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# Gauge Invariance and Scaling in Resonance Models of the Vertex

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*Abstract.* We analyze the vertex function associated with the interaction of a virtual photon of mass $q^2$, a scalar field of mass $k^2$ and a scalar particle sitting on its mass shell. The singularities in $k^2$ and in $q^2$ are taken to be simple poles characterized by linear trajectories. The behaviour of the vertex for high virtual masses is assumed to be governed by a scaling law which corresponds to a canonical singularity on the light cone. We construct the general gauge invariant vertex function consistent with these requirements.

## 1. Introduction

Landshoff and Polkinghorne [1] have proposed a dual model for the virtual Compton amplitude. Their model provides a realistic phenomenological narrow-resonance approximation to the amplitude in question; in particular, the model exhibits Regge behaviour, it possesses the appropriate pole structure and it scales in the Bjorken limit.

One essential shortcoming of the model is its failure to satisfy gauge invariance. It is the purpose of the present paper to show how to take care of this defect in a much simpler problem: the vertex. We will show in a subsequent investigation that the method allows immediate generalization to dual four-point-functions.

## 2. Kinematics

We consider the electromagnetic current $j_\mu(x)$ and a charged scalar field $\phi(x)$. The matrix element of the retarded commutator $i\theta(x^0)[\phi(x), j_\mu(0)]$ between the vacuum and a scalar one particle state $|p\rangle$ is denoted by $\tilde{R}_\mu$:

$$\tilde{R}_\mu(x,p) = i\theta(x^0)\langle 0|[\phi(x), j_\mu(0)]|p\rangle \tag{2.1}$$

whereas $R_\mu$ stands for its Fourier transform

$$R_\mu(k,p) = \int dx\, e^{ikx}\, \tilde{R}_\mu(x,p).$$

This matrix element describes the electromagnetic transition between the (on-shell) scalar particle $|p\rangle$ of momentum $p$, $p^2 = M^2$ and the scalar field $\phi$ which carries momentum $k$. The momentum of the virtual photon is given by

$$q = p - k.$$

Current conservation implies a Ward identity whose right hand side is determined by the equal time commutator

$$[\phi(0,\mathbf{x}), j_0(0)] = \delta^3(\mathbf{x})\phi(0). \tag{2.2}$$

We assume that the matrix element $\langle 0|\phi|p\rangle$ does not vanish and normalize the field by

$$\langle 0|\phi(0)|p\rangle = 1.$$

In this normalization the Ward identity reads

$$q^\mu R_\mu(k,p) = 1 \tag{2.3}$$

and in terms of the invariant decomposition

$$R_\mu(k,p) = (k_\mu + p_\mu)A(k^2,q^2) + (k_\mu - p_\mu)B(k^2,q^2) \tag{2.4}$$

$$\tilde{R}_\mu(x,p) = (i\partial_\mu + p_\mu)\tilde{A}(x^2,px) + (i\partial_\mu - p_\mu)\tilde{B}(x^2,px)$$

we have

$$(M^2 - k^2)A - q^2 B = 1. \tag{2.5}$$

## 3. Singularities in Momentum Space

The invariants $A(k^2,q^2)$ and $B(k^2,q^2)$ contain two sets of singularities arising from the two terms in the commutator $[\phi, j_\mu]$. The first set corresponds to intermediate states $|N\rangle$ which contribute to $\langle 0|\phi|N\rangle\langle N|j_\mu|p\rangle$. This includes, in particular, the Born term

$$R_\mu(k,p) = \frac{k_\mu + p_\mu}{M^2 - k^2}G(q^2) + \cdots \tag{3.1}$$

whose residue is the electromagnetic form factor of the external scalar particle. In a realistic description the matrix element $R_\mu$ must contain cuts in the variable $k^2$ corresponding to intermediate states with two or more particles. We approximate the entire spectrum of these intermediate states by narrow resonances and correspondingly replace the cuts by an infinite set of simple poles at

$$\alpha(k^2) = m; \quad m = 0, 1, 2, \ldots$$

More specifically we consider a linear trajectory $\alpha(k^2)$, i.e. a spectrum which is equally spaced in $M^2$:

$$\alpha(k^2) = \alpha_0 + k^2. \tag{3.2}$$

In this expression we have chosen mass units such that the slope of the trajectory is equal to one. The value of $\alpha_0$ is determined by the lowest mass on this trajectory which we identify with the mass of the external scalar particle, $\alpha_0 = -M^2$. Note that the residue of the pole at $\alpha = m$ is given by $\langle 0|\phi|m\rangle\langle m|j_\mu|p\rangle$, where $|m\rangle$ denotes the corresponding one particle state of momentum $k$. Since $\phi$ is a scalar field, only scalar particles contribute. The entire trajectory $\alpha$ therefore describes a sequence of scalar particles; the value of $\alpha$ is not the spin of the particle in question.

The second set of singularities of $R_\mu$ arises from matrix elements of the type $\langle 0|j_\mu|N\rangle\langle N|\phi|p\rangle$. In this case, since the current $j_\mu$ is conserved, only states of spin one contribute. We again approximate these contributions by an infinite set of zero width resonances at

$$\beta(q^2) = n; \quad n = 0, 1, 2, \ldots.$$

The trajectory $\beta$ is again taken linear and, although this is not crucial for the following, we also give it unit slope

$$\beta(q^2) = \beta_0 + q^2. \tag{3.3}$$

In our model the simple poles at non-negative integer values of $\alpha$ and $\beta$ are the only singularities in momentum space, and we assume that the representation of the amplitudes $A(k^2, q^2)$ and $B(k^2, q^2)$ in terms of their singularities

$$\begin{aligned} A(k^2, q^2) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{mn}}{(m-\alpha)(n-\beta)} \\ B(k^2, q^2) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{b_{mn}}{(m-\alpha)(n-\beta)} \end{aligned} \tag{3.4}$$

is convergent.

## 4. Singularities in Coordinate Space

By virtue of causality the quantity $\tilde{R}_\mu(x, p)$ vanishes outside the forward light cone. From the equal time commutation rule (2.2) we infer that $\tilde{R}_\mu$ possesses a singularity proportional to $\partial_\mu \Delta^{\text{ret}}(x)$ on the tip of the light cone, where

$$\Delta^{\text{ret}}(x) = \frac{\theta(x^0)}{2\pi} \delta(x^2)$$

The invariants $\tilde{A}(x^2, px)$ and $\tilde{B}(x^2, px)$ must therefore contain a singularity proportional to $\Delta^{\text{ret}}$ on the *tip* of the light cone. We now assume that the behaviour on the whole light cone is characterized by a canonical singularity of this type:

$$\begin{aligned} \tilde{A}(x^2, px) &= \Delta^{\text{ret}}(x) A^*(px) + \text{less singular terms} \\ \tilde{B}(x^2, px) &= \Delta^{\text{ret}}(x) B^*(px) + \text{less singular terms.} \end{aligned} \tag{4.1}$$

Equivalently we may express this requirement as a condition on the asymptotic behaviour in momentum space. Consider the limit

$$k^\mu = k_0^\mu + E n^\mu; \quad n^2 = 0; \quad E \to \pm \infty \tag{4.2}$$

where the (arbitrary) vector $k_0$ and the (lightlike) vector $n$ are kept fixed. We have shown elsewhere [2] that the asymptotic behaviour of a causal amplitude in this limit of momentum space is determined by its singularities on the light cone in coordinate space. For a canonical singularity of the type (4.1) the asymptotic behaviour in momentum space is given by a power law $A \sim E^{-1}$, more precisely

$$A(k_0 + En, p) \to \frac{i}{2E} \int_0^\infty d\lambda A^*(\lambda pn) e^{i\lambda k_0 n}. \tag{4.3}$$

This relation shows that the coefficient of the leading power in momentum space is the Fourier transform of the coefficient of the leading light cone singularity.

The limit (4.2) is the so-called Bjorken limit [3], as can be seen from the behaviour of the invariants $k^2$, $q^2$:

$$\left.\begin{aligned} k^2 &= k_0^2 + 2Ek_0 n \to \infty \\ q^2 &= (p-k_0)^2 - 2E(p-k_0)n \to \infty \end{aligned}\right\} \frac{k^2}{q^2} \to \text{const.} \tag{4.4}$$

In dealing with invariant amplitudes it is convenient to replace the parameter $E$ by the invariant

$$\nu = pk = pk_0 + Epn$$

and to denote by $\xi$ the ratio

$$\xi = \frac{k_0 n}{pn}$$

In this notation the limit (4.2) may be rewritten as

$$\left.\begin{aligned} k^2 &= 2\nu\xi - \eta \\ q^2 &= 2\nu(\xi - 1) - \eta + M^2 \end{aligned}\right\} \nu \to \infty, \tag{4.5}$$

where $\xi$ and $\eta$ are to be kept fixed, and the scaling law (4.3) takes the form

$$A(k^2, q^2) \to \frac{1}{2\nu} F_A^{\text{ret}}(\xi) \tag{4.6}$$

$$F_A^{\text{ret}}(\xi) = i \int_0^\infty dz A^*(z) e^{iz\xi}. \tag{4.7}$$

The canonical light cone singularity of the amplitude $B$ is of course reflected by an analogous scaling law in the Bjorken limit.

Let us note at this point that the scaling law (4.6) for the amplitudes $A$ and $B$ is compatible with the Ward identity (2.5) provided

$$-\xi F_A^{\text{ret}}(\xi) - (\xi - 1) F_B^{\text{ret}}(\xi) = 1. \tag{4.8}$$

The Ward identity is not compatible with a less singular behaviour of the amplitudes $A$ and $B$ on the light cone, but it does not, of course, prevent them from being more singular. Our model thus exhibits the smoothest possible light cone singularity consistent with the Ward identity.

What restriction does the scaling law impose on the residues $a_{mn}$ and $b_{mn}$ of the amplitudes $A$ and $B$? We first note that a single term $(m-\alpha)^{-1}(n-\beta)^{-1}$ in the representation (3.4) tends to zero with the second power of $\nu$ as $\nu \to \infty$. If the sum $\sum_{m,n} a_{mn}$ was convergent, the entire Amplitude $A$ would scale with the second power rather than the first and we would have a contradiction with the Ward identity. The scaling law therefore requires the sum $\sum_{m,n} a_{mn}$ to diverge. In this case only large values of $m$ and $n$ contribute to the asymptotic behaviour and scaling is guaranteed, provided

$$a_{mn} \to \frac{1}{m} g_A\left(\frac{n}{m}\right); \quad \left.\begin{aligned} m &\to \infty \\ n &\to \infty \end{aligned}\right\} \frac{n}{m} \text{ fixed.} \tag{4.9}$$

The corresponding scaling function is given by

$$F_A^{\text{ret}}(\xi) = \int_0^\infty d\lambda g_A(\lambda) \frac{1}{\xi - \lambda\xi - 1} \ln \frac{\lambda\xi}{\xi - 1}. \tag{4.10}$$

As a consistency check we show in the Appendix that the behaviour (4.9) of the coefficients indeed implies a canonical leading light cone singularity of the amplitude $\tilde{A}(x^2, px)$ of the type (4.1).

Needless to say that relations analogous to (4.9) and (4.10) also apply to the residues $b_{mn}$ and to the scaling function $F_B^{\text{ret}}$ associated with the amplitude $B$.

## 5. Mellin Transform

It is convenient to introduce the *generating function* associated with the residues $a_{mn}$ (we again focus on the amplitude $A$ since the analysis of the amplitude $B$ is entirely analogous):

$$a(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} (1-x)^m (1-y)^n. \tag{5.1}$$

The asymptotic behaviour (4.9) of the coefficients $a_{mn}$ guarantees that $a(x,y)$ is analytic in the interior of the unit circle $|1-x| < 1, |1-y| < 1$. The representation (3.4) of the amplitude $A$ $(k^2, q^2)$ as a sum of poles in $\alpha(k^2)$ and $\beta(q^2)$ may then be rewritten in the form of a Mellin transform:

$$A(k^2, q^2) = \int_0^1 dx \int_0^1 dy (1-x)^{-1-\alpha} (1-y)^{-1-\beta} a(x,y). \tag{5.2}$$

In contrast to the expansion in terms of poles this representation converges only for $\mathrm{Re}\,\alpha < 0$, $\mathrm{Re}\,\beta < 0$.

The asymptotic behaviour of the residues $a_{mn}$ for large values of $m$ and $n$ is reflected by the behaviour of the generating function in the vicinity of $x = y = 0$. Since the sum $\sum_{m,n} a_{mn}$ does not converge, $a(x,y)$ diverges for $x,y \to 0$. One easily works out the behaviour of $a(x,y)$ for small values of $x$ and $y$ from (4.9) with the result

$$a(x,y) \to \frac{1}{x+y} f_A\left(\frac{x}{y}\right); \quad \left.\begin{matrix} x \to 0 \\ y \to 0 \end{matrix}\right\} \frac{x}{y} \text{ fixed} \tag{5.3}$$

where

$$f_A(z) = \int_0^\infty d\lambda g_A(\lambda) \frac{z+1}{z+\lambda}. \tag{5.4}$$

Let us briefly discuss the asymptotic behaviour of the amplitude $A$ in the Bjorken limit by using the Mellin representation (5.2). In order not to leave the region of validity of this representation we consider the limit $\nu \to +\infty$, $\xi < 0$. In this limit the factor $(1-x)^{-1-\alpha}$ $(1-y)^{-1-\beta}$ tends to zero exponentially except at the point $x = y = 0$, and only the immediate neighbourhood of this point contributes significantly to the integral. If $a(x,y)$ was continuous at $x = y = 0$, the amplitude $A(k^2, q^2)$ would tend to zero with the second power of $\nu$ rather than with the first. It is not difficult to verify that the singular behaviour (5.3) indeed leads to the proper scaling law

$$A(k^2, q^2) \to \frac{1}{2\nu} F_A^{\text{ret}}(\xi)$$

with

$$F_A^{\text{ret}}(\xi) = \int_0^1 \frac{d\xi'}{\xi' - \xi - i\epsilon} f_A\left(\frac{1-\xi'}{\xi'}\right). \tag{5.5}$$

This expression for the scaling function of course agrees with our previous result (4.10). The absorptive part $F_A(\xi)$ of the scaling function, defined by

$$F_A^{\text{ret}}(\xi) = \int_0^1 \frac{d\xi'}{\xi' - \xi - i\epsilon} F_A(\xi'). \tag{5.6}$$

is determined by the function $f_A(z)$ which describes the singularity of the generating function at the origin:

$$F_A(\xi) = f_A\left(\frac{1-\xi}{\xi}\right). \tag{5.7}$$

It is interesting to observe that, in view of (5.4), our model leads to absorptive parts $F_A(\xi)$ that are analytic in the interval $0 < \xi < 1$.

## 6. Construction of the Model

We are now ready to set up a model for the amplitudes $A$ and $B$. Let us first recapitulate the conditions to be satisfied by these amplitudes:

1. Simple poles at non-negative integer values of the trajectories $\alpha(k^2)$ and $\beta(q^2)$.
2. Canonical light cone singularity of the type $\Delta^{\text{ret}}(x)$.
3. Ward identity.

We find it most convenient to work with the Mellin representation introduced above

$$A(k^2, q^2) = \int_0^1 dx\, dy (1-x)^{-1-\alpha} (1-y)^{-1-\beta} a(x, y)$$
$$\tag{6.1}$$
$$B(k^2, q^2) = \int_0^1 dx\, dy (1-x)^{-1-\alpha} (1-y)^{-1-\beta} b(x, y).$$

As we have seen in section 5, the conditions (1) and (2) are satisfied if the generating functions $a(x,y)$ and $b(x,y)$ are analytic in the interior of the unit circle $|1-x|<1$, $|1-y|<1$, and possess a singularity on the edge of this circle at the point $x=y=0$:

$$\begin{aligned} a(x,y) &\to \frac{1}{x+y} f_A\left(\frac{x}{y}\right) \\ & \qquad\qquad\qquad (x,y\to 0) \\ b(x,y) &\to \frac{1}{x+y} f_B\left(\frac{x}{y}\right) \end{aligned} \tag{6.2}$$

What remains to be done is to solve the Ward identity

$$-\alpha A-(\beta-\beta_0)B=1. \tag{6.3}$$

We rewrite this condition in the form

$$\int_0^1 dx\,dy\{-a\partial_x(1-x)-b\partial_y(1-y)+\beta_0 b\}(1-x)^{-1-\alpha}(1-y)^{-1-\beta}=1$$

and, integrating by parts (with $\mathrm{Re}\,\alpha<0$, $\mathrm{Re}\,\beta<0$)

$$\int_0^1 dx\,dy(1-x)^{-1-\alpha}(1-y)^{-1-\beta}\{(1-x)\partial_x a+(1-y)\partial_y b+\beta_0 b\}$$

$$=1-\int_0^1 dy(1-y)^{-1-\beta}a(0,y)-\int_0^1 dx(1-x)^{-1-\alpha}b(x,0)$$

or, equivalently

$$(1-x)\partial_x a+(1-y)\partial_y b+\beta_0 b=\delta(x)\,\delta(y)-\delta(x)a(0,y)-\delta(y)b(x,0). \tag{6.4}$$

In the interior of the range of integration we must therefore require

$$(1-x)\partial_x a+(1-y)\partial_y b+\beta_0 b=0$$

The solution of this condition reads:

$$\begin{aligned} a(x,y) &= -\{(1-y)\partial_y+\beta_0\}c(x,y) \\ b(x,y) &= (1-x)\partial_x c(x,y). \end{aligned} \tag{6.5}$$

The function $c(x,y)$ must again be analytic in the unit circle $|1-x|<1$, $|1-y|<1$, and at the point $x=y=0$ it must possess a singularity of the type

$$c(x,y)\to f\left(\frac{x}{y}\right) \quad (x,y\to 0) \tag{6.6}$$

Since the function $c(x,y)$ is singular, its derivatives with respect to $x$ and $y$ cannot be interchanged at $x=y=0$. [A simple example which illustrates this feature is $c(x,y)=x/(x+y)$.] In general, for a singularity of the type (6.6) one finds

$$\{\partial_x\partial_y-\partial_y\partial_x\}c(x,y)=\{f(0)-f(\infty)\}\delta(x)\,\delta(y) \tag{6.7}$$

The solution (6.5) therefore satisfies the Ward identity (6.4) provided

$$a(0,y) = b(x,0) = 0 \tag{6.8}$$

$$f(0) - f(\infty) = -1. \tag{6.9}$$

The condition (6.8) is solved by

$$c(x,0) = c_0 \tag{6.10}$$

$$c(0,y) = 0. \tag{6.11}$$

In the limit $y \to 0$ the result (6.11) shows that $f(0)$ vanishes, whereas from (6.10) we get $f(\infty) = c_0$. Finally, on account of (6.9) we find $c_0 = 1$.

To summarize: The amplitudes $A$ and $B$ defined by (6.1) satisfy the conditions (1), (2) and (3) above, provided the generating functions $a(x,y)$ and $b(x,y)$ are given by (6.5) in terms of a single function $c(x,y)$ which is analytic in the unit circle $|1-x| < 1$, $|1-y| < 1$, exhibits the singular behaviour (6.6) at $x = y = 0$ and satisfies

$$c(x,0) = f(\infty) = 1$$
$$c(0,y) = f(0) = 0. \tag{6.12}$$

## 7. Properties of the Model

Let us now suppose that we are given a function $c(x,y)$ which is analytic in the unit circle, such that the expansion

$$c(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn} (1-x)^m (1-y)^n. \tag{7.1}$$

converges for $|1-x| < 1$, $|1-y| < 1$, and which, furthermore, satisfies the conditions (6.6) and (6.12). We want to work out the properties of the corresponding amplitudes $A$ and $B$. For the amplitude $A$

$$A(k^2,q^2) = \int_0^1 dx\, dy (1-x)^{-1-\alpha} (1-y)^{-1-\beta} \{-(1-y)\, \partial_y - \beta_0\}\, c(x,y) \tag{7.2}$$

we obtain, integrating by parts and using the boundary conditions (6.12):

$$A(k^2,q^2) = -\frac{1}{\alpha} + (\beta - \beta_0) C(k^2,q^2) \tag{7.3}$$

where

$$C(k^2,q^2) = \int_0^1 dx\, dy (1-x)^{-1-\alpha} (1-y)^{-1-\beta} c(x,y). \tag{7.4}$$

Analogously,

$$B(k^2,q^2) = -\alpha C(k^2,q^2). \tag{7.5}$$

These representations for $A$ and $B$ show explicitly that the Ward identity is satisfied. In fact, (7.3) and (7.5) satisfy the Ward identity for any $C(k^2,q^2)$. What is not evident from these representations, however, is that $A$ and $B$ may be expressed in terms of their singularities according to (3.4). In particular, $A$ appears to contain a single pole term $1/\alpha$, in contradiction to (3.4). Actually this term is cancelled by a corresponding term in $C$. The representation

$$C(k^2, q^2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{c_{mn}}{(m-\alpha)(n-\beta)} \tag{7.6}$$

of the amplitude $C$ in terms of its singularities converges even more quickly than the corresponding representations for $A$ and $B$ and, using the relations

$$\begin{aligned} &\sum_{m=0}^{\infty} c_{mn} = 0 \\ &\sum_{n=0}^{\infty} c_{mn} = \delta_m^0, \end{aligned} \tag{7.7}$$

which follow from the boundary conditions (6.12), one finds that the representation (3.4) is indeed valid, with

$$\begin{aligned} a_{mn} &= (n - \beta_0) c_{mn} \\ b_{mn} &= -m c_{mn}. \end{aligned} \tag{7.8}$$

Let us, in particular, determine the residue of the pole at $\alpha = 0$ which, according to (3.1) is the electromagnetic form factor of the scalar external particle. The pole is absent in $B$ and we get

$$G(q^2) = \sum_{n=0}^{\infty} \frac{a_{0n}}{n-\beta} \tag{7.9}$$

or, in terms of the generating function

$$G(q^2) = -\int_0^1 dy (1-y)^{\beta_0 - \beta} \partial_y \{(1-y)^{-\beta_0} c(1, y)\}. \tag{7.10}$$

Since $q^2 = 0$ corresponds to $\beta = \beta_0$, this last expression shows explicitly that the particle carries unit charge, $G(0) = 1$.

Next, we determine the scaling properties of the model. According to our discussion in section 5, the scaling functions $F_A^{\mathrm{ret}}$ and $F_B^{\mathrm{ret}}$ are determined by the behaviour of the generating functions $a(x,y)$ and $b(x,y)$ in the vicinity of $x = y = 0$. From (6.5) and (6.6) we find

$$\begin{aligned} a(x,y) &\to \frac{x}{y^2} f'\left(\frac{x}{y}\right) \\ b(x,y) &\to \frac{1}{y} f'\left(\frac{x}{y}\right). \end{aligned}$$

Comparison with (6.2) shows that

$$\begin{aligned} f_A(z) &= z(1+z) f'(z) \\ f_B(z) &= (1+z) f'(z) \end{aligned} \tag{7.11}$$

and, in view of (5.7), the absorptive parts of the scaling functions are therefore given by

$$F_A(\xi) = \frac{1-\xi}{\xi^2} f'\left(\frac{1-\xi}{\xi}\right)$$

$$F_B(\xi) = \frac{1}{\xi} f'\left(\frac{1-\xi}{\xi}\right). \tag{7.12}$$

It is straightforward to verify that the corresponding retarded scaling functions $F_A^{\mathrm{ret}}$ and $F_B^{\mathrm{ret}}$ satisfy the Ward identity (4.8). The absorptive parts obey

$$\xi F_A(\xi) + (\xi - 1) F_B(\xi) = 0 \tag{7.13}$$

and, furthermore, satisfy the sum rules

$$\int_0^1 \frac{d\xi}{1-\xi} F_A(\xi) = \int_0^1 \frac{d\xi}{\xi} F_B(\xi) = 1 \tag{7.14}$$

which also follow immediately from (4.8).

It is instructive to work out the scaling properties of the amplitude $C$ defined by (7.4). Since the generating function $c(x,y)$ is less singular at the origin than the functions $a(x,y)$ and $b(x,y)$, the amplitude $C$ scales one power faster than $A$ or $B$:

$$C(k^2, q^2) \to \frac{1}{(2\nu)^2} F_C^{\mathrm{ret}}(\xi)$$

$$F_C^{\mathrm{ret}}(\xi) = \int_0^\infty dz \frac{f(z)}{(1-\xi-\xi z)^2}. \tag{7.15}$$

The corresponding absorptive part is given by

$$F_C(\xi) = \delta(\xi) - \frac{1}{\xi^2} f'\left(\frac{1-\xi}{\xi}\right).$$

The $\delta$-function contribution precisely cancels the term $1/\alpha$ in (7.3), but drops out in (7.5).

## 8. Model Determined by the Leading Light Cone Singularity

In this section we discuss a more specific model. We assume that the generating function $c(x,y)$ is a function of the ratio $x/y$ not only in the vicinity of the origin, but everywhere:

$$c(x,y) = f\left(\frac{x}{y}\right). \tag{8.1}$$

As we have seen, the coefficient of the leading light cone singularity of the model determines the singularity of the function $c(x,y)$ at the origin, i.e. determines the function $f(z)$. We are therefore considering a situation where the entire amplitude is determined by the leading light cone singularity.

Let us characterize the leading light cone singularity by the absorptive part $F_A(\xi)$ of the scaling function associated with the amplitude $A$. We assume that $F_A(\xi)$ is analytic in the interval $0 < \xi < 1$, vanishes at the endpoints $\xi = 0,1$ like $F_A(\xi) \sim \xi^{p_0}$, $p_0 > 0$ and $F_A(\xi) \sim (1-\xi)^{1+p_1}$, $p_1 > 0$ respectively [4] and obeys the sum rule

$$\int_0^1 \frac{d\xi}{1-\xi} F_A(\xi) = 1. \tag{8.2}$$

From (7.12) we infer that the function $f(z)$ which corresponds to this leading light cone singularity is given by

$$f(z) = \int_0^z \frac{dx}{x(1+x)} F_A\left(\frac{1}{1+x}\right). \tag{8.3}$$

It is convenient to introduce the Mellin transform of the scaling function by

$$\tilde{F}(\lambda) = \int_0^\infty \frac{dx}{1+x} F_A\left(\frac{1}{1+x}\right) x^{-\lambda-1} \tag{8.4}$$

with the inverse

$$\frac{1}{1+x} F_A\left(\frac{1}{1+x}\right) = \frac{1}{2\pi i} \int_{\rho-i\infty}^{\rho+i\infty} d\lambda\, \tilde{F}(\lambda) x^\lambda. \tag{8.5}$$

The function $\tilde{F}$ is analytic in the strip $-(1+p_0) < \mathrm{Re}\lambda < (1+p_1)$. Accordingly, the inversion formula is valid for any value of $\rho$ in this strip. Inserting the formula (8.5) in (8.3) one finds that the function $f(z)$ may also be represented in the form of a Mellin transform:

$$f(z) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} d\lambda \frac{\tilde{F}(\lambda)}{\lambda} z^\lambda \tag{8.6}$$

with $0 < \mathrm{Re}\,\sigma < (1+p_1)$.

Using this representation, the integral (7.4) defining $C(k^2, q^2)$ may be worked out explicitly with the result

$$C(k^2, q^2) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} d\lambda \frac{\tilde{F}(\lambda)}{\lambda} B[-\alpha, 1+\lambda] B[-\beta, 1-\lambda]. \tag{8.7}$$

The corresponding expressions for the amplitudes $A$ and $B$ are given by (7.3) and (7.5) respectively. In particular, one finds an explicit expression for the form factor of the external scalar particle in terms of the scaling function $F_A$ (compare (7.10)):

$$G(q^2) = \frac{\beta_0}{\beta} + \frac{\beta-\beta_0}{\beta} \int_0^1 \frac{dy}{1+y} (1-y)^{-\beta} F_A\left(\frac{y}{1+y}\right). \tag{8.8}$$

Note that the representation (8.7) is valid for arbitrary $k^2$ and $q^2$; one easily verifies that the corresponding amplitudes $A$ and $B$ scale properly in the Bjorken limit for any direction of momentum space.

The simple model discussed in this section shows that we may prescribe the scaling functions $F_A(\xi)$ and $F_B(\xi)$ associated with the absorptive parts of the amplitudes $A$ and $B$ essentially as we like, provided only they satisfy the relations (7.13) and (7.14) and are analytic in the interval $0 < \xi < 1$.

There always exists a pair of amplitudes $A$ and $B$ which satisfy the Ward identity, possess the required singularity structure in momentum space and scale properly in the Bjorken limit. Of course, the amplitudes $A$ and $B$ are not determined uniquely by these requirements. It is however not difficult to show that the general solution of these requirements differs from the standard solution constructed above by

$$A' = A + (\beta_0 - \beta)D$$
$$B' = B + \alpha D \tag{8.9}$$

where the singularities of the amplitude $D$ in momentum space are again simple poles at non-negative integer values of $\alpha$ and $\beta$ and, since the scaling functions are not affected, $D$ vanishes faster than $\nu^{-2}$ in the Bjorken limit. In other words, the amplitude $D$ is continuous on the light cone in coordinate space.

An extension of the model presented in this paper to dual four-point functions will be published elsewhere.

## APPENDIX: DGS REPRESENTATION

To determine the behaviour of an amplitude defined by

$$A(k^2, q^2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{mn}}{(m-\alpha)(n-\beta)} \tag{A.1}$$

on the light cone it is convenient to work out the corresponding Deser-Gilbert-Sudarshan [5] representation:

$$A(k^2, q^2) = \int_0^\infty ds \int_0^1 d\xi\, \sigma_A(s, \xi) \frac{1}{s - (k - \xi p)^2 - i\epsilon(k^0 - \xi p^0)} \tag{A.2}$$

The spectral function is easily found if one applies Feynman's parametrization

$$\frac{1}{(m-\alpha)(n-\beta)} = \int_0^1 d\xi \frac{1}{N^2}$$

with

$$N = (1-\xi)(m-\alpha) + \xi(n-\beta) = s_{mn}(\xi) - (k - \xi p)^2$$

and

$$s_{mn}(\xi) = m(1-\xi) + n\xi - \alpha_0 (1-\xi)^2 - \beta_0 \xi. \tag{A.3}$$

The spectral function is therefore given by

$$\sigma_A(s, \xi) = \sum_{m,n} a_{mn} \delta'[s - s_{mn}(\xi)]. \tag{A.4}$$

In coordinate space, the DGS representation reads

$$\tilde{A}(x^2, px) = \int_0^\infty ds \int_0^1 d\xi\, \sigma_A(s, \xi) e^{-i\xi px} \Delta^{\mathrm{ret}}(x; s) \tag{A.5}$$

with

$$\Delta^{\mathrm{ret}}(x; s) = \frac{\theta(x^0)}{2\pi} \left\{ \delta(x^2) - \theta(x^2) \frac{1}{2} \sqrt{\frac{s}{x^2}}\, J_1(\sqrt{sx^2}) \right\}.$$

The character of the leading light cone singularity is controlled by the behaviour of the spectral function for large values of $s$. It is easy to verify that if the spectral function behaves like

$$\sigma_A(s, \xi) \sim s^{-1-\epsilon}, \quad (s \to \infty) \tag{A.6}$$

with $0 < \epsilon < 1$ then the leading light cone singularity is proportional to $\delta(x^2)$:

$$\tilde{A}(x^2, px) = \Delta^{\mathrm{ret}}(x) A^*(px) + R(x) \tag{A.7}$$

where

$$\Delta^{\mathrm{ret}}(x) = \Delta^{\mathrm{ret}}(x; 0) = \frac{1}{2\pi} \theta(x^0) \delta(x^2).$$

The remainder $R(x)$ behaves like $(x^2)^{-1+\epsilon}$ as $x^2 \to 0$. Finally, the coefficient of the leading light cone singularity, $A^*(px)$, is an entire function of $px$, determined by the integral

$$F_A(\xi) = \int_0^\infty ds\, \sigma_A(s, \xi) \tag{A.8}$$

explicitly,

$$A^*(px) = \int_0^1 d\xi\, F_A(\xi) e^{-i\xi px}. \tag{A.9}$$

It remains to be shown that the spectral function indeed satisfies (A.6). To demonstrate this property we take a smooth test function $\phi(\xi)$ and consider

$$\sigma_A(s, \phi) = \int_0^1 d\xi\, \sigma_A(s, \xi) \phi(\xi) = \frac{d}{ds} \sum_{m,n} a_{mn} \{ |s'_{mn}(\xi)|^{-1} \phi(\xi) \}_{\xi = \xi_{mn}} \tag{A.10}$$

where $\xi_{mn} = \xi_{mn}(s)$ is determined by the equation

$$s_{mn}(\xi_{mn}) = s. \tag{A.11}$$

In the limit of large $s$, large $m$ and large $n$ this equation may be solved by iteration. In the following we consider only the first iteration, since the corrections are suppressed by one power of $s$. To first order:

$$\xi^0_{mn} = \frac{s-m}{n-m} \tag{A.12}$$

and we get

$$\sigma^0_A(s,\phi) = \sum_{m,n} a_{mn} \frac{\epsilon(n-m)}{(n-m)^2}\, \phi'\left(\frac{s-m}{n-m}\right). \tag{A.13}$$

To estimate this sum we need the asymptotic behaviour of the coefficients $a_{mn}$ which was discussed in section 4. We need a more precise form:

$$a_{mn} = \frac{1}{m}\, g_A\left(\frac{n}{m}\right) + 0(m^{-1-\epsilon}) \quad (m, n \to \infty).$$

It is evident that the correction term $0(m^{-1-\epsilon})$ gives rise to a contribution satisfying (A.6) and we therefore concentrate on the leading term. In this term we replace the sum over $m$ and $n$ by an integral, thereby committing an error of relative order $s^{-2}$. Finally, the integrand is a total derivative and hence integrates to zero. This completes our sketch of the proof that the spectral function of our model satisfies (A.6).

As a further check one easily works out the integral $\int ds\, \sigma_A$ from (A.10):

$$\int_0^\infty ds\, \sigma_A(s,\phi) = \lim_{s\to\infty} \sum_{m,n} a_{mn} \{|s'_{mn}(\xi)|^{-1} \phi(\xi)\}_{\xi=\xi_{mn}}.$$

Using the same methods as above this sum may be transformed into an integral of the form

$$\int_0^\infty ds\, \sigma_A(s,\phi) = \int_0^1 d\xi \int_0^\infty d\lambda\, \frac{\phi(\xi) g_A(\lambda)}{1+\lambda\xi-\xi}.$$

Since $\phi$ is an arbitrary test function, we conclude

$$F_A(\xi) = \int_0^\infty d\lambda\, \frac{g_A(\lambda)}{1+\lambda\xi-\xi}. \tag{A.14}$$

From $F_A(\xi)$ one computes $A^*(px)$ by means of (A.9) and, inserting the result in (4.7), one finds

$$F_A^{\mathrm{ret}}(\xi) = \int_0^1 \frac{d\xi'}{\xi'-\xi-i\epsilon}\, F_A(\xi') \tag{A.15}$$

The explicit expression (A.14) for $F_A$ therefore leads to an independent evaluation of $F_A^{\mathrm{ret}}$ which is in agreement with (4.10).

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REFERENCES

[1] P. V. LANDSHOFF and J. C. POLKINGHORNE, Nucl. Phys. [B] *19*, 432 (1970). For a review of related work see P. V. LANDSHOFF and J. C. POLKINGHORNE, report presented at the International Summer Institute in Theoretical Physics (Hamburg 1971).

[2] H. LEUTWYLER and J. STERN, Nucl. Phys. [B] *20*, 77 (1970). See also R. A. BRANDT, Phys. Rev. Letters *23*, 1260 (1969); L. S. BROWN, Lectures given at the Summer Institute for Theoretical Physics (University of Colorado, Boulder 1969); R. JACKIW, R. VAN ROYEN, and G. WEST, Phys. Rev. [D] *2*, 2473 (1970).

[3] J. D. BJORKEN, Phys. Rev. *179*, 1547 (1969).

[4] We do not discuss the special problems which arise if the scaling functions $F_A$, $F_B$ are singular at the points $\xi = 0, 1$.

[5] S. DESER, W. GILBERT and E. C. G. SUDARSHAN, Phys. Rev. *115*, 731 (1959).