_3 - U_1) \\ \frac{\sqrt{2}}{3} (U_3 - U_1) & \frac{2}{3} U_3 + \frac{1}{3} U_1 \end{pmatrix} \right] \begin{pmatrix} R_-^{TOT} \\ R_0^{TOT} \end{pmatrix} = 0\,. \tag{7}$$

Here U_3 and U_1 are the s-wave diagonal values of U_{op} in the total isospin basis of isospin space. Defining wave numbers corresponding to the relative linear momenta of the two channels by

$$k_-^2 \equiv 2\, m_- \,(E - m_p - \mu_-$$

$$k_0^2 \equiv 2\, m_0 \,(E - m_n - \mu_0)$$

we can rewrite (7) in the form

$$\left[\begin{pmatrix} \frac{d^2}{dr^2} + k_-^2 & 0 \\ 0 & \frac{d^2}{dr^2} + k_0^2 \end{pmatrix} - \begin{pmatrix} 2\,m_- & 0 \\ 0 & 2\,m_0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} U_3 + \frac{2}{3} U_1 - V_c & \frac{\sqrt{2}}{3} (U_3 - U_1) \\ \frac{\sqrt{2}}{3} (U_3 - U_1) & \frac{2}{3} U_3 + \frac{1}{3} U_1 \end{pmatrix}\right] \begin{pmatrix} R_-^{TOT} \\ R_0^{TOT} \end{pmatrix} = 0 \,. \tag{8}$$

We can alternatively transform this equation to the total isospin basis when we obtain the corresponding equation for R_{2T}^{TOT}, the s-wave radial wave functions in the total isospin basis.

Defining

$$m_- - m_0 \equiv \Delta m$$

$$k_0^2 - k_-^2 \equiv \Delta k^2 \tag{9}$$

the transformed equation becomes

$$\left[\begin{pmatrix} \frac{d^2}{dr^2} + k_-^2 + \frac{2}{3} \Delta k^2 & \frac{\sqrt{2}}{3} \Delta k^2 \\ \frac{\sqrt{2}}{3} \Delta k^2 & \frac{d^2}{dr^2} + k_-^2 + \frac{1}{3} \Delta k^2 \end{pmatrix} \right.$$

$$\left. - \begin{pmatrix} 2\,m_-\,U_3 - \frac{2}{3} m_-\,V_c - \frac{4}{3} \Delta m U_3 & \frac{2\sqrt{2}}{3} m_-\,V_c - \frac{2\sqrt{2}}{3} \Delta m U_1 \\ \frac{2\sqrt{2}}{3} m_-\,V_c - \frac{2\sqrt{2}}{3} \Delta m\,U_3 & 2\,m_-\,U_1 - \frac{4}{3} m_-\,V_c - \frac{2}{3} \Delta m U_1 \end{pmatrix}\right] \begin{pmatrix} R_3^{TOT} \\ R_1^{TOT} \end{pmatrix} = 0 \,. \tag{10}$$

3. The Inner Corrections

We first consider the functions R_{2T}^{IN} which are solutions of the equation obtained from (10) when mass differences and the Coulomb potential are only included for $r \leqslant r_0$, where

$$r_0 = \max\,(r_N, r_c) \,.$$

Here r_N is the range of the nuclear interaction and r_c is the charge radius beyond which the $\pi^- p$ Coulomb potential behaves like a point charge potential. Thus we have

$$\begin{pmatrix} \frac{d^2}{dr^2} + k_-^2 & 0 \\ 0 & \frac{d^2}{dr^2} + k_-^2 \end{pmatrix} \begin{pmatrix} R_3^{IN} \\ R_1^{IN} \end{pmatrix} = 0 \quad r \geqslant r_0 \tag{11}$$

while for $r \leqslant r_0$ the functions R_{2T}^{IN} obey (10).

We now introduce solutions R_{2T}^{N} of the purely nuclear problem i.e.

$$\begin{bmatrix} \frac{d^2}{dr^2} + k_-^2 - 2\,m_-\,U_3 & 0 \\ 0 & \frac{d^2}{dr^2} + k_-^2 - 2\,m_-\,U_1 \end{bmatrix} \begin{pmatrix} R_3^N \\ R_1^N \end{pmatrix} = 0 \tag{12}$$

and choose the two linearly independent solutions which have the behaviour

$$\begin{aligned} \begin{pmatrix} R_{3\alpha}^N \\ R_{1\alpha}^N \end{pmatrix} &= \begin{pmatrix} \sin(k_-\,r + \delta_3) \\ 0 \end{pmatrix} \qquad r \geqslant r_N \\ \begin{pmatrix} R_{3\beta}^N \\ R_{1\beta}^N \end{pmatrix} &= \begin{pmatrix} 0 \\ \sin(k_-\,r + \delta_1) \end{pmatrix} \qquad r \geqslant r_N \end{aligned} \tag{13}$$

where δ_{2T} are the purely nuclear charge independent s-wave phases. We have arbitrarily defined the charge independent phases to correspond to scattering with the $\pi^\pm$ and proton masses so as to avoid mass corrections in $\pi^+ p$ scattering.

Since (11) and (12) coincide for $r \geqslant r_0$ we can write the general solution of (11) in terms of four constants α_3, α_1, β_3, β_1, in the form

$$\begin{aligned} R_3^{IN} &= \alpha_3\,R_{3\alpha}^N + \beta_3\,R_{1\beta}^N\,, \\ R_1^{IN} &= \alpha_1\,R_{3\alpha}^N + \beta_1\,R_{1\beta}^N\,, \end{aligned} \qquad r \geqslant r_0 \quad . \tag{14}$$

Performing the same manipulations as in Ref. [1] we arrive at implicit relations between these constants of the form

$$\begin{aligned} \chi_3 \equiv \beta_3 \sin(\delta_3 - \delta_1) &= \\ = -\frac{1}{3\,k_-} \int_0^{r_0} dr\,R_{3\alpha}^N\,&[2\,m_-\,V_c\,(R_3^{IN} - \sqrt{2}\,R_1^{IN}) \\ &+ 2\sqrt{2}\,\Delta m\,(\sqrt{2}\,U_3\,R_3^{IN} + U_1\,R_1^{IN}) \\ &+ \sqrt{2}\,\Delta k^2\,(\sqrt{2}\,R_3^{IN} + R_1^{IN})] \end{aligned} \tag{15a}$$

$$\begin{aligned} \chi_1 \equiv \alpha_1 \sin(\delta_3 - \delta_1) &= \\ = -\frac{1}{3\,k_-} \int_0^{r_0} dr\,R_{1\beta}^N\,&[2\sqrt{2}\,m_-\,V_c\,(R_3^{IN} - \sqrt{2}\,R_1^{IN}) \\ &- 2\,\Delta m\,(\sqrt{2}\,U_3\,R_3^{IN} + U_1\,R_1^{IN}) \\ &- \Delta k^2\,(\sqrt{2}\,R_3^{IN} + R_1^{IN})]\,. \end{aligned} \tag{15b}$$

Thus combining (14) and (15) we have

$$\begin{aligned} R_3^{IN} &= \alpha_3\,R_{3\alpha}^N + \frac{\chi_3}{\sin(\delta_3 - \delta_1)}\,R_{1\beta}^N \\ & \qquad\qquad\qquad\qquad r \geqslant r_0\,. \\ R_1^{IN} &= \frac{\chi_1}{\sin(\delta_3 - \delta_1)}\,R_{3\alpha}^N + \beta_1\,R_{1\beta}^N \end{aligned}$$

Finally we select two linearly independent solutions $R^{IN}_{2T,\alpha}$ and $R^{IN}_{2T,\beta}$ corresponding to the choices $\alpha_3 = 1$, $\beta_1 = 0$ and $\alpha_3 = 0$, $\beta_1 = 1$ i.e.

$$\left.\begin{aligned} R^{IN}_{3\alpha} &= R^N_{3\alpha} + \frac{\chi_{3\alpha}}{\sin(\delta_3 - \delta_1)} R^N_{1\beta} \\ R^{IN}_{1\alpha} &= \frac{\chi_{1\alpha}}{\sin(\delta_3 - \delta_1)} R^N_{3\alpha} \\ R^{IN}_{3\beta} &= \frac{\chi_{3\beta}}{\sin(\delta_3 - \delta_1)} R^N_{1\beta} \\ R^{IN}_{1\beta} &= \frac{\chi_{1\beta}}{\sin(\delta_3 - \delta_1)} R^N_{3\alpha} + R^N_{1\beta} \end{aligned}\right\} \quad r \geqslant r_0 \tag{16}$$

4. The Outer Corrections

We now have to take into account the tail of the Coulomb potential and the effect of the mass differences for $r \geqslant r_0$. It is convenient to work in the charge basis so we transform (16) to obtain

$$\left.\begin{aligned} R^{IN}_{-\alpha} &= \sqrt{\frac{1}{3}}\, R^N_{3\alpha} + \frac{\sqrt{\frac{1}{3}}\,\chi_{3\alpha} R^N_{1\beta} - \sqrt{\frac{2}{3}}\,\chi_{1\alpha} R^N_{3\alpha}}{\sin(\delta_3 - \delta_1)} \\ R^{IN}_{-\beta} &= -\sqrt{\frac{2}{3}}\, R^N_{1\beta} + \frac{\sqrt{\frac{1}{3}}\,\chi_{3\beta} R^N_{1\beta} - \sqrt{\frac{2}{3}}\,\chi_{1\beta} R^N_{3\alpha}}{\sin(\delta_3 - \delta_1)} \\ R^{IN}_{0\alpha} &= \sqrt{\frac{2}{3}}\, R^N_{3\alpha} + \frac{\sqrt{\frac{2}{3}}\,\chi_{3\alpha} R^N_{1\beta} + \sqrt{\frac{1}{3}}\,\chi_{1\alpha} R^N_{3\alpha}}{\sin(\delta_3 - \delta_1)} \\ R^{IN}_{0\beta} &= \sqrt{\frac{1}{3}}\, R^N_{1\beta} + \frac{\sqrt{\frac{2}{3}}\,\chi_{3\beta} R^N_{1\beta} + \sqrt{\frac{1}{3}}\,\chi_{1\beta} R^N_{3\alpha}}{\sin(\delta_3 - \delta_1)} \end{aligned}\right\} \quad r \geqslant r_0 \,. \tag{17}$$

From (8) we see that R^{TOT}_c and R^{IN}_c satisfy the same coupled equations for $r \leqslant r_0$ and so we can write

$$\begin{pmatrix} R^{TOT}_- \\ R^{TOT}_0 \end{pmatrix} = A_\alpha \begin{pmatrix} R^{IN}_{-\alpha} \\ R^{IN}_{0\alpha} \end{pmatrix} + A_\beta \begin{pmatrix} R^{IN}_{-\beta} \\ R^{IN}_{0\beta} \end{pmatrix} \qquad r \leq r_0 \tag{18}$$

where A_α and A_β are arbitrary constants.

Also from (8) we see that for $r \geqslant r_0$, R^{TOT}_c satisfy the equations

$$\left(\frac{d^2}{dr^2} + k^2_-\right) R^{TOT}_- + 2\, m_-\, V_c\, R^{TOT}_- = 0 \tag{19a}$$

$$\left(\frac{d^2}{dr^2} + k^2_0\right) R^{TOT}_0 = 0 \,. \tag{19b}$$

We now define two independent solutions of (19a), u_α and u_β, with asymptotic behaviour

$$u_\alpha \underset{r \to \infty}{\sim} \sin(k_- r + \tau_\alpha^{(-)} - \sigma_0 + \eta \ln 2 k_- r)$$
$$u_\beta \underset{r \to \infty}{\sim} \sin(k_- r + \tau_\beta^{(-)} - \sigma_0 + \eta \ln 2 k_- r) \tag{20}$$

where the Coulomb parameter η is given by

$$\eta = \frac{e^2 m_-}{k_-} \qquad (e^2 = \alpha = 137.0388^{-1})$$

and the s-wave Coulomb phase is given by

$$\sigma_0 = \arg \Gamma(1 + i \eta) .$$

The phases $\tau_\alpha^{(-)}$ and $\tau_\beta^{(-)}$ are fixed by imposing the condition

$$\left.\frac{u'_\alpha}{u_\alpha}\right|_{r = r_0} = \left.\frac{R_{-\alpha}^{IN\prime}}{R_{-\alpha}^{IN}}\right|_{r = r_0}$$
$$\left.\frac{u'_\beta}{u_\beta}\right|_{r = r_0} = \left.\frac{R_{-\beta}^{IN\prime}}{R_{-\beta}^{IN}}\right|_{r = r_0} \tag{21}$$

where the prime denotes differentiation with respect to r. For $r \geqslant r_0$, R_-^{TOT} obeys the same equation as u_α and u_β so we can write

$$R_-^{TOT} = B_\alpha u_\alpha + B_\beta u_\beta \qquad r \geqslant r_0 . \tag{22}$$

Matching value and derivative of R_-^{TOT} at r_0 as given by (18) and by (22) we obtain

$$B_\alpha = X_\alpha A_\alpha , \quad B_\beta = X_\beta A_\beta \tag{23}$$

where the constants X_α and X_β are given by

$$u_\alpha(r_0) = X_\alpha^{-1} R_{-\alpha}^{IN}(r_0)$$
$$u_\beta(r_0) = X_\beta^{-1} R_{-\beta}^{IN}(r_0) . \tag{24}$$

In the same spirit we define two independent solutions of (19b), v_α and v_β with asymptotic behaviour

$$v_\alpha \underset{r \to \infty}{\sim} \sin(k_0 r + \tau_\alpha^{(0)})$$
$$v_\beta \underset{r \to \infty}{\sim} \sin(k_0 r + \tau_\beta^{(0)}) , \tag{25}$$

the phases $\tau_\alpha^{(0)}$ and $\tau_\beta^{(0)}$ being determined by

$$\left.\frac{v'_\alpha}{v_\alpha}\right|_{r = r_0} = \left.\frac{R_{0\alpha}^{IN\prime}}{R_{0\alpha}^{IN}}\right|_{r = r_0}$$
$$\left.\frac{v'_\beta}{v_\beta}\right|_{r = r_0} = \left.\frac{R_{0\beta}^{IN\prime}}{R_{0\beta}^{IN}}\right|_{r = r_0} . \tag{26}$$

We also define two constants Y_α and Y_β by

$$v_\alpha(r_0) = Y_\alpha^{-1} R_{0\alpha}^{IN}(r_0)$$
$$v_\beta(r_0) = Y_\beta^{-1} R_{0\beta}^{IN}(r_0) . \tag{27}$$

For $r \geqslant r_0$, R_0^{TOT} obeys the same equation as v_α and v_β and so we arrive at the analogous result to (22) and (23),

$$R_0^{TOT} = A_\alpha\, Y_\alpha\, v_\alpha + A_\beta\, Y_\beta\, v_\beta \qquad r \geqslant r_0 \,. \tag{28}$$

Thus our general solution of the two channel $\pi^- p$, $\pi^0 n$ problem with Coulomb interaction and mass differences included has the form

$$\begin{pmatrix} R_-^{TOT} \\ R_0^{TOT} \end{pmatrix} = A_\alpha \begin{pmatrix} X_\alpha\, u_\alpha \\ Y_\alpha\, v_\alpha \end{pmatrix} + A_\beta \begin{pmatrix} X_\beta\, u_\beta \\ Y_\beta\, v_\beta \end{pmatrix} \qquad r \geqslant r_0 \tag{29}$$

where A_α and A_β are arbitrary constants.

Using the known asymptotic behaviour of the u's and the v's we can proceed from (29) to obtain the S-matrix elements in the charge basis, taking care to include the correct normalisation factors since the $\pi^- p$ and $\pi^0 n$ systems have different relative velocities. The resulting expressions are:

$$\begin{aligned}
S_{TOT}^{--} &= \left[X_\alpha\, Y_\beta\, e^{i\left(\tau_\alpha^{(-)} - \tau_\beta^{(0)}\right)} - X_\beta\, Y_\alpha\, e^{i\left(\tau_\beta^{(-)} - \tau_\beta^{(0)}\right)}\right] D_{TOT}^{-1} \\
S_{TOT}^{-0} &= \left(\frac{k_-\, m_0}{k_0\, m_-}\right)^{1/2} \left[-2\, i\, X_\alpha\, X_\beta \sin\left(\tau_\alpha^{(-)} - \tau_\beta^{(-)}\right)\right]\; D_{TOT}^{-1} \\
S_{TOT}^{0-} &= \left(\frac{k_0\, m_-}{k_-\, m_0}\right)^{1/2} \left[2\, i\, Y_\alpha\, Y_\beta \sin\left(\tau_\alpha^{(0)} - \tau_\beta^{(0)}\right)\right]\; D_{TOT}^{-1} \\
S_{TOT}^{00} &= \left[X_\alpha\, Y_\beta\, e^{-i\left(\tau_\alpha^{(-)} - \tau_\beta^{(0)}\right)} - X_\beta\, Y_\alpha\, e^{-i\left(\tau_\beta^{(-)} - \tau_\alpha^{(0)}\right)}\right]\; D_{TOT}^{-1}
\end{aligned} \tag{30}$$

where

$$D_{TOT} = X_\alpha\, Y_\beta\, e^{-i\left(\tau_\alpha^{(-)} + \tau_\beta^{(0)}\right)} - X_\beta\, Y_\alpha\, e^{-i\left(\tau_\beta^{(-)} + \tau_\alpha^{(0)}\right)} \quad . \tag{31}$$

In Appendix I we show how the scattering amplitudes are constructed from these S-matrix elements and in Appendix II we check that the values given by (30) and (31) correspond to the elements of a symmetric and unitary matrix.

5. Perturbation Expressions

We have shown in the previous section how the full S-matrix elements can be obtained once the functions R_c^{IN} are known. These R_c^{IN} can be obtained by numerical solution of the coupled equations for $0 \leqslant r \leqslant r_0$ or alternatively they can be approximated by replacing the R_{2T}^{IN} by R_{2T}^{N} in (15a) and (15b) thus obtaining approximate values of the χ's to substitute into (16). This latter procedure amounts to treating the Coulomb interaction and the mass differences as perturbations. The first order values obtained in this way are

$$\chi_{3\alpha} = -\frac{1}{3\,k_-} \int_0^{r_0} dr (R_{3\alpha}^N)^2\, (2\, m_-\, V_c + 4\, \Delta m\, U_3 + 2\, \Delta k^2)$$

$$\chi_{3\beta} = \frac{1}{3\,k_-} \int_0^{r_0} dr (R_{3\alpha}^N\, R_{1\beta}^N)\, (2\sqrt{2}\; m_-\, V_c - 2\sqrt{2}\; \Delta m\, U_1 - \sqrt{2}\; \Delta k^2)$$

$$\chi_{1\alpha} = -\frac{1}{3k_-}\int\limits_0^{r_0} dr(R^N_{3\alpha}\, R^N_{1\beta})\,(2\sqrt{2}\, m_-\, V_c - 2\sqrt{2}\,\Delta m\, U_3 - \sqrt{2}\,\Delta k^2)$$

$$\chi_{1\beta} = \frac{1}{3k_-}\int\limits_0^{r_0} dr(R^N_{1\beta})^2\,(4\, m_-\, V_c + 2\,\Delta m\, U_1 + \Delta k^2)\,. \tag{32}$$

The nuclear potentials U_3 and U_1 can be eliminated by using (12) to obtain

$$U_{2T} = \left(\frac{d^2 R^N_{2T}}{dr^2} + k^2_-\, R^N_{2T}\right)\cdot(2\, m_-\, R^N_{2T})^{-1}\,.$$

Having obtained the R^{IN}_c, either numerically or in the approximation described above, we then have to calculate values for the X's, $\tau^{(-)}$'s, Y's and $\tau^{(0)}$'s. Using (20) and (24) and matching value and derivative at r_0 we obtain

$$\tan\tau^{(-)}_\alpha = \left.\frac{F_0(-\eta, k_- r)\, R^{IN\prime}_{-\alpha}(r) - F_0(-\eta, k_- r)'\, R^{IN}_{-\alpha}(r)}{G_0'(-\eta, k_- r)\, R^{IN}_{-\alpha}(r) - G_0(-\eta, k_- r)\, R^{IN\prime}_{-\alpha}(r)}\right|_{r=r_0}$$

$$X_\alpha = \left.\frac{R^{IN}_{-\alpha}(r)}{\cos\tau^{(-)}_\alpha\, F_0(-\eta, k_- r) + \sin\tau^{(-)}_\alpha\, G_0(-\eta, k_- r)}\right|_{r=r_0}\,. \tag{33}$$

Analogous expressions are obtained for $\tau^{(-)}_\beta$ and X_β by replacing $R^{IN}_{-\alpha}$ by $R^{IN}_{-\beta}$. In a similar way using (25) and (27) we obtain

$$\tan\tau^{(0)}_\alpha = \left.\frac{[k_0\, r\, j_0(k_0\, r)]\, R^{IN\prime}_{0\alpha}(r) - [k_0\, r\, j_0(k_0\, r)]'\, R^{IN}_{0\alpha}(r)}{[k_0\, r\, n_0(k_0\, r)]\, R^{IN\prime}_{0\alpha}(r) - [k_0\, r\, n_0(k_0\, r)]'\, R^{IN}_{0\alpha}(r)}\right|_{r=r_0}$$

$$Y_\alpha = \left.\frac{R^{IN}_{0\alpha}(r)}{\cos\tau^{(0)}_\alpha\, [k_0\, r\, j_0(k_0\, r)] - \sin\tau^{(0)}_\alpha\, [k_0\, r\, n_0(k_0\, r)]}\right|_{r=r_0}\,. \tag{34}$$

$\tau^{(0)}_\beta$ and Y_β being obtained by replacing $R^{IN}_{0\alpha}$ by $R^{IN}_{0\beta}$. The notation suggests the obvious generalisation to $l \neq 0$.

Having obtained the values for the X's, Y's and τ's the S-matrix elements can be calculated directly from (30) and (31). Alternatively values can be obtained for the eigenphases τ_3 and τ_1 and the mixing parameter ω as defined in Ref. [1]. We have from Ref. [1]

$$S^{--}_{TOT} + S^{00}_{TOT} = e^{2i\tau_3} + e^{2i\tau_1} \tag{35a}$$

$$S^{0-}_{TOT} = S^{-0}_{TOT} = \frac{\sqrt{2}}{3}\,(e^{2i\tau_3} - e^{2i\tau_1}) + \frac{\tan\omega}{3}\,(e^{2i\delta_3} - e^{2i\delta_1}) \tag{35b}$$

$$+ \text{ higher order terms.}$$

If we define Δ's and ε's by

$$\tau^{(-)}_\alpha = \delta_3 + \Delta^{(-)}_3 \qquad \tau^{(0)}_\alpha = \delta_3 + \Delta^{(0)}_3$$

$$\tau^{(-)}_\beta = \delta_1 + \Delta^{(-)}_1 \qquad \tau^{(0)}_\beta = \delta_1 + \Delta^{(0)}_1$$

$$X_\alpha = \sqrt{\frac{1}{3}}\,(1 + \varepsilon^{(-)}_3) \qquad Y_\alpha = \sqrt{\frac{2}{3}}\,(1 + \varepsilon^{(0)}_3)$$

$$X_\beta = -\sqrt{\frac{2}{3}}\,(1 + \varepsilon^{(-)}_1) \qquad Y_\beta = \sqrt{\frac{1}{3}}\,(1 + \varepsilon^{(0)}_1) \tag{36}$$

then from (30) and (31) we obtain the first order results

$$S_{TOT}^{--} + S_{TOT}^{00} = e^{2i\delta_3}\left(1 + 2\,i\,\frac{\Delta_3^{(-)} + 2\,\Delta_3^{(0)}}{3}\right) + e^{2i\delta_1}\left(1 + 2\,i\,\frac{2\,\Delta_1^{(-)} + \Delta_1^{(0)}}{3}\right) \tag{37a}$$

$$S_{TOT}^{-0} = \frac{\sqrt{2}}{3}\left(\left(\frac{k_-\,m_0}{k_0\,m_-}\right)^{1/2} + \frac{2\,\varepsilon_3^{(-)} - 2\,\varepsilon_3^{(0)} + \varepsilon_1^{(-)} - \varepsilon_1^{(0)}}{3} + i\,\frac{\Delta_3^{(-)} + 2\Delta_3^{(0)} + 2\Delta_1^{(-)} + \Delta_1^{(0)}}{3}\right) \cdot \left\{e^{2i\delta_3}[1 + i\,(\Delta_3^{(-)} - \Delta_1^{(-)})] - e^{2i\delta_1}\,[1 - i\,(\Delta_3^{(-)} - \Delta_1^{(-)})]\right\}. \tag{37b}$$

Since the decomposition of a complex number into the sum of two complex numbers each of modulus 1 is unique, we have on comparing (35a) and (37a),

$$\tau_3 = \delta_3 + \frac{\Delta_3^{(-)} + 2\,\Delta_3^{(0)}}{3}$$

$$\tau_1 = \delta_1 + \frac{2\,\Delta_1^{(-)} + \Delta_1^{(0)}}{3}\,. \tag{38}$$

Using (38) and comparing (35b) and (37b) we have

$$\tan\,\omega = \sqrt{2}\left[\left(\frac{k_-\,m_0}{k_0\,m_-}\right)^{1/2} - 1\right] + \frac{\sqrt{2}}{3}\left[2\,(\varepsilon_3^{(-)} - \varepsilon_3^{(0)}) + (\varepsilon_1^{(-)} - \varepsilon_1^{(0)}) + \frac{2\,(\Delta_3^{(-)} - \Delta_3^{(0)}) - (\Delta_1^{(-)} - \Delta_1^{(0)})}{\tan\,(\delta_3 - \delta_1)}\right]. \tag{39}$$

APPENDIX I

Differential Cross Section in Terms of S-Matrix Elements

The connection between the S-matrix elements and the differential cross sections proceeds in a similar way to that outlined in Section 3 of Ref. [1]. The asymptotic form of the radial and isotopic part of the wave function for a $\pi^- p$ incoming state of given l value is

$$\left[\sin\,(k_-\;r - \frac{l\pi}{2} - \sigma_l + \eta\,\ln\,2\,k_-\,r) + \frac{1}{2\,i}(S_{l\pm}^{--} - 1)\;e^{i(k_- r\; l\pi/2 - \sigma_l + \eta\ln 2k_- r)}\;|-\rangle\right] + \left[\frac{1}{2\,i}\left(\frac{k_-\,m_0}{k_0\,m_-}\right)^{1/2} S_{l\pm}^{0-}\;e^{i(k_0 r - l\pi/2)}\;|0\rangle\right] \tag{A1.1}$$

where the $S_{l\pm}^{c'\,c}$ are the full S-matrix elements i.e. such that

$$S_{0+}^{c'\,c} = S_{TOT}^{c'\,c}\,.$$

Defining the quantities

$$f_{l\pm}^{--} = \frac{1}{2\,i}\,(S_{l\pm}^{--} - 1)\;e^{-2i\sigma_l}$$

$$f_{l\pm}^{0-} = \frac{1}{2\,i}\;S_{l\pm}^{0-}\;e^{-i\sigma_l}$$

the same treatment as in Ref. [1] shows that the scattering amplitude can be written in the form

$$F^{c'-} = f^{c'-} + i\, \underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{n}}\, g^{c'-} \qquad c' = -, 0$$

where

$$f^{c'-} = \delta_{c'-}\, F_{c_p} + \frac{1}{k_-} \sum_l [(l+1)\, f_{l\pm}^{c'-} + l\, f_{l-}^{c'-}]\, P_l(\cos\theta)$$

$$g^{c'-} = \frac{1}{k_-} \sum_l [f_{l+}^{c'-} - f_{l-}^{c'-}]\, P_l^1(\cos\theta)\,.$$

F_{c_p} is the point charge pure Coulomb scattering amplitude and $\hat{\underset{\sim}{n}}$ is the normal to the scattering plane.

In terms of $f^{c'-}$ and $g^{c'-}$ the differential cross section is given by

$$\frac{d\sigma}{d\Omega}_{-\to c'} = |\, f^{c'-}\,|^2 + |\, g^{c'-}\,|^2\,.$$

It should be noted that the usual treatments of $\pi^- p$ charge exchange scattering approximate the expression

$$\frac{1}{2i} \left(\frac{k_- m_0}{k_0 m_-}\right)^{1/2} S_{l\pm}^{0-}$$

in equation (A1.1) by the charge independent limiting value in neglecting the mass differences and putting $\eta = 0$. Making this approximation the mass difference effects are only allowed for in the kinematical factor which comes in, if one goes from the amplitude to cross section. One then can write

$$\frac{d\sigma}{d\Omega}_{-\to 0} \cong \frac{v_0}{v_-} [|\, f_N^{0-}\,|^2 + |\, g_N^{0-}\,|^2]\,.$$

Here v_0 and v_- are the relative c.m. velocities of the $\pi^0 n$ and $\pi^- p$ systems; f_N^{0-} and g_N^{0-} are the purely nuclear amplitudes, neglecting all isospin violating effects [2, 3].

APPENDIX II

Symmetry and Unitarity of the Full S-Matrix

As a check on our calculation we show explicitly that the S-matrix we obtain is symmetric and unitary.

a) *Symmetry*

Since u_α and u_β are two independent solutions of the same differential equation their Wronskian is independent of r and from (20) we have

$$W[u_\alpha, u_\beta] = k_- \sin(\tau_\alpha^{(-)} - \tau_\beta^{(-)})\,.$$

Using (24) then gives

$$k_- X_\alpha X_\beta \sin(\tau_\alpha^{(-)} - \tau_\beta^{(-)}) = W[R_{-\alpha}^{IN}, R_{-\beta}^{IN}]_{r = r_0}.$$

Similarly

$$k_0 Y_\alpha Y_\beta \sin(\tau_\alpha^{(0)} - \tau_\beta^{((0)}) = W[R_{0\alpha}^{IN}, R_{0\beta}^{IN}]_{r=r_0}$$

and so from (30) we have

$$D_{TOT} \cdot (S_{TOT}^{0-} - S_{TOT}^{-0}) =$$

$$2i\left(\frac{m_- m_0}{k_- k_0}\right)^{1/2}\left(\frac{1}{m_-} W[R_{-\alpha}^{IN}, R_{-\beta}^{IN}]_{r=r_0} + \frac{1}{m_0} W[R_{0\alpha}^{IN}, R_{0\beta}^{IN}]_{r=r_0}\right). \qquad (A2.1)$$

However, $R_{c\alpha}^{IN}$ and $R_{c\beta}^{IN}$ are regular solutions of (8) for $r \leqslant r_0$. Using this system of coupled differential equations one obtains

$$\frac{1}{m_-} W[R_{-\alpha}^{IN}, R_{-\beta}^{IN}] + \frac{1}{m_0} W[R_{0\alpha}^{IN}, R_{0\beta}^{IN}] = 0 \quad , \quad 0 \leqslant r \leqslant r_0 .$$

Thus the right hand side of (A2.1) vanishes and so

$$S_{TOT}^{0-} = S_{TOT}^{-0} . \qquad (A2.2)$$

b) *Unitarity*

We have to prove the three relations

$$|S_{TOT}^{--}|^2 + |S_{TOT}^{0-}|^2 = 1 , \qquad (A2.3)$$

$$|S_{TOT}^{00}|^2 + |S_{TOT}^{-0}|^2 = 1 , \qquad (A2.4)$$

$$S_{TOT}^{--} S_{TOT}^{0-*} + S_{TOT}^{00*} S_{TOT}^{-0} = 0 . \qquad (A2.5)$$

Writing (A2.3) in the form

$$|D_{TOT} \cdot S_{TOT}^{0-}|^2 = |D_{TOT}|^2 - |D_{TOT} \cdot S_{TOT}^{--}|^2$$

we insert the explicit expressions from (30) and (31) to obtain

$$4\frac{k_0 m_-}{k_- m_0} Y_\alpha^2 Y_\beta^2 \sin^2(\tau_\alpha^{(0)} - \tau_\beta^{(0)}) = -4 X_\alpha Y_\beta X_\beta Y_\alpha \sin(\tau_\alpha^{(-)} - \tau_\beta^{(-)}) \sin(\tau_\alpha^{(0)} - \tau_\beta^{(0)}).$$

Thus we must prove

$$\frac{k_0}{m_0} Y_\alpha Y_\beta \sin(\tau_\alpha^{(0)} - \tau_\beta^{(0)}) + \frac{k_-}{m_-} X_\alpha X_\beta \sin(\tau_\alpha^{(-)} - \tau_\beta^{(-)}) = 0$$

but this follows at once from the results of the previous section.

To prove (A2.4) we note from (30) that

$$(D_{TOT} \cdot S_{TOT}^{00})^* = (D_{TOT} \cdot S_{TOT}^{--}) \qquad (A2.6)$$

and so (A2.4) follows at once from (A2.3) which we have just proved. Finally we use (A2.2) and (A2.6) to rewrite (A2.5) as

$$(D_{TOT} \cdot S_{TOT}^{--}) \cdot 2\, Re\, (D_{TOT} \cdot S_{TOT}^{-0}) = 0 .$$

However, from (30), we see that $D_{TOT} S_{TOT}^{-0}$ is pure imaginary and so this relation is also proved to be true.

REFERENCES

[1] G. C. OADES and G. RASCHE, Helv. phys. Acta, *44*, 5 (1971).
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[2] See p. 82, H. A. BETHE and F. DE HOFFMANN, *Mesons and Fields*, Vol. II (Row, Peterson + Co., New York 1955).

[3] See Section 4 of J. HAMILTON, Phys. Rev. *110*, 1134 (1958).