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Rigged Hilbert Spaces in Quantum Field Theory: a Lesson Drawn from Charge Operators[1])

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Abstract. Motivated by the problem of defining a charge operator, a systematic study is made of several classes of rigged Hilbert spaces suitable for Quantum Field Theory. The result is that (essentially) only one of them satisfies all the requirements, namely $\mathscr{H}_{ql} \subset \mathscr{H} \subset (\mathscr{H}_{ql})'$, where $\mathscr{H}_{ql}$ is the well-known space of quasi-local states. It is then shown that, even in the case of a non-conserved current, the charge operator always exists as a continuous operator from $\mathscr{H}_{ql}$ into $(\mathscr{H}_{ql})'$.

1. Introduction

Ever since the pioneering work of Wightman [1], [2], the theory of distributions [3] has been an essential ingredient of Quantum Field Theory, although it was soon realized that other kinds of generalized functions might be also useful [4]. All of these fall in the general scheme of rigged Hilbert spaces (RHS) developed by Gelfand et al. [5]. It was shown indeed explicitly by Borchers [6] (see also [7], [8]) that the Wightman approach fits into this framework perfectly.

More recently, when the concept of current algebra became popular [9], the problem arose of defining a charge operator as the space integral of a density, itself a current operator (operator-valued distribution):

$$Q(t) = \int d^3\underset{\sim}{x}\, j_0(\underset{\sim}{x}, t) \,. \tag{1}$$

Of course this integral does not make sense as it stands, but a suitable definition was given by Kastler et al. [10]:

$$Q(t) = \lim_{T\to 0} \lim_{R\to\infty} j_0(f_R f_T) \,, \tag{2}$$

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1) A preliminary version of this work was presented at the Conference on 'Special Topics in Quantum Field Theory', University of Missouri, St. Louis, July 27–31, 1970.

where $f_R \in \mathscr{D}(\boldsymbol{R}^3)$, $f_T \in \mathscr{D}(\boldsymbol{R})$ are suitable test functions[2]) such that, in the sense of tempered distributions:

$$\lim_{R\to\infty} f_R(\underset{\sim}{x}) = 1\,, \quad \lim_{T\to 0} f_T(x^0) = \delta(x^0 - t)\,.$$

With this definition and pure Hilbert space methods, fundamental results were obtained by Kastler et al. [10] and also Schroer and Stichel [11]. But Katz [12] and de Mottoni [13] later suggested recasting the results in the RHS language, for the following reasons. The definition (2) is still incomplete; one has to specify in which topology the limit is taken. From [10] and [11] one sees that both strong and weak operator topologies are excluded. Then one considers the following sesquilinear form over $\Phi \times \Phi$:

$$Q(\varphi_1, \varphi_2) = \lim_{R\to\infty} \langle \varphi_1 \mid j_0(f_R f_T) \mid \varphi_2 \rangle\,, \tag{3}$$

where Φ is some dense domain of the Hilbert space $\mathscr{H}$ containing all local and quasi-local states. In the general case (nonconserved current), this sesquilinear form is not separately continuous in the Hilbert space topology, and therefore does not derive from an (unbounded) operator in $\mathscr{H}$ [10]. This is the standard situation leading to a RHS [14], [15]. One just has to put on Φ a stronger topology in order to get the RHS: $\Phi \subset \mathscr{H} \subset \Phi'$, ($\Phi'$ is the strong dual of Φ), in such a way that $Q(.\,,.)$ becomes a *continuous* sesquilinear form over $\Phi \times \Phi$, or, equivalently, there exists a continuous operator $\widehat{Q}_{op}$ from Φ into Φ' such that:

$$Q(\varphi_1, \varphi_2) = \langle \widehat{Q}_{op}\, \varphi_1, \varphi_2 \rangle\,,$$

where $\langle .\,,. \rangle$ is the canonical bilinear form over $\Phi' \times \Phi$. This was exactly the suggestion of Katz [12] and de Mottoni [13]. However, their treatment is partly formal, because they neglected the topological aspects of the problem. We will study these in the sequel: continuity of the embedding $\Phi \to \mathscr{H}$, nuclearity of Φ, action of field operators and Poincaré generators on Φ, etc. But, in fact, the problem of charge operators merely serves as a motivation. What is really at stake is the question of the usefulness of the whole RHS approach to field theory. The answer we will find is essentially negative, but the analysis yields some interesting by-products.

The material is organized as follows. In Sections 2 and 4, we study various candidates for the space Φ in a general field theory. Section 3 is devoted to a comparison of the respective merits of the so-called differentiable states and the quasi-local states as building blocks. We return in Section 5 to the problem of charge operators in the essentially unique RHS available and give some general conclusions in Section 6. The differentiable states are studied systematically in Appendix A, whereas Appendix B lists the relevant properties of the various functional spaces used in the text.

Throughout the paper, we assume the usual axioms of a Wightman field theory for a neutral, scalar field $A(x)$, [1], [2], including locality and the *strong* spectral condition (presence of a mass gap). We denote by $U(a, \Lambda)$ the representation of the Poin-

[2]) Here, and throughout the paper, we follow the standard notation of Schwartz [3] for the various spaces of test functions or distributions; for the convenience of the reader, we have listed in Appendix B the properties of these spaces used in the text. We will not consider other spaces than those described in Schwartz' book.

caré group underlying the theory and by P^μ, $M^{\mu\nu}$ (μ, $\nu = 0, 1, 2, 3$) the representatives of the infinitesimal generators of translations and homogeneous Lorentz transformations, respectively.

2. RHS built from Differentiable States

The first RHS introduced in field theory, by Borchers [6] (see also [2], [7]), was based on quasi-local states:

$$\mathscr{H}_{ql} \subset \mathscr{H} \subset (\mathscr{H}_{ql})' , \tag{4}$$

where $\mathscr{H}_{ql}$ is the linear span of all states of the form:

$$\begin{aligned} | \varphi^{(n)} > &\equiv A^n(\varphi^{(n)}) \,|\, 0\rangle , \quad \varphi^{(n)} \in \mathscr{S}(\boldsymbol{R}^{4n}) \\ &= \int dx_1 \ldots dx_n \, \varphi^{(n)}(x_1 \ldots x_n) \, A(x_1) \ldots A(x_n) \,|\, 0\rangle . \end{aligned}$$

Through the identification:

$$A^n(\varphi^{(n)}) \,|\, 0\rangle \leftrightarrow \varphi^{(n)} \in \mathscr{S}(\boldsymbol{R}^{4n}) ,$$

$\mathscr{H}_{ql}$ is given the topology of $\underline{\mathscr{S}}/\underline{\mathscr{S}}_0$, where $\underline{\mathscr{S}} \equiv \sum_{n=0}^{\infty} \mathscr{S}(\boldsymbol{R}^{4n})$ (topological direct sum) and $\underline{\mathscr{S}}_0$ is the closed subspace of $\underline{\mathscr{S}}$ corresponding to all states of vanishing $\mathscr{H}$-norm. This space was recently studied in detail by Wyss [8].

In their paper [11], Schroer and Stichel already introduced a larger set of states, namely, states of the following form:

$$| v\rangle = \int d^3\underset{\sim}{x} \, h(\underset{\sim}{x}) \, U(\underset{\sim}{x}) \,|\, \varphi\rangle , \tag{5}$$

where $\varphi \in \mathscr{H}_{ql}$, $U(\underset{\sim}{x})$ is the restriction to space translations of the Poincaré representation U, and h is a smooth function of $\underset{\sim}{x}$ such that

$$\lim_{|\underset{\sim}{x}| \to \infty} | \underset{\sim}{x} |^2 \, h(\underset{\sim}{x}) = 0 . \tag{6}$$

Similar states were later proposed in [12], [13] for building a RHS. However, one can do better if one remarks that the argument of [11] actually holds true for a much larger set of states. Indeed:

(i) The smoothness of h is never used, whereas the behaviour at ∞, equation (6), is crucial.

(ii) The quasi-local character of $|\varphi\rangle$ is used in two ways; first, it ensures the validity of *cluster properties* in the space variables, namely [2]:

$$\langle \varphi_1 \,|\, U(\underset{\sim}{x}) \,|\, \varphi_2\rangle \in \mathscr{S}[\underset{\sim}{x}] \quad \text{for any} \quad \varphi_i \in \mathscr{H}_{ql} , \langle 0 \,|\, \varphi_i\rangle = 0 , \quad i = 1, 2;$$

second, the inverse of the Hamiltonian, $(P^0)^{-1}$, is defined on all quasi-local states orthogonal to the vacuum: if $\varphi \in \mathscr{H}_{ql}$, $\langle 0 \,|\, \varphi\rangle = 0$, the strong spectral condition implies the existence of $\psi \in \mathscr{H}_{ql}$ such that $|\varphi\rangle = P^0 \,|\, \psi\rangle$, i.e. $|\psi\rangle = (P^0)^{-1} \,|\, \varphi\rangle$.

But it is known [16] that cluster properties are mere consequences of the structure of the representation U (strong spectral condition), and that they hold true for arbitrary states φ_1, φ_2 belonging to $\mathscr{H}^\infty$, the manifold of all C^∞-vectors for U (cf. Appendix A). Also, $\varphi \in \mathscr{H}^\infty$, $\langle 0 \mid \varphi \rangle = 0$, implies that $(P^0)^{-1} \mid \varphi\rangle \in \mathscr{H}^\infty$, as it is readily seen. Thus we are led to choose as a candidate for a space Φ the linear span of the states $\mid v\rangle = U(h) \mid \varphi\rangle$ of equation (5), where now $h \in \mathscr{K}$, some space of distributions to be determined later, and $\varphi \in \mathscr{H}^\infty$, the elements of which we will call *differentiable states*. $\mathscr{H}^\infty$ is understood with its natural Fréchet space topology and the structure (cf. Appendix A):

$$\mathscr{H}^\infty = \mathscr{S}(\boldsymbol{R}^3) \hat{\otimes}_\pi \mathscr{F}(\boldsymbol{R}_+) \tag{7}$$

(completed projective tensor product [17], [18]).

Proposition 1. – The map u defined by $u(h, \varphi) = U(h) \mid \varphi\rangle$ is a continuous bilinear map of $\mathscr{K} \times \mathscr{H}^\infty$ into $\mathscr{H}$, with dense range, iff $\mathscr{K} \subseteq \mathscr{D}'_{L^2}$ (continuous embedding).

Proof. – First $U(h) \mid \varphi\rangle \in \mathscr{H}$ iff $h \in \mathscr{D}'_{L^2}$; indeed:

$$\| U(h) \mid \varphi\rangle \|^2 = \langle \varphi \mid U(\bar{h} * h) \mid \varphi\rangle$$

$$= \int d^3\underset{\sim}{x}\, g(\underset{\sim}{x})\, \eta(\underset{\sim}{x})$$

with:

$$g(\underset{\sim}{x}) \equiv (\bar{h} * h)(\underset{\sim}{x}) \quad \text{(convolution)}$$

$$\eta(\underset{\sim}{x}) \equiv \langle \varphi \mid U(\underset{\sim}{x}) \mid \varphi\rangle .$$

The matrix element η being in $\mathscr{S}$, this norm is finite whenever $g = \bar{h} * h \in \mathscr{S}'$, i.e. $h \in \mathscr{D}'_{L^2}$.

Second, the range of u is already dense in $\mathscr{H}$ if $\mathscr{K} = \mathscr{D}$; indeed $\mathscr{H}^\infty$ is dense in $\mathscr{H}$, any $\varphi \in \mathscr{H}^\infty$ can be identified with $U(\delta) \mid \varphi\rangle$ and δ can be approached by a sequence of elements of $\mathscr{D}$; the property holds a fortiori for any space $\mathscr{K}$ containing $\mathscr{D}$ (algebraically and topologically), in particular for all the spaces we will consider.

In order to establish the continuity of u, we note that:

$$\| U(h) \mid \varphi\rangle \|^2 = \langle g, \eta\rangle \leqslant p'(g)\, p(\eta) ,$$

where p', p are continuous seminorms over $\mathscr{S}'$ and $\mathscr{S}$ respectively, and $\langle . , . \rangle$ is the canonical bilinear form over $\mathscr{S}' \times \mathscr{S}$, which is separately continuous, hence continuous ([17], Theorem 41.1). Furthermore, the convolution map is continuous from $\mathscr{D}'_{L^2} \times \mathscr{D}'_{L^2}$ into $\mathscr{D}'_{L^\infty}$, a fortiori into $\mathscr{S}'$, i.e. there exists a continuous seminorm r' over $\mathscr{D}'_{L^2}$ such that:

$$p'(g) \leqslant [r'(h)]^2 .$$

Finally the matrix element $\eta(\underset{\sim}{x})$ is the Fourier transform of the function (see Appendix A for the notation):

$$\hat{\eta}(\underset{\sim}{p}) = \int_{M^2}^{\infty} d\varrho(m^2) \quad (\underset{\sim}{p}^2 + m^2)^{-1/2} \mid \varphi(\underset{\sim}{p}, m^2) \mid^2$$

and standard estimates show that there exists a continuous seminorm w over $\mathscr{H}^\infty$ such that:

$$p(\eta) \leqslant [w(\varphi)]^2 .$$

Altogether we get, with s a continuous seminorm over $\mathscr{K}$:

$$\| U(h) \mid \varphi\rangle \|^2 \leqslant r'(h)\, w(\varphi)$$
$$\leqslant s(h)\, w(\varphi) . \quad \text{QED.}$$

Obviously the manifold generated by the states $\mid v\rangle$ has the structure of a tensor product $\mathscr{K} \otimes \mathscr{H}^\infty$. We topologize it as the projective tensor product $\mathscr{K} \otimes_\pi \mathscr{H}^\infty$ [17], [18], and assume from now on that $\mathscr{K} \subseteq \mathscr{D}'_{L^2}$; thus continuity of u is equivalent to the continuity of the corresponding mapping $\bar{u}$: $\mathscr{K} \otimes_\pi \mathscr{H}^\infty \to \mathscr{H}$. Hence the kernel of $\bar{u}$, $\mathscr{N}_{\mathscr{K}}$, is closed in the above topology. After division by this kernel and completion of the quotient, we get:

$$\Phi^\infty[\mathscr{K}] \equiv \left(\frac{\mathscr{K} \otimes_\pi \mathscr{H}^\infty}{\mathscr{N}_{\mathscr{K}}}\right)^{\wedge} , \tag{8}$$

a complete TVS with continuous embedding into $\mathscr{H}$ ($\hat{}$ denotes completion).

Proposition 2. – The TVS $\Phi^\infty[\mathscr{K}]$ has the following properties:

(i) $\Phi^\infty[\mathscr{K}]$ is nuclear iff $\mathscr{K}$ is nuclear and $\mathscr{F}$ finite-dimensional.

(ii) $\Phi^\infty[\mathscr{K}]$ is invariant under $U(a, \Lambda)$, for all $(a, \Lambda) \in \mathfrak{P}$.

(iii) $\Phi^\infty[\mathscr{K}]$ is a Gårding domain for the operators P^μ; the same holds for $M^{\mu\nu}$ iff $\underset{\sim}{x}$ is a continuous multiplier of $\mathscr{K}$.

(iv) The following inclusions hold, with all embeddings continuous:

$$\Phi^\infty[\mathscr{S}] \subset \mathscr{H}^\infty \subseteq \Phi^\infty[\mathscr{O}'_c] \subset \Phi^\infty[\mathscr{D}'_{Lp}] \subset \mathscr{H}, \quad 1 \leqslant p \leqslant 2 \tag{9}$$

where the sign '$\subseteq$' means that the two spaces coincide as vector spaces, but $\mathscr{H}^\infty$ has a stronger topology.

Proof:

(i) $\mathscr{H}^\infty$ is nuclear iff $\mathscr{F}$ is finite-dimensional; $E \otimes_\pi F$ is nuclear if E and F are, and so are its quotient by a *closed* subspace and the completion of this quotient.

(ii) Using the group law of $\mathfrak{P}$ one finds, for any $(a, \Lambda) \in \mathfrak{P}$:

$$U(a, \Lambda)\, U(h) \mid \varphi\rangle = U(h_\Lambda)\, U(a, \Lambda) \mid \varphi\rangle \in \Phi^\infty [\mathscr{K}]$$

since $h_\Lambda \in \mathscr{K}$ whenever $h \in \mathscr{K}$, and $U(a, \Lambda) \mid \varphi\rangle \in \mathscr{H}^\infty$.

(iii) The commutation relations of the Poincaré Lie algebra imply:

$$P^\mu\, U(h) \mid \varphi\rangle = U(h)\, P^\mu \mid \varphi\rangle , \quad \mu = 0, 1, 2, 3,$$

$$M^{jl}\, U(h) \mid \varphi\rangle = \int d^3\underset{\sim}{x}\, h(\underset{\sim}{x})\, U(\underset{\sim}{x})\, \{M^{jl} + x^l P^j - x^j P^l\} \mid \varphi\rangle ,$$

$$M^{0l}\, U(h) \mid \varphi\rangle = \int d^3\underset{\sim}{x}\, h(\underset{\sim}{x})\, U(\underset{\sim}{x})\, \{M^{0l} + x^l P^0\} \mid \varphi\rangle , \quad j, l = 1, 2, 3.$$

By definition, P^μ, $M^{\mu\nu}$ map $\mathscr{H}^\infty$ into itself continuously. Therefore, P^μ maps $\Phi^\infty[\mathscr{K}]$ into itself continuously, but M^{jl}, M^{0l} will do it iff multiplication by x^l, i.e. the map: $\mathscr{K} \ni h \to x^l h \in \mathscr{K}$ is continuous on $\mathscr{K}$. This is fulfilled for $\mathscr{K} = \mathscr{S}$ or $\mathscr{O}'_c$ but *not* for $\mathscr{D}_{Lp}$ or $\mathscr{D}'_{Lp}$.

(iv) The only part deserving a proof is the position of $\mathscr{H}^\infty$ among the various $\Phi^\infty[\mathscr{K}]$. As said before, $\mathscr{H}^\infty$ may be identified with the set $\{\delta\} \otimes \mathscr{H}^\infty \subset \Phi^\infty[\mathscr{O}_c']$. On the other hand, a vector $|\,\psi\rangle = U(h)\,|\,\varphi\rangle$ belongs to $\mathscr{H}^\infty$ iff $h \in \mathscr{O}_c'$; indeed, in the $\{\underset{\sim}{p}, m^2\}$-representation of C^∞-vectors (cf. Appendix A), one has:

$$\psi(\underset{\sim}{p}, m^2) = \int d^3\underset{\sim}{x}\, h(\underset{\sim}{x})\, e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, \varphi(\underset{\sim}{p}, m^2) = \hat{h}(\underset{\sim}{p})\, \varphi(\underset{\sim}{p}, m^2)\,,$$

so that $\psi \in \mathscr{H}^\infty \equiv \mathscr{S} \hat{\otimes}_\pi \mathscr{F}$ iff $\hat{h} \in \mathscr{O}_M$, i.e. $h \in \mathscr{O}_c'$. Moreover, by the identification $|\,\varphi\rangle \leftrightarrow U(\delta)\,|\,\varphi\rangle$, $\mathscr{H}^\infty$ has the topology of separately continuous bilinear maps, (since the multiplication $\mathscr{O}_M \times \mathscr{S} \to \mathscr{S}$ is separately continuous, but not continuous [3]), i.e. the topology of the inductive tensor product $\mathscr{O}_c' \bar{\otimes} \mathscr{H}^\infty$ (in the notation of Grothendieck [18]), which is strictly stronger than the topology of $\mathscr{O}_c' \hat{\otimes}_\pi \mathscr{H}^\infty$, i.e. that of $\Phi^\infty[\mathscr{O}']$. QED.

Borchers has shown [19], [20], that a field smeared in the time variable only is a C^∞-function of $\underset{\sim}{x}$, bounded together with all its derivatives; more precisely:

$$A(\underset{\sim}{x}, f_t)\,|\,\psi\rangle \in \mathscr{B}(\mathscr{H})\,, \quad \text{for any} \quad f_t \in \mathscr{S},\ \psi \in \mathscr{H}_{ql}\,,$$

or, equivalently:

$$\langle h\,|\,A(\underset{\sim}{x}, f_t)\,|\,\psi\,\rangle \in \mathscr{B} \quad \text{for any} \quad h \in \mathscr{H},\ \psi \in \mathscr{H}_{ql}\,.$$

The properties of differentiable states allow us to improve this result slightly.

Proposition 3. – For any $h \in \mathscr{H}$, $\psi \in \mathscr{H}_{ql}$, the matrix element $\langle h\,|\,A(\underset{\sim}{x}, f_t)\,|\,\psi\rangle$ is a C^∞-function of $\underset{\sim}{x}$, square integrable with all its derivatives:

$$\langle h\,|\,A(\underset{\sim}{x}, f_t)\,|\,\psi\rangle \in \mathscr{D}_{L^2}\,.\ ^{3)}$$

Proof. – Let $|\,\psi\rangle \equiv A(\psi)\,|\,0\rangle \in \mathscr{H}_{ql}$. Then:

$$\begin{aligned}\langle h\,|\,A(\underset{\sim}{x}, f_t)\,|\,\psi\rangle &= \langle h\,|\,A(\underset{\sim}{x}, f_t)\,A(\psi)\,|\,0\rangle \\ &= \langle h\,|\,A(\psi)\,A(\underset{\sim}{x}, f_t)\,|\,0\rangle + \langle h\,|\,[A(\underset{\sim}{x}, f_t), A(\psi)]\,|\,0\rangle\,.\end{aligned}$$

The first term equals $\langle h\,|\,A(\psi)\,U(\underset{\sim}{x})\,A(0, f_t)\,|\,0\rangle$ and belongs to $\mathscr{D}_{L^2}$ (Proposition 8 in Appendix A), since $\langle 0\,|\,A(0, f_t)\,|\,0\rangle = 0$ in all reasonable cases (an exception to this would be a current operator generating a spontaneously broken symmetry). As for the second term, it follows from locality and standard estimates that it belongs to $\mathscr{S}$, thus to $\mathscr{D}_{L^2}$.

3. Differentiable vs. Quasi-Local States

So far we have used $\mathscr{H}^\infty$ as the natural domain of cluster properties, but this choice has several drawbacks. First, it is true that $\mathscr{H}_{ql} \subset \mathscr{H}^\infty$ (see Proposition 4 below). But one cannot, in general, express $\varphi \in \mathscr{H}_{ql}$ *explicitly* in the $\{\underset{\sim}{p}, m^2\}$-representation

3) Contrary to the corresponding result of [19], [20], the same property does *not* hold for matrix elements of $A^{(n)}(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n; f_t)$, $f_t \in \mathscr{S}(R^n)$, $n > 1$.

characteristic of differentiable states. Let $|\varphi\rangle \equiv A^{(n)}(\varphi^{(n)})\,|\,0\rangle \in \mathscr{H}_{ql}$. One can show that the representative $\varphi(p, m^2)$ can be obtained from $\varphi^{(n)}$ if, and only if, A is a free field or a generalized free field. Indeed in order to compute $\tilde{\varphi}(p, m^2) \equiv \langle \tilde{p}, m^2 \,|\, \varphi\rangle$ (we use the suggestive Dirac notation, which may always be made rigorous with a RHS [15]), we need the kernel $\langle p, m^2 \,|\, p_1, \ldots, p_n\rangle$, and this requires some information about the Wightman function $\tilde{W}^{(2n)}$, which is unknown in general. All of this stems, of course, from the form of the scalar product in $\mathscr{H}$: $\langle\varphi \,|\, \psi\rangle = \langle W^{(2n)}, \varphi^+ \times \psi\rangle$, for $\varphi, \psi \in \mathscr{H}_{ql}$ [2]. The only exception is the case of a (generalized) free field, where the above kernel essentially reduces to $\delta^{(4)}(p - p_1 - \cdots - p_n)$.

Second, $\mathscr{H}^\infty$ is in general not nuclear (whereas $\mathscr{H}_{ql}$ always is), so that $\Phi^\infty[\mathscr{K}]$ is not nuclear either. But nuclearity is essential in a RHS approach (kernel theorem, generalized spectral theorem [5], [15]).

Third, and more serious, we don't know how the field operators $A(\varphi)$ act on $\mathscr{H}^\infty$. It is actually quite possible that $\mathscr{H}^\infty$ would not be invariant under $A(\varphi)$. All of this becomes clearer if one compares the present situation with the general construction of a RHS in the presence of a symmetry group [15]. There one also ends with a space Φ of the form $\mathscr{K} \hat{\otimes}_\pi \Psi$, where $\mathscr{K}$ takes care of those operators derived from the symmetry, and Ψ, which is nuclear, takes care of the other ones. Here, of course, the other ones are precisely the field operators. $\mathscr{H}^\infty$ is by definition adapted to U, P^μ, $M^{\mu\nu}$ (and in this respect $\mathscr{K}$ is redundant), but not to field operators. In order to remedy this defect, we need a space Ψ which is nuclear and mapped continuously into itself by all field operators; $\mathscr{K}$ then will take care of U, P^μ, $M^{\mu\nu}$. The obvious candidate for such a space is $\mathscr{H}_{ql}$ itself. We will, therefore, replace $\mathscr{H}^\infty$ by $\mathscr{H}_{ql}$ is our previous arguments and study the corresponding spaces Φ.

4. RHS Built from Quasi-Local States

The space of quasi-local states $\mathscr{H}_{ql}$ was described in Section 2. But here again, obvious generalizations are at hand, which were considered by Kolm [20], From the fact that vacuum expectation values are C^∞-functions of the space variables, bounded with all their derivatives, one sees that it is possible to smear the fields in the space variables with any distribution from $\mathscr{D}'_{L^1}$, in particular from $\mathscr{O}'_c$ or $\mathscr{S}$[4]. $\mathscr{D}'_{L^1}$ is discarded for being neither nuclear, nor invariant under $M^{\mu\nu}$; we are thus left with two spaces:

$$\mathscr{H}_{ql} \simeq \frac{\underline{\mathscr{S}}}{\underline{\mathscr{S}_0}}\,, \quad \underline{\mathscr{S}} \equiv \sum_{n=0}^{\infty} \mathscr{S}(R^{4n})\,,$$

$$\tilde{\mathscr{H}}_{ql} \simeq \frac{\underline{\mathscr{O}'_c \otimes \mathscr{S}}}{\underline{\mathscr{N}_0}}\,, \quad \underline{\mathscr{O}'_c \otimes \mathscr{S}} \equiv \sum_{n=0}^{\infty} [\mathscr{O}'_c(R^{3n}) \hat{\otimes}_\pi \mathscr{S}(R^n)]\,,$$

where $\underline{\mathscr{S}_0}$, $\underline{\mathscr{N}_0}$ are the closed subspaces corresponding to states of vanishing $\mathscr{H}$-norm. $\tilde{\mathscr{H}}_{ql}$ has the same $*$-algebra structure as $\mathscr{H}_{ql}$ [8].

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4) For a single field operator ($n = 1$) one can go further up to $\mathscr{D}'_{L^2}$, in view of Prop. 3.

Proposition 4. – One has the following inclusions, algebraically and topologically:

$$\mathscr{H}_{ql} \subseteq \tilde{\mathscr{H}}_{ql} \subset \mathscr{H}^{\infty} \subset \mathscr{H} \,, \tag{10}$$

where $\subseteq$, as usual, means 'algebraically identical'.

Proof:

(i) $\mathscr{H}_{ql} \subseteq \tilde{\mathscr{H}}_{ql}$

The inclusion, both algebraically and topologically, is obvious. The algebraic equality follows from the spectral condition, by an extension of an argument due to Kolm [20]: for any $\varphi_s(x) \in \mathscr{O}'_c(\boldsymbol{R}^3)$, $f_t(x^0) \in \mathscr{S}(\boldsymbol{R})$, there exists $\underset{\sim}{\eta}(x) \in \mathscr{S}(\boldsymbol{R}^4)$ such that:

$$A(\varphi_s, f_t) \mid 0\rangle = A(\eta) \mid 0\rangle \,.$$

(ii) $\tilde{\mathscr{H}}_{ql} \subset \mathscr{H}$

This follows immediately from the structure of $\tilde{\mathscr{H}}_{ql}$ [8]:

$$\begin{aligned} \| A^{(n)}(\varphi^{(n)}) \mid 0\rangle \|^2 &= \langle W^{(2n)}, \varphi^{(n)+} \times \varphi^{(n)}\rangle \,, \\ &\leqslant c\, p(\varphi^{(n)+} \times \varphi^{(n)}) \,, \\ &\leqslant c'[q(\varphi^{(n)})]^2 \,, \end{aligned}$$

where p, q are continuous seminorms over $\mathscr{O}'_c(\boldsymbol{R}^{6n}) \hat{\otimes}_\pi \mathscr{S}(\boldsymbol{R}^{2n})$, $\mathscr{O}'_c(\boldsymbol{R}^{3n}) \hat{\otimes}_\pi \mathscr{S}(\boldsymbol{R}^{n})$, respectively.

(iii) $\tilde{\mathscr{H}}_{ql} \subset \mathscr{H}^{\infty}$

By definition, $\mathscr{H}^{\infty}$ is the largest domain of $\mathscr{H}$ invariant under any element L of the algebra generated by $\{P^\mu, M^{\mu\nu}\}$, with the corresponding projective topology [14], i.e. the coarsest topology that makes all the mappings L: $\mathscr{H}^{\infty} \to \mathscr{H}$ continuous. But $\tilde{\mathscr{H}}_{ql}$ has the same property. Indeed, L is defined on $\tilde{\mathscr{H}}_{ql}$ by:

$$L\big(A(\varphi) \mid 0\rangle\big) = A(L\,\varphi) \mid 0\rangle \,, \quad \varphi \in \frac{\overline{\mathscr{O}'_c \otimes \mathscr{S}}}{\underline{\mathscr{N}_0}}$$

and this is a continuous map from $\tilde{\mathscr{H}}_{ql}$ into $\mathscr{H}$, because the map $\varphi \to L\,\varphi$ and the embedding $\tilde{\mathscr{H}}_{ql} \to \mathscr{H}$ are continuous. Thus the embedding $\tilde{\mathscr{H}}_{ql} \to \mathscr{H}^{\infty}$ must also be continuous.

As a trivial consequence of this result we have:

Proposition 5. – The statement of Proposition 1 holds true with $\mathscr{H}_{ql}$ or $\tilde{\mathscr{H}}_{ql}$ instead of $\mathscr{H}^{\infty}$.

As before, we may now build appropriate spaces Φ. Define

$$\overset{(\sim)}{\Phi}_{ql}[\mathscr{K}] \equiv \left(\frac{\mathscr{K} \otimes_\pi \overset{(\sim)}{\mathscr{H}}_{ql}}{\mathscr{N}_{\mathscr{K}}}\right)^{\wedge} \,, \tag{11}$$

where $\mathscr{N}_{\mathscr{K}}$ is again the kernel of $\bar{u}$: $\mathscr{K} \otimes_\pi \overset{(\sim)}{\mathscr{H}}_{ql} \to \mathscr{H}$. Then:

Proposition 6

(i) The TVS $\overset{(\sim)}{\Phi}_{ql}$ is nuclear iff $\mathscr{H}$ is nuclear and it has the properties (ii)–(iii) of Proposition 2.

(ii) The following inclusions hold, with all embeddings continuous and '$\subseteq$' standing for 'equal as vector spaces':

$$\Phi_{ql}[\mathscr{S}] \subseteq \tilde{\Phi}_{ql}[\mathscr{S}] \subset \mathscr{H}_{ql} \subseteq \Phi_{ql}[\mathscr{O}_c'] \subseteq \tilde{\mathscr{H}}_{ql} = \tilde{\Phi}_{ql}[\mathscr{O}_c'] . \tag{12}$$

(iii) For any $\varphi \in \mathscr{S}$, $A(\varphi)$ maps continuously $\tilde{\mathscr{H}}_{ql}$ into itself and the four other spaces into $\mathscr{H}_{ql}$; for any $\varphi \in \underline{\overline{\mathscr{O}_c'} \otimes \mathscr{S}}$, $A(\varphi)$ maps all five spaces continuously into $\tilde{\mathscr{H}}_{ql}$.

Proof:

(i) The proof is identical to the one given in Proposition 2, using the fact that $\mathscr{H}_{ql}$, $\tilde{\mathscr{H}}_{ql}$ are Gårding domains for P^μ, $M^{\mu\nu}$.

(ii) For any test 'function' $h(x)$, one has immediately:

$$U(h) \mid \varphi \rangle = \mid h * \varphi \rangle , \quad \mid \varphi \rangle \in \overset{(\sim)}{\mathscr{H}}_{ql} .$$

Therefore, it follows from the continuity properties of the convolution [3] that $u: \{h, \mid \varphi\rangle\} \to U(h) \mid \varphi\rangle$ is a separately continuous map $\mathscr{S} \times \tilde{\mathscr{H}}_{ql} \to \mathscr{H}_{ql}$ $(\mathscr{O}_c' \times \mathscr{H}_{ql} \to \mathscr{H}_{ql})$, and a continuous map $\mathscr{S} \times \mathscr{H}_{ql} \to \mathscr{H}_{ql}$ $(\mathscr{O}_c' \times \tilde{\mathscr{H}}_{ql} \to \tilde{\mathscr{H}}_{ql})$. Using Proposition 4 and the argument of Proposition 2, one gets all the inclusions (12). The last equality follows from the fact that the bilinear map $\mathscr{O}_c' \times \tilde{\mathscr{H}}_{ql} \to \tilde{\mathscr{H}}_{ql}$ is continuous, thus also the linear maps $\mathscr{O}_c' \otimes_\pi \tilde{\mathscr{H}}_{ql} \to \tilde{\mathscr{H}}_{ql}$, $\tilde{\Phi}_{ql}[\mathscr{O}_c'] \to \tilde{\mathscr{H}}_{ql}$; this is not the case with $\mathscr{H}^\infty$ or $\mathscr{H}_{ql}$, since the bilinear maps $\mathscr{O}_c' \times \mathscr{H}^\infty \to \mathscr{H}^\infty$, $\mathscr{O}_c' \times \mathscr{H}_{ql} \to \mathscr{H}_{ql}$ are *not* continuous (cf. Appendix B).

(iii) Obvious by direct check, from the relation:

$$A^{(m)}(\varphi^{(m)}) \; U(h) \mid \psi^{(n)} \rangle = \mid \varphi^{(m)} \otimes (h * \psi^{(n)}) \rangle . \quad \text{QED.}$$

In conclusion, if we want a space Φ which is nuclear, continuously embedded into $\mathscr{H}$ and a Gårding domain for P^μ, $M^{\mu\nu}$, and all field operators, our choice is rather limited. The relations (12) tell us that only one manifold of states will do the job, namely $\mathscr{H}_{ql}$ (with its various topologies). In other words, without any additional information on the field operators, there is essentially (i.e. up to unnecessary complications) only one RHS available, namely the well-known Borchers algebra $\mathscr{H}_{ql} \subset \mathscr{H} \subset (\mathscr{H}_{ql})'$!

5. Properties of Charge Operators

We will return now to the problem of charge operators raised in [12], [13]. Thus we consider a Wightman field theory, with the strong spectral condition, in which there is a distinguished quadruplet of fields $j_\mu(x)$ $(\mu = 0, 1, 2, 3)$[5], local and relatively local with respect to all other fields in the theory. The divergence, $D(x) \equiv \partial^\mu j_\mu(x)$,

[5]) j_μ need not be a Lorentz four-vector [10].

has the same properties, but nothing more is assumed, in particular $D(x)$ does not necessarily vanish. Then we consider the following sesquilinear form over $\mathscr{H}_{ql} \times \mathscr{H}_{ql}$ (see Section 1):

$$Q(\varphi_1, \varphi_2) = \lim_{R \to \infty} \langle \varphi_1 | j_0(f_R f_T) | \varphi_2 \rangle, \quad \varphi_1, \varphi_2 \in \mathscr{H}_{ql}, \tag{13}$$

where f_R, f_T are the usual test functions of [10] such that $\lim_{R \to \infty} f_R = 1$, $\lim_{T \to 0} f_T = \delta$ as tempered distributions[6]).

Proposition 7. – $Q(.,.)$ is a continuous sesquilinear form over $\mathscr{H}_{ql} \times \mathscr{H}_{ql}$ or, equivalently:

$$Q(\varphi_1, \varphi_2) = \langle \hat{Q}_{op} \varphi_1, \varphi_2 \rangle, \quad \varphi_1, \varphi_2 \in \mathscr{H}_{ql}, \tag{14}$$

where $\hat{Q}_{op}$ is a continuous (antilinear) operator from $\mathscr{H}_{ql}$ into $(\mathscr{H}_{ql})'$ and $\langle .,. \rangle$ is the canonical bilinear form over $(\mathscr{H}_{ql})' \times \mathscr{H}_{ql}$.

Proof:

Write

$$| \varphi_i \rangle = A(\varphi_i) | 0 \rangle, \quad i = 1, 2$$

$$Q(\varphi_1, \varphi_2) = Q_1(\varphi_1, \varphi_2) + Q_2(\varphi_1, \varphi_2)$$

with

$$Q_1(\varphi_1, \varphi_2) = \lim_{R \to \infty} \langle \varphi_1 | [j_0(f_R f_T), \; A(\varphi_2)] | 0 \rangle,$$

$$Q_2(\varphi_1, \varphi_2) = \lim_{R \to \infty} \langle \varphi_1 | A(\varphi_2) j_0(f_R f_T) | 0 \rangle.$$

For the first term, we write $| \chi(\underset{\sim}{x}) \rangle \equiv [j_0(\underset{\sim}{x}, f_T), A(\varphi_2)] | 0 \rangle$. Again $\| | \chi(\underset{\sim}{x}) \rangle \| \in \mathscr{S}(\boldsymbol{R}^3)$ by locality. Then, using standard estimates, one gets

$$\left| \int d^3\underset{\sim}{x} f_R(\underset{\sim}{x}) \langle \varphi_1 | \chi(\underset{\sim}{x}) \rangle \right| \leqslant \| \varphi_1 \| \int d^3\underset{\sim}{x} f_R(\underset{\sim}{x}) \| | \chi(\underset{\sim}{x}) \rangle \|$$
$$\leqslant c_R \| \varphi_1 \| p(\| \chi \|)$$

and therefore:

$$| Q_1(\varphi_1, \varphi_2) | \leqslant c \| \varphi_1 \| p(\| \chi \|)$$
$$\leqslant c' \| \varphi_1 \| q(\varphi_2),$$

where p, q are continuous norms over $\mathscr{S}$ and $\mathscr{H}_{ql}$ respectively, and $c = \lim_{R \to \infty} c_R$.

For the second term, the argument of [11] gives immediately:

$$| \langle \varphi_1 | A(\varphi_2) j_0(f_R f_T) | 0 \rangle | \leqslant | \langle \psi | j_r(f'_R f_T) | 0 \rangle | + | \langle \psi | D(f_R f_T) | 0 \rangle |,$$

---

6) The limit $T \to 0$ is superfluous in the conserved current case [10]; we will not be concerned with that problem here and will concentrate on the infinite volume limit.

where $|\psi\rangle \equiv (P^0)^{-1} A(\varphi_1) A(\varphi_2) |0\rangle \in \mathscr{H}_{ql}$ and j_r denotes the radial component of j_k. The first term vanishes as $R \to \infty$; then:

$$\langle\psi | D(f_R f_T) | 0\rangle = \int d^3\underset{\sim}{x}\, f_R(\underset{\sim}{x}) \langle\psi | U(\underset{\sim}{x})\, D(0, f_T) | 0\rangle = \langle f_R, \eta\rangle_{\mathscr{S}' \times \mathscr{S}}$$

with

$$\eta(\underset{\sim}{x}) \equiv \langle\psi | U(\underset{\sim}{x}) | D\rangle \in \mathscr{S}(\boldsymbol{R}^3),$$

$$|D\rangle \equiv D(0, f_T) | 0\rangle \in \mathscr{H}_{ql}.$$

Therefore, since $f_R \to 1$ in $\mathscr{S}'$, we have:

$$\lim_{R\to\infty} |\langle f_R, \eta\rangle| = |\langle \lim_{R\to\infty} f_R, \eta\rangle|$$
$$\leqslant p(\eta)$$
$$\leqslant d\, q(\psi)\, q'(D)$$
$$\leqslant d'\, q'(D)\, q''(\varphi_1)\, q'''(\varphi_2),$$

where $q, q' \ldots$ are continuous norms over $\mathscr{H}_{ql}$.

Finally:

$$|Q(\varphi_1, \varphi_2)| \leqslant c' \| \varphi_1 \| q(\varphi_2) + d'\, q'(D)\, q''(\varphi_1)\, q'''(\varphi_2). \tag{15}$$

First case: $D(x) \equiv 0$.

Equation (15) then becomes

$$|Q(\varphi_1, \varphi_2)| \leqslant c \| \varphi_1 \|,$$

for any fixed φ_2 (or vice-versa). From this follows [10] the existence of an (unbounded) operator Q_{op} in $\mathscr{H}$, with domain $D(Q_{op}) \supset \mathscr{H}_{ql}$, such that:

$$Q(\varphi_1, \varphi_2) = \langle\varphi_1 | Q_{op} | \varphi_2\rangle.$$

Second case: $D(x) \not\equiv 0$.

Equation (15) then reads:

$$|Q(\varphi_1, \varphi_2)| \leqslant c\, r(\varphi_1)\, r'(\varphi_2),$$

where r, r' are continuous norms over $\mathscr{H}_{ql}$, i.e. $Q(.\,,.)$ is a continuous sesquilinear form over $\mathscr{H}_{ql} \times \mathscr{H}_{ql}$, or, what is the same, there exists a continuous operator $\hat{Q}_{op}$ from $\mathscr{H}_{ql}$ into $(\mathscr{H}_{ql})'$ such that:

$$Q(\varphi_1, \varphi_2) = \langle\hat{Q}_{op}\, \varphi_1, \varphi_2\rangle.$$

6. Discussion

Trying to build a RHS appropriate for a general Quantum Field Theory, we have found that the solution is virtually unique, namely $\mathscr{H}_{ql} \subset \mathscr{H} \subset (\mathscr{H}_{ql})'$. Other RHS's are more complicated but not essentially different (Proposition 6), or, like those suggested by Katz [12] and de Mottoni [13], they lack one or another vital property: continuous embedding into $\mathscr{H}$, invariance under $M^{\mu\nu}$, nuclearity (thus the kernel theorem and the general spectral theorem, for P^{μ}, e.g. [20]), . . . etc.

On the other hand, when we look for a reasonable definition of the charge operator (in the case of a nonconserved current) in the RHS, we find that the corresponding sesquilinear form $Q(.\,,.)$ is already continuous in the worst case, i.e. when we assume nothing of $D(x)$ besides its being a Wightman field. This means, should we have some additional information about $D(x)$ (e.g. smoothness in momentum space, as in the PCAC condition [9]), there is no room in the formalism to exploit it. More precisely, there is no way of correlating those properties of $D(x)$ with the behaviour of the operator $\hat{Q}_{op}$ (larger domain, for instance); the formalism is not flexible enough, for it offers choice only between two spaces, $\mathscr{H}$ and $\mathscr{H}_{ql}$.

In conclusion, a systematic use of RHS's in Quantum Field Theory does not look very promising, at least in the original form of Gelfand et al. [5]. However, the last remark points towards a possible improvement. An answer to the above criticism would be to enrich the RHS further, by providing, so to speak, a way of interpolating between $\mathscr{H}_{ql}$ and $\mathscr{H}$:

$$\mathscr{H}_{ql} \subset \cdots \subset \mathscr{H}_{\beta} \subset \mathscr{H}_{\alpha} \subset \cdots \subset \mathscr{H} \subset \cdots \subset (\mathscr{H}_{\alpha})' \subset (\mathscr{H}_{\beta})' \subset \cdots \subset (\mathscr{H}_{ql})' .$$

An example of such a structure (which is built into the RHS, but not used explicitly) is afforded by the nested Hilbert spaces of Grossmann [21]. If one does not want to go into this, the only alternative is to stick to Hilbert space techniques (which can never be dispensed of totally, since some concepts, such as selfadjointness of an operator, are defined only in a Hilbert space). For the definition of charge operators and related problems, a detailed review of the Hilbert space approach was given recently by Orzalesi [22].

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Appendix A: Poincaré C^{∞}-vectors

Let G be a connected Lie group, U a strongly continuous representation of G into a Hilbert space $\mathscr{H}$. A *C^{∞}-vector for U* is a vector $\varphi \in \mathscr{H}$ such that the $\mathscr{H}$-valued function $g \to U(g)\,\varphi$ is C^{∞} on G. The manifold $\mathscr{H}^{\infty}$ of C^{∞}-vectors is then given by:

$$\mathscr{H}^{\infty} = \bigcap_{j=1}^{d} \bigcap_{n=1}^{\infty} D(A_j^n) \,, \tag{A.1}$$

where $A_1 \dots A_d$ $(d < \infty)$ are the representatives of a basis of the Lie algebra of G (usually noted $dU(A_j)$) and $D(A) \subset \mathscr{H}$ is the domain of the operator A [23]. On $\mathscr{H}^\infty$ there exists a natural topology, given (for instance) by the countable set of seminorms:

$$p_{n_1 \dots n_d}(\varphi) = \| A_1^{n_1} \dots A_d^{n_d} \varphi \| , \quad \varphi \in \mathscr{H}^\infty , \tag{A.2}$$

$$n_i = 0, 1, 2 \dots \quad \text{for} \quad i = 1, 2, \dots, d$$

where $\| \cdot \|$ is the Hilbert space norm of $\mathscr{H}$. This topology makes $\mathscr{H}^\infty$ into a complete metric space, i.e. a Fréchet space [17].

Note. This is of course the projective topology with respect to all the mappings $A : \mathscr{H}^\infty \to \mathscr{H}$, where A is any element of the algebra generated by the restriction to $\mathscr{H}^\infty$ of $A_1 \dots A_d$ [14].

In our case, the physical representation of the Poincaré group $\mathfrak{P}$ underlying the theory is characterized by the following norm:

$$\| h \|^2 = | \langle 0 | h \rangle |^2 + \int_{M^2}^{\infty} d\varrho(m^2) \int d^3\underset{\sim}{p} (\underset{\sim}{p}^2 + m^2)^{-1/2} | h(\underset{\sim}{p}, m^2) |^2 , \tag{A.3}$$

where $| 0 \rangle$ is the unique vacuum, $M^2 > 0$ (mass gap), ϱ is a tempered measure, and we have written:

$$h(\underset{\sim}{p}, m^2) \equiv h(\underset{\sim}{p}, p^0) |_{p^2 = m^2} \quad \text{(restriction to the mass shell)} .$$

Remark. For the sake of simplicity, we leave out all spin degrees of freedom (see also equation (A.4) below). Since each irreducible representation $[s, m]$ of $\mathfrak{P}$ corresponds to a finite, discrete spin, their inclusion is trivial (discrete direct sum of finite dimensional representations of $S\,U(2)$) and does not change the argument.

In this representation, the Lie algebra of $\mathfrak{P}$ is represented by the following operators [2]:

$$P^k \to p^k , \quad k = 1, 2, 3 ,$$

$$P^0 \to (\underset{\sim}{p}^2 + m^2)^{1/2} ,$$

$$M^{kl} \to p^k \frac{\partial}{\partial p^l} - p^l \frac{\partial}{\partial p^k} , \quad k, l = 1, 2, 3 ,$$

$$M^{0l} \to (\underset{\sim}{p}^2 + m^2)^{1/2} \frac{\partial}{\partial p^l} , \quad l = 1, 2, 3 . \tag{A.4}$$

Taking arbitrary powers of these operators, according to equation (A.2) we find that:

$$\varphi \in \mathscr{H}^\infty \leftrightarrow \| \mathfrak{P}_1(m^2) \, \mathfrak{P}_2\left(p^k, \frac{\partial}{\partial p^l}\right) \varphi \| < \infty ,$$

for arbitrary polynomials $\mathfrak{P}_1$, $\mathfrak{P}_2$.

This implies for $\mathscr{H}^\infty$ the following structure:

$$\mathscr{H}^\infty \simeq \mathscr{S}^3[\underset{\sim}{p}] \mathbin{\hat{\otimes}} \mathscr{F}[m^2] , \tag{A.5}$$

where $\mathscr{S}^3[\underset{\sim}{p}] \equiv \mathscr{S}(\boldsymbol{R}^3)$ is the usual Schwartz space of fast decreasing C^∞-functions [3] and $\mathscr{F}[m^2] \equiv \mathscr{F}(\boldsymbol{R}_+)$ is a Fréchet space of fast decreasing, continuous functions of m^2 (no differentiability required); $\hat{\otimes}$ denotes a completed tensor product (the three usual topologies of the tensor product coincide here since $\mathscr{S}$ and $\mathscr{F}$ are both Fréchet and $\mathscr{S}$ is nuclear [17]); finally $\simeq$ denotes an isomorphism of topological vector spaces.

Proposition 8. – The topological vector space $\mathscr{H}^\infty$ has the following properties:

(i) $\mathscr{H}^\infty$ is nuclear iff $\mathscr{F}$ is finite dimensional.

(ii) $(\mathscr{H}^\infty)' \simeq \mathscr{S}'[\underset{\sim}{p}] \hat{\otimes}_\pi \mathscr{F}'[m^2]$, where $\mathscr{S}'$ is the space of tempered distributions, $\mathscr{F}'$ the space of tempered measures, in the indicated variables, and $\hat{\otimes}_\pi$ denotes a completed projective tensor product [17].

(iii) For any $\varphi \in \mathscr{H}^\infty$, $h \in \mathscr{H}$, the matrix element $\langle h \mid U(\underset{\sim}{x})\,(1 - E_0) \mid \varphi\rangle$ belongs to the space $\mathscr{D}_{L^2}[\underset{\sim}{x}]$ (C^∞-functions, square integrable together with all their derivatives); $E_0 = \mid 0\rangle\langle 0 \mid$ is the projection on the unique vacuum $\mid 0\rangle$.

(iv) $\varphi \in \mathscr{H}^\infty$, $\langle 0 \mid \varphi\rangle = 0$, implies the existence of $\psi \in \mathscr{H}^\infty$ such that $\mid \varphi\rangle = P^0 \mid \psi\rangle$.

Proof:

(i) The isomorphism (A.5) implies that $\mathscr{H}^\infty$ is nuclear iff $\mathscr{F}$ is nuclear [17], [18]; but this cannot happen, unless $\mathscr{F}$ is finite-dimensional; indeed, no nuclear space is known which consists of functions satisfying only a condition of decrease at ∞; a condition on differentiability is required in all cases.

(ii) This follows from the fact that both $\mathscr{S}$ and $\mathscr{F}$ are Fréchet spaces [17].

(iii) The matrix element can be computed in a straightforward way in the $\{\underset{\sim}{p}, m^2\}$-representation:

$$\eta(\underset{\sim}{x}) \equiv \langle h \mid U(\underset{\sim}{x})\,(1 - E_0) \mid \varphi\rangle = \int d^3\underset{\sim}{p}\; e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\; \chi(\underset{\sim}{p})$$

with

$$\chi(\underset{\sim}{p}) = \int_{M^2}^{\infty} d\varrho(m^2)\; (\underset{\sim}{p}^2 + m^2)^{-1/2}\; \overline{h(\underset{\sim}{p}, m^2)}\; \varphi(\underset{\sim}{p}, m^2)\,.$$

Since $\varphi \in \mathscr{H}^\infty$, the following estimate holds, uniformly in $\underset{\sim}{p}$:

$$(\underset{\sim}{p}^2 + m^2)^{-1/2} \mid \varphi(\underset{\sim}{p}, m^2) \mid^2 \leqslant B\, \psi(m^2) \tag{A.6}$$

with ψ a fast decreasing function. Using the Schwarz inequality in $L^2_\varrho[m^2]$, we get immediately:

$$\int d^3\underset{\sim}{p} \mid \chi(\underset{\sim}{p}) \mid^2 \leqslant c(\| h \|^2 - \mid \langle 0 \mid h\rangle \mid^2) < \infty$$

with

$$c^{1/2} = B \int_{M^2}^{\infty} d\varrho(m^2)\; \psi(m^2)\,.$$

Thus $\chi \in L^2$. Similarly $\mathfrak{P}(.)\,\chi \in L_2$ for any polynomial $\mathfrak{P}(\underset{\sim}{p})$, since the estimate (A.6) holds for $|\,\mathfrak{P}(\underset{\sim}{p})\,|\,|\,\varphi(\underset{\sim}{p}, m^2)\,|^2$ as well. In other words, $\tilde{\chi} \in \hat{\mathscr{D}}_{L^2}$, i.e. χ is the Fourier transform of a function from $\tilde{\mathscr{D}}_{L^2}$, namely η.

(iv) The proof is identical to the one given by Kastler et al. [10], the essential ingredients being the presence of a mass gap and the $\{\underset{\sim}{p}, m^2\}$-representation of P^0 as $(\underset{\sim}{p}^2 + m^2)^{1/2}$:

$$\varphi \in \mathscr{S}[\underset{\sim}{p}] \,\hat{\otimes}\, \mathscr{F}[m^2] \Rightarrow (\underset{\sim}{p}^2 + m^2)^{1/2}\,\varphi \in \mathscr{S}[\underset{\sim}{p}] \,\hat{\otimes}\, \mathscr{F}[m^2]\,.$$

Remarks

1. Using, instead of the above, $\chi \in L^1$, $\mathfrak{P}(.)\,\chi \in L^1$, one gets: $\eta \in \mathscr{D}_{L^\infty} \equiv \mathscr{B}$, a well-known result of Borchers [19].
2. If $h \in \mathscr{H}^\infty$ also, then a similar argument yields:

$$\mathfrak{P}\left(\frac{\partial}{\partial q^l}\right)\chi \in L^2\,,$$

i.e.

$$\chi \in \mathscr{D}_{L^2} \cap \hat{\mathscr{D}}_{L^2} = \mathscr{S}$$

and thus $\eta \in \mathscr{S}$, also a well-known result [2], [16].

Appendix B: Schwartz' Spaces

For the convenience of the reader, we list here the various functional spaces used in the paper, with the notations of Schwartz [3]. They can all be presented in the following diagram, where all embeddings are continuous:

$$\begin{array}{ccccccc} \mathscr{S} & \subset & \mathscr{D}_{L^p} & \subset & \mathscr{D}_{L^q} & \subset & \mathscr{O}_M\,, \\ \cap & & \cap & & \cap & & \cap \\ \mathscr{O}'_c & \subset & \mathscr{D}'_{L^p} & \subset & \mathscr{D}'_{L^q} & \subset & \mathscr{S}' \end{array} \quad (1 \leqslant p \leqslant q \leqslant \infty)\,. \tag{B.1}$$

The upper line contains spaces of C^∞-functions, which are, together with all their derivatives, fast decreasing ($\mathscr{S}$), p-integrable ($\mathscr{D}_{L^p}$) or at most polynomially increasing ($\mathscr{O}_M$), respectively. $\mathscr{D}_{L^\infty}$ is also called $\mathscr{B}$ [19]. The lower line contains spaces of distributions: fast decreasing distributions ($\mathscr{O}'_c$); tempered distributions ($\mathscr{S}'$, dual of $\mathscr{S}$); $\mathscr{D}'_{L^p}$ is the dual of $\mathscr{D}'_{L^r}$ $[(1/r) + (1/p) = 1]$. Then:

(i) *Fourier transform*:

$$\mathscr{S} \leftrightarrow \mathscr{S}\,,\ \mathscr{S}' \leftrightarrow \mathscr{S}'\,,\ \mathscr{O}'_c \leftrightarrow \mathscr{O}_M\,.$$

(ii) *Metrizability*: $\mathscr{S}$ and $\mathscr{D}_{L^p}$ are Fréchet spaces, the others are not.

(iii) *Nuclearity*: $\mathscr{S}$, $\mathscr{S}'$, $\mathscr{O}_M$, $\mathscr{O}'_c$ are nuclear; $\mathscr{D}_{L^p}$, $\mathscr{D}'_{L^p}$ are not.

(iv) *Convolution*:

$$\mathscr{S} \times \mathscr{S} \to \mathscr{S}: \text{ continuous },$$

$$\mathscr{O}'_c \times \mathscr{S} \to \mathscr{S}: \text{ separately continuous },$$

$$\mathscr{O}'_c \times \mathscr{O}'_c \to \mathscr{O}'_c: \text{ continuous },$$

$$\mathscr{D}'_{L^p} \times \mathscr{D}'_{L^q} \to \mathscr{D}'_{L^r}: \text{ continuous },$$

$$\left(\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1\right).$$

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