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# A Generalization of the Hopf Bifurcation Theorem

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*Zusammenfassung.* Mit Methoden, die z.T. den klassischen Methoden der analytischen Mechanik nachgebildet sind, z.T. von M. W. Hirsch und C. C. Pugh stammen, wird eine Verallgemeinerung der E. Hopfschen «Abzweigung einer periodischen Lösung von einer stationären Lösung eines Differentialsystemes» behandelt. Insbesondere wird gezeigt, wie bei der Explosion einer stabilen Gleichgewichtslage im $R^4$ anziehende zweidimensionale Tori entstehen können. Die Resultate stehen in den Theoremen 1, 2 und 3.

## Introduction

In reading the paper by David Ruelle and Floris Takens: *On the Nature of Turbulence* [2] it appeared that their discussion of normal forms for families of diffeomorphisms with a fixed point is very closely related to the classical work by G. D. Birkhoff on area-preserving analytic diffeomorphisms with fixed point in the plane [3] §21. It seemed to us that these old-fashioned methods could possibly be of some pedagogical advantage especially for physicists. For this reason we present them here.

We restrict ourselves to the special case of four dimensions. Generalizations are of course possible and in many situations straightforward. Since the case of two dimensions has been dealt with by Ruelle and Takens, our discussion is a direct continuation of their work.

The justification for our restriction to an even number of dimensions is similar to the justification given in [2] §5 for the reduction to two dimensions (cf. proposition (5.2)).

Section 1 contains the proof of Theorem 1, which describes normal forms for our family of diffeomorphisms. The family of diffeomorphisms is parametrized by two real parameters. Following E. Hopf [4] we are interested in the situation where the two pairs of complex conjugate eigenvalues cross the unit circle (explosion of a stable point of equilibrium). Our normal forms, however, are in general not unique. They decompose up to terms of higher order, naturally into a dissipative part and into a measure preserving part. The special truncated normal form which underlies sections 2 and 3 is unique.

In section 2 we analyze the explosion of the stable equilibrium point for a truncated family of diffeomorphism from which the given family of diffeomorphisms is obtained by the addition of a small perturbation. This discussion is elementary and we restrict ourselves to a special case. As a result of the explosion certain invariant circles and two dimensional tori appear. Our interest is limited to the cases where these invariant manifolds are attractive.

Section 3 finally discusses the effect of the perturbation which leads from the truncated diffeomorphism to the actual diffeomorphism. As was to be expected the perturbation does not change the nature of the invariant attractive manifolds which result from the explosion. The results are stated in theorem 2. Section 3 leans heavily on the work of Hirsch and Pugh [1].

We thank David Ruelle for sending us his manuscript prior to publication and for his kind encouragement in the course of this work.

It is a special pleasure to dedicate this paper to our teacher, colleague and friend Markus Fierz on the occasion of his 60th birthday.

## 1. Normal Forms

### 1.0. *Statement of Theorem* 1

We consider a two-parameter family $\{\phi(\mu)|\mu \in I \subset R^2\}$, $I$ open, of diffeomorphisms $U \to R^4$, $U \subset R^4$ open. Each $\phi(\mu)$ has the origin $0 \in U$ of $R^4$ as a fixed point. Define $\phi: U \times I \to R^4 \times R^2$ by

$$\phi:(x,\mu) \mapsto (\phi(\mu)(x),\mu). \tag{1.0.1}$$

We assume

$$\phi \in C^K(U \times I), \quad K \geqslant 5. \tag{1.0.2}$$

Of the spectrum of $(D\phi(\mu))(0) = \Lambda(\mu)$ we require

$$\sigma(\Lambda(\mu)) = \{\lambda_1(\mu), \overline{\lambda_1(\mu)}, \lambda_2(\mu), \overline{\lambda_2(\mu)}\} \tag{1.0.3}$$

and

$$\lambda_\sigma(\mu) \neq \overline{\lambda_\sigma(\mu)}, \quad \sigma \in \{1,2\}. \tag{1.0.4}$$

In addition $0 \in I$ and

$$|\lambda_\sigma(0)| = 1. \tag{1.0.5}$$

We write

$$\lambda_\sigma(\mu) = \rho_\sigma(\mu) e^{i\alpha_\sigma(\mu)}$$

$$\rho_\sigma(\mu) \in R_+, \quad \alpha_\sigma(\mu) \in (0, 2\pi). \tag{1.0.6}$$

Let $R: I \to R_+^2$ be defined by

$$R: \mu \mapsto (\rho_1(\mu), \rho_2(\mu)). \tag{1.0.7}$$

We finally assume $(DR)$ (0) to be non-singular. It is then no essential restriction to put

$$\rho_\sigma(\mu) = 1 + \mu_\sigma. \tag{1.0.8}$$

According to these assumptions $\phi(\mu)$ can be expanded into powers of $x$:

$$\phi(\mu)(x) = \Lambda(\mu)x + \sum_{k=2}^{K} f^{(k)}(\mu)(x) + \theta_K, \tag{1.0.9}$$

with $f^{(k)}(\mu)$ a vectorvalued homogeneous polynomial of degree $k$ in $x$ and coefficients from $C^{K-k}(I)$. $\theta_K$ is of order $K$ in $x$.

After a suitable $\mu$-dependent linear coordinate transformation in $x$, $\Lambda(\mu)$ will be of the following *normal form*: $\Lambda(\mu): x \mapsto x'$

$$x_1' \pm i x_2' = (1 + \mu_1) e^{\pm i \alpha_1(\mu)} (x_1 \pm i x_2)$$
$$x_3' \pm i x_4' = (1 + \mu_2) e^{\pm i \alpha_2(\mu)} (x_3 \pm i x_4). \qquad (1.0.10)$$

Theorem 1 generalizes this normal form to higher powers of $x$.

Theorem 1. *Assume that*

$$s_1 \alpha_1(0) + s_2 \alpha_2(0) = 2\pi m; \quad s_\sigma \in \mathbb{Z}, \quad m \in \mathbb{Z}$$
$$\|s\| \equiv |s_1| + |s_2| \leqslant S + 1, \quad S \in \mathbb{N}_0, \quad S < K \qquad (1.0.11)$$

*implies* $\|s\| = 0$, *then the following statement holds*:

$I_S \subset I$, $I_S$ *open*, $0 \in I_S$ *and a two-parameter family* $\{T(\mu) | \mu \in I_S\}$ *of coordinate transformations exist with the properties*

i) $T(\mu): V \to U' \subset U, U'$ *open* $0 \mapsto 0$ (1.0.12)

ii) $T(\mu) \in C^\omega(R^4)$ (1.0.13)

iii) $T: (x, \mu) \mapsto (T(\mu)(x), \mu)$ *satisfies* $T \in C^{K-S}(R^4 \times I_S)$ (1.0.14)

iv) $\psi(\mu) = T(\mu)^{-1} \circ \phi(\mu) \circ T(\mu): x \mapsto x'$

*is of the form*

$$(x_1' \pm i x_2') = P_1(\mu)(\omega) \cdot e^{\pm i Q_1(\mu)(\omega)} (x_1 \pm i x_2) + \theta_{S+1}.$$
$$(x_3' \pm i x_4') = P_2(\mu)(\omega) \cdot e^{\pm i Q_2(\mu)(\omega)} (x_3 \pm i x_4) + \theta_{S+1} \qquad (1.0.15)$$

*with real polynomials* $P_\sigma(\mu)$ *and* $Q_\sigma(\mu)$ *of degree* $[S/2]$ *in* $\omega_1, \omega_2$

$$\omega_1 = x_1^2 + x_2^2 \quad \omega_2 = x_3^2 + x_4^2. \qquad (1.0.16)$$

*and*

$$P_\sigma(\mu)(0) = 1 + \mu_\sigma, \quad Q_\sigma(\mu)(0) = \alpha_\sigma(\mu). \qquad (1.0.17)$$

v) $\psi: (x, \mu) \mapsto (\psi(\mu)(x), \mu)$ (1.0.18)

*satisfies*

$$\psi \in C^{K-S}(V \times I_S). \qquad (1.0.19)$$

## 1.1 *Introduction of complex coordinates*

The theorem suggests the introduction of the complex coordinates:

$$u_1 = x_1 + i x_2 \quad v_1 = x_1 - i x_2$$
$$u_2 = x_3 + i x_4 \quad v_2 = x_3 - i x_4. \qquad (1.1.1)$$

$\phi(\mu)$ is only defined on *real points* which correspond to complex coordinates satisfying

$$v_\sigma = \overline{u_\sigma}. \qquad (1.1.2)$$

$\phi(\mu)$ is *real*, i.e. it transforms real points into real points. The expansion (1.0.9) now takes the form

$$u'_\sigma = \lambda_\sigma u_\sigma + \sum_{k=2}^{K-1} p_\sigma^{(k)}(\mu)(u,v) + \theta_K \tag{1.1.3}$$

$$v'_\sigma = \bar{\lambda}_\sigma v_\sigma + \sum_{k=2}^{K-1} \overline{p_\sigma^{(k)}(\mu)(\bar{v},\bar{u})} + \theta_K,$$

with $\theta_K$ only defined on real points and $p_\sigma^{(k)}(\mu)$ a homogeneous polynomial of degree $k$ in $(u,v)$ with coefficients from $C^{K-k}(I)$. Correspondingly we write

$$\phi(\mu) = \Lambda(\mu) + \sum_{k=2}^{K-1} \mathfrak{p}^{(k)}(\mu) + \theta_K, \tag{1.1.4}$$

$\mathfrak{p}^{(k)}(\mu): \mathbb{C}^4 \to \mathbb{C}^4$ is defined by $(u,v) \mapsto (u',v')$

$$u'_\sigma = p_\sigma^{(k)}(\mu)(u,v)$$
$$v'_\sigma = \overline{p_\sigma^{(k)}(\mu)(\bar{v},\bar{u})} \tag{1.1.5}$$

and is itself a *real* mapping.

## 1.2. *The coordinate transformations*

We admit coordinate transformations $T(\mu):(U,V) \mapsto (u,v)$ which are generated by

i) $\Delta(\mu): u_\sigma = \tau_\sigma(\mu) U_\sigma$

$$v_\sigma = \overline{\tau_\sigma(\mu)} V_\sigma, \tag{1.2.1}$$

where $\tau_\sigma \in C^{K-S}(I)$, $\tau_\sigma: I \to \mathbb{C}\backslash\{0\}$.

ii) $E_S(\mu)$, $2 \leqslant S < K$

$$u_\sigma = U_\sigma + q_\sigma^{(S)}(\mu)(U,V)$$
$$v_\sigma = V_\sigma + \overline{q_\sigma^{(S)}(\mu)(\bar{V},\bar{U})} \tag{1.2.2}$$

with a homogeneous polynomial $q_\sigma^{(S)}(\mu)$ of degree $S$ and coefficients in $C^{K-S}(I)$.

In analogy to (1.1.5) we write

$$E_S(\mu) = 1 + \mathfrak{q}^{(S)}(\mu) \tag{1.2.3}$$

All these transformations admit local inverses and are *real*, i.e. satisfy $(U,\bar{U}) \mapsto (u,\bar{u})$. The same is true for arbitrary products. These also satisfy i), ii), iii) of Theorem 1.

## 1.3. *The lattice*

Let $s \in \mathbb{Z}^2$, $(s,\alpha(\mu)) = s_1\alpha_1(\mu) + s_2\alpha_2(\mu)$, and $\|s\|$ as in (1.0.11). The lattice $\mathfrak{g}(\mu)$ is defined by

$$\mathfrak{g}(\mu) = \{s \mid (s,\alpha(\mu)) \equiv 0(2\pi)\}. \tag{1.3.1}$$

We write

$$\mathfrak{g} = \mathfrak{g}(0). \tag{1.3.2}$$

For $S \in \mathbb{N}_0$ let

$$\mathfrak{k}_S = \{s \mid \|s\| \leqslant S+1\}. \tag{1.3.3}$$

The assumption of Theorem 1 states

$$\mathfrak{g} \cap \mathfrak{k}_S = \{0\}. \tag{1.3.4}$$

Define $I_S$ in Theorem 1 by

$$I_S = \{\mu \mid \mu \in I,\ \mathfrak{g}(\mu) \cap \mathfrak{k}_S = \{0\}\}. \tag{1.3.5}$$

$I_S$ is open and $0 \in I_S$.

The following Lemma 1 is trivial

Lemma 1. *Let* $m \in \mathbb{Z}_+^2$, $n \in \mathbb{Z}_+^2$, $\|m\| + \|n\| \leqslant S$, $\mu \in I_S$ *then*

$$\lambda^m(\mu)\,\overline{\lambda(\mu)^n} = \lambda_\sigma(\mu) \tag{1.3.6}$$

*implies*

$$m_\sigma = n_\sigma + 1 \quad \text{and} \quad m_{\sigma'} = n_{\sigma'} \text{ for } \sigma' \neq \sigma. \tag{1.3.7}$$

*Corollary:* (1.3.6) is possible only if $\|m\| + \|n\| \equiv 1\ (2)$.

## 1.4. *The operator* $\mathscr{L}_S(\mu)$

Take $\Lambda(\mu)$ from (1.1.3) and (1.1.4) and define in analogy to (1.1.5) $\mathfrak{q}^{(S)}:(u,v) \mapsto (u',v')$

$$u'_\sigma = q_\sigma^{(S)}(u,v)$$

$$v'_\sigma = \overline{q_\sigma^{(S)}(\bar{v},\bar{u})}, \tag{1.4.1}$$

with homogeneous polynomials $q_\sigma^{(S)}$ of degree $S \geqslant 2$.

$\{\mathfrak{q}^{(S)}\} = \mathfrak{Q}^{(S)}$ is in a natural way a finite dimensional vector space over $\mathbb{C}$. The linear mapping $\mathscr{L}_S(\mu): \mathfrak{Q}^{(S)} \to \mathfrak{Q}^{(S)}$ is defined by

$$\mathscr{L}_S(\mu): \mathfrak{q}^{(S)} \mapsto \mathfrak{q}^{(S)} = \Lambda(\mu) \circ \mathfrak{q}^{(S)} - \mathfrak{q}^{(S)} \circ \Lambda(\mu). \tag{1.4.2}$$

$\mathscr{L}_S(\mu)$ induces a mapping on the homogeneous vector valued polynomials $q^{(S)}$ of degree $S$. Let

$$q_\sigma^{(S)}(u,v) = \sum_{\|m\|+\|n\|=S} b_{\sigma,mn} u^m v^n, \tag{1.4.3}$$

then

$$\mathscr{L}_S(\mu): q^{(S)} \mapsto q^{(S)\prime}$$

$$q_\sigma^{(S)\prime}(u,v) = \sum_{\|m\|+\|n\|=S} (\lambda_\sigma(\mu) - \lambda(\mu)^m\,\overline{\lambda(\mu)^n})\, b_{\sigma,mn} u^m v^n. \tag{1.4.4}$$

From this formula and Lemma 1 we read off

Lemma 2. *Under the assumptions of Theorem* 1, $\mu \in I_S$*:*

i) kernel $\mathscr{L}_S(0) = 0 \in \mathfrak{Q}^{(S)}$ for $S \equiv 0(2)$ (1.4.5)

ii) $S \equiv 1(2)$

$$\text{kernel } \mathscr{L}_S(0) = \{q^{(S)} \mid q_\sigma^{(S)} = \sum_{2(\alpha+\beta)=S-1} b_{\sigma,\alpha\beta} (u_1 v_1)^\alpha (u_2 v_2)^\beta u_\sigma\} \tag{1.4.6}$$

iii) kernel $\mathscr{L}_S(\mu) \subset$ kernel $\mathscr{L}_S(0)$ (1.4.7)

iv) kernel $\mathscr{L}_S(0)$ + range $\mathscr{L}_S(0) = \mathfrak{Q}^{(S)}$ (1.4.8)

v) $(\mathscr{L}_S(\mu))^{-1}$: range $\mathscr{L}_S(0) \to$ range $\mathscr{L}_S(0)$ *is injective*

*The coefficients of* $(\mathscr{L}_S(\mu))^{-1} q^S$, $q^S \in$ range $\mathscr{L}_S(0)$ are $C^{K-S}(I_S)$.

## 1.5. *S-normal forms*

*Definition:*

$$\phi(\mu) = \Lambda(\mu) + \sum_{k=2}^{K-1} \mathfrak{p}^{(k)}(\mu) + \theta_K \tag{1.5.1}$$

*is an S-normal form if*

$$\mathfrak{p}^{(k)}(\mu) \in \text{kernel } \mathscr{L}_k(0), \quad k \leqslant S < K.$$

*for* $\mu \in I_S$.

In any case (1.1.4) is a 1-normal form. The existence of $S$-normal forms, $S < K$ follows from

Lemma 3. *Let* $S < K$ *and* $\phi(\mu)$ *an* $(S-1)$ *normal form. A coordinate-transformation* $E_S(\mu)$ (1.2.3) *exists such that* $E_S(\mu)^{-1} \circ \phi(\mu) \circ E_S(\mu)$ *is an S-normal form.*

*Proof:* $\psi(\mu) = E_S(\mu)^{-1} \circ \phi(\mu) \circ E_S(\mu)$ is defined by

$$E_S(\mu) \circ \psi(\mu) = \phi(\mu) \circ E_S(\mu). \tag{1.5.2}$$

Substitution of (1.5.1) and (1.2.3) yields

$$\psi(\mu) = \Lambda(\mu) + \sum_{k=2}^{S-1} \mathfrak{p}^{(k)}(\mu)$$

$$+ \mathfrak{p}^{(S)}(\mu) + (\Lambda(\mu) \circ \mathfrak{q}^{(S)}(\mu) - \mathfrak{q}^{(S)}(\mu) \circ \Lambda(\mu)) + \theta_{S+1}$$

$$= \Lambda(\mu) + \sum_{k=2}^{S-1} \mathfrak{p}^{(k)}(\mu) + (\mathfrak{p}^{(S)}(\mu) + \mathscr{L}_S(\mu)\mathfrak{q}^{(S)}(\mu)) + \theta_{S+1} \tag{1.5.3}$$

According to Lemma 2 (iv) for $\mu \in I_S$ uniquely

$$\mathfrak{p}^{(S)}(\mu) = \mathfrak{p}_1^{(S)}(\mu) + \mathfrak{p}_2^{(S)}(\mu) \tag{1.5.4}$$

$\mathfrak{p}_1^{(S)}(\mu) \in$ kernel $\mathscr{L}_S(0)$, $\mathfrak{p}_2^{(S)}(\mu) \in$ range $\mathscr{L}_S(0)$ and from Lemma 2 (v)

$$\mathfrak{q}^{(S)}(\mu) = -(\mathscr{L}_S(\mu))^{-1} \mathfrak{p}_2^{(S)}(\mu). \tag{1.5.5}$$

$E_S(\mu)$ is a coordinate-transformation from 1.2 and the polynomial defining $\mathfrak{p}_1^{(S)}(\mu)$ has coefficients from $C^{K-S}(I_S)$. ■

Now we use Lemma 2i) and ii) to obtain

Lemma 4. *Under the assumptions of theorem* 1 *with the definition* (1.3.5) *of* $I_S$ *there exists* $T(\mu)$ *satisfying i), ii), iii) of theorem* 1 *such that* $T(\mu)^{-1} \circ \phi(\mu) \circ T(\mu) = \psi(\mu)$: $(u,v) \mapsto (u',v')$ *is of the form*

$$u'_\sigma = \hat{P}^{(S)}_\sigma(\mu)(\omega)u_\sigma + \theta_{S+1} \tag{1.5.6}$$

$$v'_\sigma = \overline{\hat{P}^{(S)}_\sigma(\mu)(\overline{\omega})}v_\sigma + \theta_{S+1},$$

$$\omega = (\omega_1, \omega_2), \quad \omega_\sigma = u_\sigma v_\sigma, \tag{1.5.7}$$

$\hat{P}^{(S)}_\sigma(\mu)$ *a polynomial of degree* $[S/2]$ *in* $\omega$ *satisfying*

$$\hat{P}^{(S)}_\sigma(\mu)(0) = \lambda_\sigma(\mu) = (1+\mu_\sigma)e^{i\alpha_\sigma(\mu)}. \tag{1.5.8}$$

### 1.6. *Proof of Theorem* 1

All we have to do to arrive from Lemma 4 to Theorem 1 is to pass to real coordinates by (1.1.1), whereby $\omega_1$, $\omega_2$ (1.5.7) take the form (1.0.16), and in expressing $\hat{P}^S_\sigma(\mu)$ by

$$\hat{P}^{(S)}_\sigma(\mu) = P^{(S)}_\sigma(\mu)e^{iQ^{(S)}_\sigma(\mu)} + \theta_{S+1}$$

with real polynomials $P^{(S)}_\sigma(\mu)$, $Q^{(S)}_\sigma(\mu)$ in $\omega_1, \omega_2$ of degree $[S/2]$, satisfying (1.0.17). This is clearly possible perhaps by suitable restriction of $U$ to $U' \subset U$.

### 1.7. *Remark about uniqueness*

The local group of $S$-normal-forms is not commutative, the normal forms are therefore not unique. It is easily seen, that if $T$ is a diffeomorphism and $\phi_1$ and $\phi_2$ two $S$-normal-forms with $T \circ \phi_1 \circ T^{-1} = \phi_2$, then $T$ must also be a $S$-normal-form.

## 2. The Qualitative Behaviour of the Truncated Normal Forms

If in the normal form (1.0.15) we omit $\theta_{S+1}$ we obtain the truncated diffeomorphism $\hat{\phi}$. This truncated diffeomorphism decomposes naturally into an isometric part and a dissipative part. The dissipative part defines in many cases invariant and attractive submanifolds. We are interested in these submanifolds and shall prove in section 3 that they persist under a $\theta_{S+1}$-perturbation of the truncated diffeomorphism. The restriction of $\phi$ to an invariant manifold will not be discussed. It is presumably involved, since the corresponding restriction of $\hat{\phi}$ is not structurally stable.

### 2.1. *The special case* $S = 3$

Introducing polar coordinates $T^2 \times R^2_+ \to R^4 \backslash \{0\}$:

$$x_1 \pm ix_2 = r_1 e^{\pm i2\pi\varphi_1}$$

$$x_3 \pm ix_4 = r_2 e^{\pm i2\pi\varphi_2}, \tag{2.1.1}$$

$r_\sigma \in R_+$, $\varphi_\sigma$ (mod 1), we obtain, according to Theorem 1 for

$$\psi(\mu): (r_1, \varphi_1, r_2, \varphi_2) \mapsto (r'_1, \varphi'_1, r'_2, \varphi'_2)$$

$$r'_\sigma = \left(1 + \mu_\sigma + \sum_{\sigma'=1}^{2} a_{\sigma\sigma'}(\mu)r^2_{\sigma'}\right) r_\sigma + \theta_4$$

$$\varphi'_\sigma = \varphi_\sigma + \tilde{\alpha}_\sigma(\mu) + \sum_{\sigma'=1}^{2} d_{\sigma\sigma'}(\mu)r^2_{\sigma'} + \theta_3 \tag{2.1.2}$$

We merely look at the generic case $a_{\sigma\sigma}(0) \neq 0$, $\sigma = 1,2$. Until now we have never used the transformations (1.2.1). Now we use (1.2.1) with $\tau_\sigma \in C^2(I)$, however, in order to transform $a_{\sigma\sigma}(\mu)$ to the values $\pm 1$. This leaves us with the following 4 types $(\pm 1, \pm 1)$:

$$r_1' = (1 + \mu_1 \pm r_1^2 + ar_2^2)r_1 + \theta_4$$

$$r_2' = (1 + \mu_2 + br_1^2 \pm r_2^2)r_2 + \theta_4, \tag{2.1.3}$$

$\phi_\sigma'$ as above; $a,b \in C^2(I)$. Additionally, by means of a transformation $E_3(\mu)$ according to (1.2.3) we get for $a$ and $b$ in (2.1.3):

$$a(\mu) = \hat{a}(\mu_1)$$

$$b(\mu) = \hat{b}(\mu_2). \tag{2.1.4}$$

An elementary discussion proves that in the normal form (2.1.3) these functions are invariants of the family of diffeomorphisms in question. In the following, we merely investigate families $(\phi(\mu))$ of the $(-1,-1)$ type. We begin with a discussion of the truncated part $\hat{\phi}(\mu)$ defined by

$$r_1' = (1 + \mu_1 - r_1^2 + ar_2^2)r_1$$

$$r_2' = (1 + \mu_2 + br_2^2 - r_2^2)r_2 \tag{2.1.5}$$

$\hat{\phi}(\mu)$:

$$\varphi_\sigma' = \varphi_\sigma + \tilde{\alpha}_\sigma(\mu) + \sum_{\sigma'=1}^{2} d_{\sigma\sigma'}(\mu) r_{\sigma'}^2$$

## 2.2. *Hopf bifurcation*

Let $\mu_1 > 0$, then $\{r_1 = \sqrt{\mu_1}, r_2 = 0\}$ defines a manifold $S_\mu$ invariant under $\hat{\phi}(\mu)$. We introduce the submanifold coordinates for $S_\mu$:

$$r_1^2 = \mu_1 + x_1, \quad |x_1| < \mu_1$$

$$x_2 \pm ix_3 = r_2 e^{\pm i2\pi\varphi_2}. \tag{2.2.1}$$

Then (2.1.5) defines a diffeomorphism $S^1 \times U_\mu \to S^1 \times R^3$; $U_\mu \subset R^3$ is an open neighbourhood of $0 \in R^3$. Let $\pi_1$ and $\pi_2$ be the projections from $S^1 \times R^3$ onto $S^1$ and $R^3$. Using the notation

$$\pi_1 \circ \phi(\mu) \equiv f_1 : S^1 \times U_\mu \to S^1$$

$$\pi_2 \circ \phi(\mu) \equiv f_2 : S^1 \times U_\mu \to R^3, \tag{2.2.2}$$

there follows for (2.1.5):

$$f_1(y,x) = T_1(\mu)(y) + \theta(x)$$

$$f_2(y,x) = T_2(\mu)x + \theta_2(x), \tag{2.2.3}$$

$T_1(\mu)$ being the translation

$$y \mapsto y + \tilde{\alpha}_1(\mu) + \mu_1 d_{11}(\mu), \tag{2.2.4}$$

and $T_2(\mu) \in \mathscr{L}(R^3)$ given by

$$T_2(\mu) = \begin{pmatrix} 1-2\mu_1 & 0 & 0 \\ 0 & \rho\cos\sigma & -\rho\sin\sigma \\ 0 & \rho\sin\sigma & \rho\cos\sigma \end{pmatrix}, \tag{2.2.5}$$

$\rho \equiv (1+\mu_2+b\mu_1)$, $\sigma \equiv (\tilde{\alpha}_2(\mu)+\mu_1 d_{21}(\mu))$. The manifold $S_\mu$, invariant under (2.2.3), is the graph (i) $\subset S^1 \times U_\mu$ of the map

$$i: S^1 \to R^3, \quad i(y) = 0 \in R^3. \tag{2.2.6}$$

This circle is attractive under (2.2.3) if and only if the spectrum $\sigma(T_2(\mu))$ is contained in $\{z \in \mathbb{C} \mid |z| < 1\}$. This is the case according to (2.2.5) iff the following holds true:

i) $\mu_1 > 0$,

ii) $\mu_2 + b\mu_1 < 0$. (2.2.7)

Two cases can be distinguished:

I. $b > 0$. In this case $\mu_2 < 0$ and $\mu_1 < b^{-1}|\mu_2|$ follows because of ii). This case corresponds to the Hopf bifurcation.

II. $b < 0$. In this case, one also gets an attractive circle for $\mu_2 \geqslant 0$, if only

$$\mu_2 < |b|\mu_1. \tag{2.2.8}$$

Analogously for $\mu_2 > 0\{r_1^2 = 0, r_2 = \sqrt{\mu_2}\}$ defines another circle $S_\mu^{(2)}$ invariant under $\hat{\phi}(\mu)$ and attractive iff

iii) $\mu_2 > 0$

iv) $\mu_1 + a\mu_2 < 0$. (2.2.9)

If $a < 0$, $b < 0$ and $ab > 1$, there exist two attractive circles $S_\mu^{(1)}, S_\mu^{(2)}$ for $\mu \in G \subset R_+^2$:

$$G = \{\mu \mid 0 < \mu_1 < |a|\mu_2 < ab\mu_1\}. \tag{2.2.10}$$

Hence we are left with the following representation for the attractive circles of $\hat{\phi}(\mu)$ in the $(\mu_1, \mu_2)$-plane:

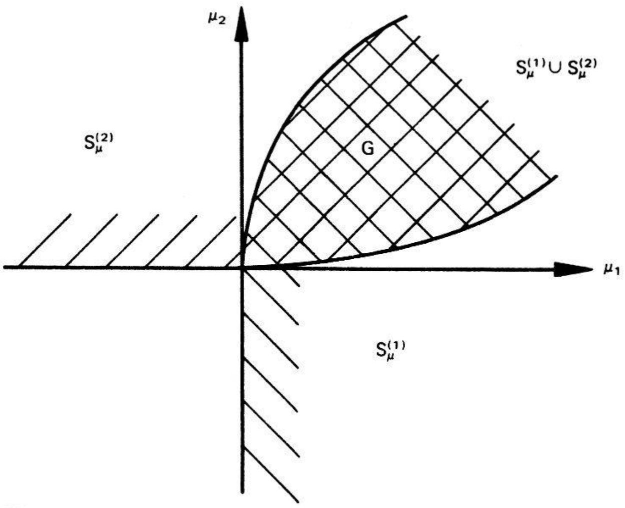


**Figure 1**
**Appearance of attractive circles in the $(\mu_1, \mu_2)$-plane.**

The fixed point of $\phi(\mu)$ is attractive for $\mu_\sigma < 0$, elliptic for $\mu_\sigma = 0$ and expansive for $\mu_\sigma > 0$, $\sigma \in \{1,2\}$.

## 2.3. *Explosion into a 2-dim. torus*

Let $\hat{\phi}(\mu): T^2 \times R^2_+ \to T^2 \times R^2_+$ be according to (2.1.5). The torus $T^2 \times (\mathring{r}_1, \mathring{r}_2)$, $\mathring{r}_i \in R_+$ is invariant under $\hat{\phi}(\mu)$, if

$$\begin{aligned} \mathring{r}_1^2 - a\mathring{r}_2^2 &= \mu_1 \\ -b\mathring{r}_1^2 + \mathring{r}_2^2 &= \mu_2. \end{aligned} \tag{2.3.1}$$

Define $A(\mu) \in \mathscr{L}(R^2)$

$$A(\mu) = \begin{pmatrix} 1 & -a \\ -b & 1 \end{pmatrix}. \tag{2.3.2}$$

We restrict $A$ by

i) $A(\mu)$ non singular

ii) $A(\mu)^{-1}: K_+ \to K_+, K_+ = R^2_+$ (2.3.3)

or equivalent to i) and ii)

iii) $a \geqslant 0, \quad b \geqslant 0, \quad ab < 1.$ (2.3.4)

Then, for all $\beta \in K_+$, the torus

$$\hat{T}^2(\beta) = T^2 \times (\beta) \subset T^2 \times \mathbb{R}^2_+, \tag{2.3.5}$$

$$\beta = A(\mu)^{-1}\mu, \tag{2.3.6}$$

is invariant under $\hat{\phi}(\mu)$. The mapping $\mu \mapsto \beta$, defined by (2.3.6), is a $C^2$-diffeomorphism in an open neighbourhood of $\mu = 0$. Therefore, the family $(\phi(\mu))$ can be parametrised by $\beta$. In the submanifoldcoordinates:

$$\begin{aligned} r_1^2 &= \beta_1 + x_1, \quad |x_1| < \beta_1 \\ r_2^2 &= \beta_2 + x_2, \quad |x_2| < \beta_2 \end{aligned} \tag{2.3.7}$$

the diffeomorphism $\hat{\phi}(\beta): T^2 \times U_\beta \to T^2 \times R^2$, $U_\beta \subset R^2$ an open neighbourhood of $0 \in R^2$, is given by $\hat{\phi}(\beta) = (\hat{f}_1, \hat{f}_2)$:

$$\begin{aligned} \hat{f}_1(y,x) &= T_1(\beta)(y) + \theta(x) \\ \hat{f}_2(y,x) &= T_2(\beta)x + \theta_2(x). \end{aligned} \tag{2.3.8}$$

$T_1(\beta): T^2 \to T^2$ being the translation

$$y_\sigma \mapsto y_\sigma + \tilde{\alpha}_\sigma(\beta) + \sum_{\sigma'=1}^{2} d_{\sigma\sigma'}(\beta) \cdot \beta_\sigma, \tag{2.3.9}$$

and $T_2(\beta) \in \mathscr{L}(R^2)$ given by

$$T_2(\beta) = \begin{pmatrix} 1 - 2\beta_1 & 2\beta_1 a \\ 2\beta_2 b & 1 - 2\beta_2 \end{pmatrix}. \tag{2.3.10}$$

$$T^2 \to T^2 \times (0) \tag{2.3.11}$$

is the embedding of the invariant torus $\hat{T}^2_\beta$ (2.3.5). These tori $\hat{T}^2_\beta$ are attractive for $\hat{\phi}(\beta)$ iff $\sigma(T_2(\beta)) \subset \{z \in \mathbb{C} \mid |z| < 1\}$. From (2.3.10) we get

$$\sigma(T_2(\beta)) = (\lambda_1, \lambda_2),$$
$$\lambda_{1,2} = 1 - (\beta_1 + \beta_2) \pm \sqrt{(\beta_1 - \beta_2)^2 + 4\beta_1\beta_2 ab}. \tag{2.3.12}$$

Hence $\hat{T}^2_\beta$ is attractive, iff

$$ab < 1 \tag{2.3.13}$$
$$(\beta_1 + \beta_2) < 2 - \sqrt{(\beta_1 - \beta_2)^2 + 4\beta_1\beta_2 ab}$$

Under the assumption (2.3.4) we have therefore the following representation for the attractive manifolds of $\hat{\phi}(\mu)$ in the $(\mu_1, \mu_2)$-plane:

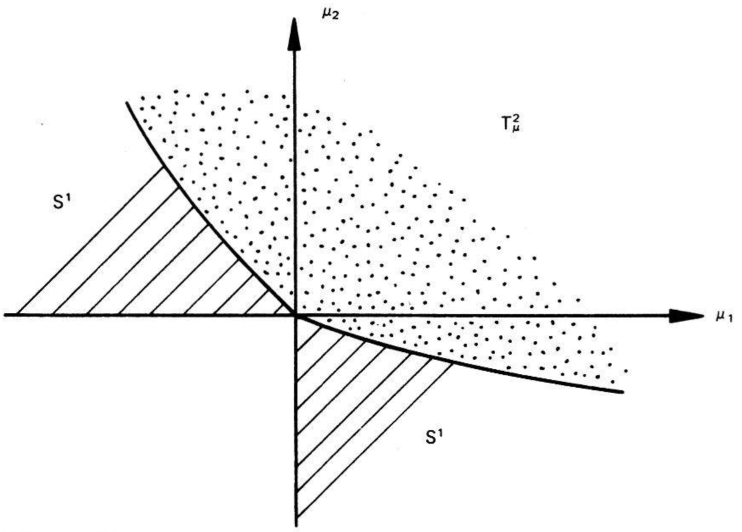


Figure 2
Appearance of an attractive circle of an attractive torus in the $(\mu_1, \mu_2)$-plane.

The attractive circles result from Hopf-bifurcation (cf. 2.2).

## 3. Perturbation of the Truncated Transformation

### 3.1. *Statement of Theorem 2 and 3*

Theorem 2. *Let* $\phi \in C^K(U \times I)$ *satisfy the hypotheses of theorem* 1 *with* $S = 3$. *Let in addition* $\phi$ *be of the* $(-1,-1)$*-type* (2.1.3) *and satisfy* (2.1.4). *We assume*

$$a(0) > 0, \quad b(0) > 0, \quad a(0) \cdot b(0) < 1. \tag{3.1.1}$$

*The following statements now hold:*

*There exists an open set* $V \subset I$ *containing the set* $\{\mu \mid \mu \in R_+ \times R_+, \theta < |\mu| < \rho\}$ *for some* $\rho > 0$, *such that for each* $\mu \in V$ *the diffeomorphism* $\phi(\mu)$ *has an invariant* $C^{K-4}$*-embedded torus* $T^2(\mu)$. *This torus* $T^2(\mu)$ *is attractive; i.e. there exists an open neighbourhood* $U_\mu \supset T^2(\mu)$ *such that for* $p \in U_\mu$

$$\phi(\mu)^k(p) \to T^2(\mu), \quad k \to \infty. \tag{3.1.2}$$

*In the coordinates* $T^2 \times R^2$ (2.3.7) *these tori are given by*

$$T^2(\mu) = \text{graph}\,(s_\mu), \tag{3.1.3}$$
$$s_\mu : T^2 \to R^2, \quad s_\mu \in C^{K-4}(T^2, R^2).$$

Theorem 3. *Let $\phi(\mu)$ satisfy the hypotheses of theorem 1 with $S = 3$ and be of the $(-1,-1)$-type and satisfy (2.1.4). Let a be arbitrary and restrict $b(0)$ by*

$$b(0) < 0. \tag{3.1.4}$$

*Then there exists a set $V_{\epsilon_1\epsilon_2} \subset I$, given by*

$$V_{\epsilon_1\epsilon_2} = \{\mu|\ |\mu_2| < \epsilon_1,\quad 0 < \mu_1 < \epsilon_2,\quad \mu_2 < |b|\mu_1\}, \tag{3.1.5}$$

*such that for each $\mu \in V_{\epsilon_1\epsilon_2}$ the diffeomorphism $\phi(\mu)$ has an attractive invariant $C^{K-4}$-embedded circle.*

An analogous statement holds true for $b$ arbitrary and $a(0) < 0$. If then in addition (3.1.4) holds and if $a(0)\,b(0) > 1$, then the diffeomorphism $\phi(\mu)$ has two attractive invariant circles for $\mu$ in a certain region $G$ as indicated in figure 1. The idea of the proof is the following. We study the diffeomorphism $\phi(\mu)$ in a neighbourhood of the invariant attractive manifolds of the normal form $\hat{\phi}(\mu)$ and consider $\phi(\mu)$ a perturbation of $\hat{\phi}(\mu)$. The perturbed invariant manifold is then constructed by means of a transversal vector field of the unperturbed invariant manifold using the contracting map principle.

### 3.2. *Preliminaries to the proof of Theorem 2*

In what follows we will use $\beta$ (2.3.6) instead of $\mu$ for parameters of the diffeomorphisms. We use the coordinates (2.3.7) and write $\phi(\beta) = (\pi_1 \circ \phi(\beta),\ \pi_2 \circ \phi(\beta) \equiv (f_1,f_2): T^2 \times U_\beta \to T^2 \times R^2$. According to theorem 1 ($S = 3$):

$$f_1(y,x) = T_1(y) + \theta(|x|) + (|\beta| + x_1 + x_2)^{3/2}\phi_{1\beta}(y,x) \tag{3.2.1}$$

$$f_2(y,x) = T_2(x) + \theta_2(|x|) + (|\beta| + x_1 + x_2)^{5/2}\phi_{2\beta}(y,x),$$

$$|\beta| = \beta_1 + \beta_2, \tag{3.2.2}$$

$y$ (mod 1); $T_1(\beta)$ and $T_2(\beta)$ are defined by (2.3.10) and (2.3.11). $\phi_1$ and $\phi_2$ are $C^{K-4}$. (3.2.1) interpreted as a mapping $R^2 \times U_\beta \to R^2 \times R^2$ satisfies

$$f_{1i}(y_j + 1, x) = f_{1i}(y_j, x) + \delta_{ij} \tag{3.2.3}$$

$$f_2(y_i + 1, x) = f_2(y, x). \tag{3.2.4}$$

In $R^2$ we use the following norm

$$|x| = \max(|x_1|, |x_2|),\quad x = (x_1, x_2) \in R^2. \tag{3.2.5}$$

*Remark:* In the norm $|\cdot|_\beta$ defined by

$$|x|_\beta = \max(\sqrt{b}|x_1|, \sqrt{a}|x_2|) \tag{3.2.6}$$

we have for $T_2(\beta) \in \mathscr{L}(R^2)$ according to (2.3.11)

$$\|T_2(\beta)\|_\beta \equiv \sup_{|x|_\beta = 1} |T_2(\beta)x|_\beta$$

$$= \max(1 - 2\beta_1[1 - \kappa], 1 - 2\beta_2[1 - \kappa]), \tag{3.2.7}$$

$\kappa^2 = ab < 1$. Hence in this norm $T_2(\beta)$ is contractive for $0 < \beta_i < \frac{1}{2}$. Instead of using the norm (3.2.7) we can introduce new coordinates

$$x_1 = \sqrt{a}\,\xi_1,\quad x_2 = \sqrt{b}\,\xi_2, \tag{3.2.8}$$

and get

$$\|\hat{T}_2(\beta)\| \equiv \sup_{|\xi|=1} |\hat{T}_2(\beta)\xi| = \|T_2(\beta)\|_\beta. \tag{3.2.9}$$

From now on we refer to these coordinates, without however changing the original notation of (3.2.1).

Let $\epsilon \in (0,\frac{1}{2})$ and define $K_\epsilon \subset R^2_+$

$$K_\epsilon = \{\beta | \beta_1 \geqslant \epsilon\beta_2, \beta_2 \geqslant \epsilon\beta_1, \beta_i \in (0,\tfrac{1}{2})\}. \tag{3.2.10}$$

For $\delta \in (0,1)$ define $U'_\delta \subset T^2 \times R^2 \times R^2_+$

$$U'_\delta = T^2 \times \{(x,\beta) | |x| \leqslant |\beta|\delta, \beta \in K_\epsilon\}. \tag{3.2.11}$$

We consider $\phi(\beta)$ in the neighbourhood $U'_\delta \cap (\beta)$ of $T^2(\beta)$, the invariant torus for $\hat{\phi}(\beta)$. With $L(f)$ we denote the Lipschitz constant of a Lipschitz map $f: M \to N$ between metric spaces. If $g$ is a function of two variables we denote by $g_x$ the function $y \mapsto g(x,y)$.

Lemma 1. *$\phi(\beta)$ satisfies the following propositions in $U'_\delta \cap (\beta)$:*

i) $|f_1(y_1,x_1) - f_1(y_2,x_2)| \leqslant (1 + |\beta|^{3/2}C_1)|y_1 - y_2| + C_2|x_1 - x_2|$.

ii) $|f_2(y_1,x_1) - f_2(y_2,x_2)| \leqslant L(f_{2x})|y_1 - y_2| + L(f_{2y})|x_1 - x_2|$,

$L(f_{2x}) \leqslant |\beta|^{5/2}\cdot C_3$

$L(f_{2y}) \leqslant \|T_2(\beta)\| + |\beta|\delta\cdot C_4$.

iii) $|f_2(y,0)| \leqslant |\beta|^{5/2}C_5$.

iv) $L(f_{2y} - T_2(\beta)) \leqslant |\beta|\delta C_4$.

*Proof:* The statements are consequences of (3.2.1), (3.2.11) and the mean value theorem. ■

According to (3.2.8) and (3.2.10) we have for $\beta \in K_\epsilon$

$$\|T_2(\beta)\| \leqslant (1 - |\beta|\gamma) < 1, \tag{3.2.14}$$

$$\gamma = \epsilon \inf_\beta (1 - \kappa) > 0.$$

Lemma 2. *There exist $\sigma_1 > 0$ and $\delta > 0$, such that*

$$\phi(U_\delta) \subset U_\delta,$$

$$U_\delta = U'_\delta \cap \{\beta | |\beta| < \sigma_1\}.$$

*Proof:* From Lemma 1

$$|f_2(y,x)| \leqslant |x|(\|T_2(\beta)\| + |\beta|\delta C_4) + |\beta|^{5/2}C_5,$$

hence with (3.2.14)

$$\leqslant |\beta|\delta(1 - |\beta|[\gamma - \delta C_4 - |\beta|^{1/2}\delta^{-1}C_5]).$$

Choose first

$$\delta < \gamma C_4^{-1} \tag{3.2.16}$$

and afterwards

$$|\beta|^{1/2} < \tfrac{1}{2}\delta(\gamma - \delta C_4)C_5^{-1} \equiv \sigma_1^{1/2}.$$

We then have for $|\beta| < \sigma_1$

$$\max_{|x| \leqslant |\beta|\delta} |f_{2\beta}(y, x)| \leqslant |\beta|\delta(1 - |\beta|C_6) \tag{3.2.17}$$

■

### 3.3. *The graph transform*

To prove Theorem 2 and 3 we use the methods of M. W. Hirsch and C. C. Pugh [1]. Accordingly we prove first the existence of a Lipschitzian manifold, which is invariant under $\phi(\beta)$. In a further step the differentiability is proved with the help of the fiber contraction theorem.

Let $B$ be the Banach space of the continuous functions $h: T^2 \to \mathbb{R}^2$

$$B = \{h | h \in C_0(R^2, R^2); \quad h \circ \phi_z = h, \quad z \in \mathbb{Z}^2\} \tag{3.3.1}$$

whereby $z \mapsto \varphi_z \in \mathrm{Diff}(R^2)$ is the action of $\mathbb{Z}^2$: $\varphi_z(y) = (y_1 + z_1, y_2 + z_2)$.

We write also $B = C^0(T^2, R^2)$. The Banach spaces $C^K(T^2, R^2)$ are to be understood correspondingly.

We shall now define a map $\Gamma_{\phi(\beta)}: B \to B$, the so called graph transform of $\phi(\beta)$, for which the following holds true:

$$h = \Gamma_{\phi(\beta)}(g) \Rightarrow \phi(\beta)(\mathrm{graph}(g)) = \mathrm{graph}(h). \tag{3.3.2}$$

Because of

$$\phi(\beta)(\mathrm{graph}(g)) = \{(\pi_1 \circ \phi(\beta) \circ (1, g)(y), \pi_2 \circ \phi(\beta) \circ (1, g)(y)) | y \in T^2\}, \tag{3.3.3}$$

we have:

$$h: \pi_1 \circ \phi(\beta) \circ (1, g)(y) \mapsto \pi_2 \circ \phi(\beta) \circ (1, g)(y) \tag{3.3.4}$$

or

$$\Gamma_{\phi(\beta)}(g) \circ f_1 \circ (1, g) = f_2 \circ (1, g). \tag{3.3.5}$$

If $f_1 \circ (1, g): T^2 \to T^2$ is bijective this means

$$\Gamma_{\phi(\beta)}(g) = f_2 \circ (1, g) \circ [f_1 \circ (1, g)]^{-1}. \tag{3.3.6}$$

(3.3.6) is the defining formula for $\Gamma_{\phi(\beta)}$.

Let $\beta$ and $\delta$ be according to Lemma 2, we then define the closed subset $B_\beta$ of Lipschitz functions

$$B_\beta = \{g \in B | \, |g| \leqslant |\beta|\delta, \; L(g) \leqslant |\beta|^{5/4}\}. \tag{3.3.7}$$

Lemma 3. *There exists $\sigma_2 < \sigma_1$, such that we have for $|\beta| < \sigma_2$*

i) $B_\beta \subset D(\Gamma_{\phi(\beta)})$.

ii) $\Gamma_{\phi(\beta)}(B_\beta) \subset B_\beta$.

*Proof:* Define

$$\psi_\beta(g) = f_1 \circ (1, g): T^2 \to T^2 \tag{3.3.8}$$

$$\varphi_\beta(g) = f_2 \circ (1, g): T^2 \to R^2. \tag{3.3.9}$$

Let $g \in B_\beta$, then (3.2.1) implies

$$L(\psi_\beta(g) - T_1) \leqslant |\beta|^{5/4} C_7. \tag{3.3.10}$$

Since $L(T_1^{-1})^{-1} = 1$, we have for $|\beta|^{5/4} < (2C_7)^{-1}$

$$L(\psi_\beta(g) - T_1) < L(T_1^{-1})^{-1}. \tag{3.3.11}$$

Thus $\psi_\beta(g)$ is injective, and

$$L(\psi_\beta(g)^{-1}) \leqslant (L(T_1^{-1})^{-1} - L(\psi_\beta(g) - T_1))^{-1}. \tag{3.3.12}$$

Because $T_1$ is a Lipschitz homöomorphism, it follows from (3.3.11) that $\psi_\beta(g)$ is also a Lipschitz homöomorphism [1], and $\Gamma_{\phi(\beta)}(g)$ is therefore well defined. We still have to show:

$$L(\Gamma_{\phi(\beta)}(g)) \leqslant |\beta|^{5/4}. \tag{3.3.13}$$

From

$$\Gamma_{\phi(\beta)}(g) = \varphi_\beta(g) \circ \psi_\beta(g)^{-1} \tag{3.3.14}$$

we have

$$L(\Gamma_{\phi(\beta)}(g)) \leqslant L(\varphi_\beta(g)) \cdot L(\psi_\beta(g)^{-1}). \tag{3.3.15}$$

From (3.3.10) and (3.3.12) we have:

$$L(\psi_\beta(g)^{-1}) \leqslant (1 - |\beta|^{5/4} C_7)^{-1}. \tag{3.3.16}$$

Lemma 1 ii implies for $g \in B_\beta$

$$L(\varphi_\beta(g)) \leqslant |\beta|^{5/4} (\|T_2(\beta)\| + |\beta|\delta C_4 + |\beta|^{5/4} C_3). \tag{3.3.17}$$

There exists, therefore, according to (3.3.15–17), $\sigma_2 > 0$, such that

$$L(\Gamma_{\phi(\beta)}(g)) \leqslant |\beta|^{5/4} (1 - |\beta| C_8), \tag{3.3.18}$$

if $|\beta| \leqslant \sigma_2$. The result follows. ■

Lemma 4. *There exists a $\sigma_3 < \sigma_2$, such that the graph transform $\Gamma_{\phi(\beta)}: B_\beta \to B_\beta$ is a contraction for $|\beta| < \sigma_3$:*

$$L(\Gamma_{\phi(\beta)}) = \lambda_\beta \leqslant (1 - |\beta| C_9) < 1.$$

*Proof:* Take $g_1, g_2 \in B_\beta$ and estimate

$$\begin{aligned}
|\Gamma_{\phi(\beta)}(g_1) - \Gamma_{\phi(\beta)}(g_2)| &\leqslant |\varphi_\beta(g_1) \circ \psi_\beta(g_1)^{-1} - \varphi_\beta(g_1) \circ \psi_\beta(g_2)^{-1}| \\
&+ |\varphi_\beta(g_1) \circ \psi_\beta(g_2)^{-1} - \varphi_\beta(g_2) \circ \psi_\beta(g_2)^{-1}| \\
&\leqslant L(\varphi_\beta(g_1)) |\psi_\beta(g_1)^{-1} - \psi_\beta(g_2)^{-1}| + |\varphi_\beta(g_1) - \varphi_\beta(g_2)| \\
&\leqslant L(\varphi_\beta(g_1)) \cdot L(\psi_\beta(g_1)^{-1}) |\psi_\beta(g_1) - \psi_\beta(g_2)| + |\varphi_\beta(g_1) - \varphi_\beta(g_2)| \\
&\leqslant (L(\varphi_\beta(g_1)) \cdot L(\psi_\beta(g_1)^{-1}) \cdot L(\psi_\beta) + L(\varphi_\beta)) |g_1 - g_2|.
\end{aligned}$$

Therefore, from Lemma 1, (3.3.17) and (3.3.16) the result follows. ■

Clearly $B_\beta$ is complete. Hence $\Gamma_{\phi(\beta)}$ has a unique fixed point $s_\beta \in B_\beta$:

$$\Gamma_{\phi(\beta)}(s_\beta) = s_\beta, \tag{3.3.20}$$

if $g \in B_\beta$, then

$$\lim_{n\to\infty} \Gamma^n_{\phi(\beta)}(g) = s_\beta. \tag{3.3.21}$$

This fixed point $s_\beta$ for the graph transform yields a Lipschitzian manifold $T^2(\beta) =$ graph $(s_\beta) \subset T^2 \times R^2$, which is invariant under $\phi(\beta)$:

$$\phi(\beta)(\text{graph}(s_\beta)) = \text{graph}(s_\beta), \tag{3.3.22}$$

and which is homöomorph to a torus $T^2$. By construction this torus $T^2(\beta)$ is attractive.

### 3.4. *The differentiability*

Next we investigate the differentiability of $s_\beta$ given by (3.3.20). Let

$$g_1 \in B_\beta \cap C^1(T^2, R^2).$$

Define

$$\Delta g_1 = (g_1, Dg_1) \in B_\beta \cap C^0(T^2, \mathscr{L}(R^2, R^2)). \tag{3.4.1}$$

If $g_1 = \Gamma_{\phi(\beta)}(g)$, we have

$$\Delta(\Gamma_{\phi(\beta)}(g)) = (\Gamma_{\phi(\beta)}(g), D\Gamma_{\phi(\beta)}(g)), \tag{3.4.2}$$

with

$$D\Gamma_{\phi(\beta)}(g)(y) = Df_2(\xi) \circ (1, Dg(\xi_1)) \circ [Df_1(\xi) \circ (1, Dg(\xi_1))]^{-1}, \tag{3.4.3}$$

$$\xi = (\xi_1, \xi_2) = (\xi_1, g(\xi_1)), \quad \xi_1 = \psi_\beta(g)^{-1}(y). \tag{3.4.4}$$

Let $\mathscr{H}^1$ be the Banach space

$$\mathscr{H}^1 = \{h \in C^0(T^2, \mathscr{L}(R^2, R^2)\} \tag{3.4.5}$$

with norm

$$|h| = \max_{y \in T^2} \|h(y)\|.$$

Define the closed subset $\mathscr{H}^1_\beta \subset \mathscr{H}^1$

$$\mathscr{H}^1_\beta = \{h \in \mathscr{H}^1 \mid |h| \leqslant |\beta|^{5/4}\}. \tag{3.4.6}$$

For $|\beta| < \sigma_2$ define the mapping

$$\begin{aligned} &\Gamma_{\Delta\phi(\beta)}: B_\beta \times \mathscr{H}^1_\beta \to B_\beta \times \mathscr{H}^1 \\ &(g, h) \mapsto (\Gamma_{\phi(\beta)}(g), \hat{h}), \\ &\hat{h}(y) = Df_2(\xi) \circ (1, h(\xi_1)) \circ [Df_1(\xi) \circ (1, h(\xi_1))]^{-1}, \end{aligned} \tag{3.4.7}$$

with $\xi$ and $\xi_1$ according to (3.4.4). $\hat{h}$ is well defined. Observe that

$$\Delta(\Gamma_{\phi(\beta)}(g) = \Gamma_{\Delta\phi(\beta)}(\Delta g), \tag{3.4.8}$$

for

$$g \in B_\beta \cap C^1(T^2, \mathscr{L}(R^2, R^2)).$$

Lemma 5. *There exists $\sigma_4 > 0$, such that for the map*

$$\Gamma_{\Delta\phi(\beta)}(g, \cdot): \mathscr{H}^1_\beta \to \mathscr{H}^1 \quad h \mapsto \pi_2 \circ \Gamma_{\phi(\beta)}(g, h), \tag{3.4.9}$$

$g \in B_\beta$ *with* $|\beta| < \sigma_4$, *the following holds:*

i) $\Gamma_{\Delta\phi(\beta)}(g, \cdot)(\mathscr{H}^1_\beta) \subset \mathscr{H}^1_\beta.$ (3.4.10)

ii) $L(\Gamma_{\Delta\phi(\beta)}(g, \cdot)) \leqslant \lambda_\beta < 1.$ (3.4.11)

*Proof:* It corresponds to the proofs of Lemma 3 and 4, but is simpler. ■

The following Lemma is an extension of the contracting map principle. The proof is elementary (cf. [1]).

Lemma 6. *Fiber contraction theorem (topological).*

*Let $X$ be a space and $f: X \to X$ a map having an attractive fixed point $p \in X$ ($\lim_{n\to\infty} f^n(x) = p$ for all $x \in X$). Let $Y$ be a metric space and $(g_x)\, x \in X$ a family of maps such that the formula $F(x,y) = (f(x), g_x(y))$ defines a continuous map $F: X \times Y \to X \times Y$. Let $q \in Y$ be a fixed point for $g_p$. Then $(p,q) \in X \times Y$ is an attractive fixed point for $F$ provided*

$$L(g_x) \leqslant \lambda < 1$$

*for all $x \in X$.*

Lemma 7. *Let* $|\beta| < \sigma_4$, $s_\beta$ *according to* (3.3.18), *then*
$s_\beta \in C^1(T^2, R^2)$.

*Proof:* Because of Lemma 6 and 7 there exists a unique fixed point

$(s_\beta, h_\beta) \in B_\beta \times \mathscr{H}^1_\beta$ for $\Gamma_{\Delta\phi(\beta)}$ and

$$\lim_{n\to\infty} (\Gamma_{\Delta\phi(\beta)})^n(g, h) = (s_\beta, h_\beta), \tag{3.4.12}$$

for all $(g,h) \in B_\beta \times \mathscr{H}^1_\beta$. Choose $(g,h) = (0,0) = \Delta g$, and there follows from (3.4.12)

$$\lim_{n\to\infty} (\Gamma_{\Delta\phi(\beta)})^n(\Delta g) = (s_\beta, h_\beta). \tag{3.4.13}$$

But

$$(\Gamma_{\Delta\phi(\beta)})^n(\Delta g) = \Delta(\Gamma^n_{\phi(\beta)}(g)), \tag{3.4.14}$$

because of (3.4.8), such that $Ds_\beta$ exists and equals $h_\beta$:

$$(s_\beta, h_\beta) = (s_\beta, Ds_\beta) = \Delta s_\beta. \tag{3.4.15}$$

■

Let $\hat{g} \in B_\beta \cap C^n(T^2, \mathbb{R}^2)$. Put

$$\Delta^n \hat{g} \equiv (\hat{g}, j^n\hat{g}) = (\hat{g}, D\hat{g}, \ldots, D^n\hat{g}), \quad D^i\hat{g} \in C^0(T^2, \mathscr{L}^i_s(R^2, R^2)). \tag{3.4.16}$$

Let

$$\Delta^n(\Gamma_{\phi(\beta)}(g)) = (\Gamma_{\phi(\beta)}(g), \epsilon(j^n g, j^n f_1, j^n f_2)). \tag{3.4.17}$$

Define the Banach space $\mathscr{H}^n$

$$\mathscr{H}^n = \mathscr{H}^1_\beta \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_n, \tag{3.4.18}$$

$$\mathscr{H}_i \in C^0(T^2, \mathscr{L}^i_s(R^2, R^2)), \tag{3.4.19}$$

and define the map $\Gamma_{\Delta^n\phi(\beta)} : B_\beta \times \mathscr{H}^n \to B_\beta \times \mathscr{H}^n$:

$$\Gamma_{\Delta^n\phi(\beta)}(g, h) = (\Gamma_{\phi(\beta)}(g), \epsilon(h, j^n f_1, j^n f_2)). \tag{3.4.20}$$

Considering the definitions we have

$$\Gamma_{\Delta^n\phi(\beta)}(\Delta^n g) = \Delta^n(\Gamma_{\phi(\beta)}(g)). \tag{3.4.21}$$

As can be easily verified by induction, we have

$$D^n(\Gamma_{\phi(\beta)}(g))(y) = (Df_2(\xi) + D(\Gamma_{\phi(\beta)}(g))(y) \circ Df_1(\xi))(0, D^n g(\xi_1)[\beta(y), \ldots, \beta(y)]) + \epsilon'(j^{n-1} g, j^n f_1, j^n f_2), \tag{3.4.22}$$

whereby $\xi$, $\xi_1$, $y$ according to (3.4.4) and

$$\beta(y) = [Df_1(\xi) \circ (1, Dg(\xi_1))]^{-1}. \tag{3.4.23}$$

If we define the map $A^n_\beta(h_{n-1}) : \mathscr{H}_n \to \mathscr{H}_n$:

$$h \mapsto \pi_n \epsilon(h_{n-1}, h, j^n f_1, j^n f_2), \tag{3.4.24}$$

$$h_{n-1} \in \mathscr{H}^1_\beta \times \cdots \times \mathscr{H}_{n-1},$$

there follows from (3.4.22)

$$|A^n_\beta(h_{n-1})(h) - A^n_\beta(h_{n-1})(\hat{h})| \leqslant A \cdot |\beta(y)|^n |h - \hat{h}|, \tag{3.4.25}$$

whereby

$$A \leqslant (1 - |\beta| C_9) \tag{3.4.26}$$

$$|\beta(y)|^n \leqslant (1 - |\beta|^{5/4} C_7)^{-n}. \tag{3.4.27}$$

There exists, therefore, for all $n$, $1 \leqslant n \leqslant K$, a $\tau_n > 0$, $\tau_i < \tau_j$ for $i > j$, such that for $|\beta| < \tau_n$

$$L(A^n_\beta(h_{n-1})) \leqslant \hat{\lambda}_\beta < 1, \tag{3.4.28}$$

$h_{n-1} \in \mathscr{H}^1_\beta \times \cdots \times \mathscr{H}_{n-1}$. Using again Lemma 6 and (3.4.28) we reach the following result by induction:

Lemma 8. *There exists* $(\tau_n)$, $1 \leqslant n \leqslant K - 4$, $0 < \tau_i < \tau_j$ *for* $i > j$, *such that*

$$s_\beta \in C^n(T^2, R^2),$$

*if* $|\beta| < \tau_n$.

From Lemma 8 and (3.3.22) Theorem 2 follows.

*Remark:* Let $\phi \in C^\infty$, then we have from (3.4.28) $\tau_n \to 0$, $n \to \infty$. Even for small $|\beta|$ we cannot expect the tori $T^2(\beta)$ to be $C^\infty$.

The proof of Theorem 3 follows the lines of the proof of Theorem 2, starting from the corresponding truncated normal form in $S^1 \times U_\mu \to S^1 \times U_\mu$. To prove the contraction of the graph transform in question, we use the inequality

$$\|T_2(\mu)\| < (1 - \epsilon_1 \mu_1) < L(T_1) = 1$$

where $T_1(\mu)$, $T_2(\mu)$ are defined by (2.2.4) and (2.2.5).

For the generalisation of the Hopf bifurcation for dimension $2n > 4$ we only have to discuss the matrix $(a_{ij}(\mu))$ at $\mu = 0$ in the dissipative part of the related truncated normal form $\hat{\phi}$

$$r_i \mapsto \left(1 + \mu_i + \sum_{j=1}^{n} a_{ij}(\mu) r_i^2\right) r_i$$

$1 \leqslant i \leqslant n$. We do not wish to carry out this discussion. However, one can get under generic assumptions bifurcation of attractive tori $T^m(\mu)$, $1 \leqslant m \leqslant n$. The restrictions of $\hat{\phi}$ onto these invariant tori are translations. As these translations are not structurally stable, we cannot from here gain any insight into the qualitative behaviour of the restrictions of $\phi(\mu)$ onto their invariant tori. For $n \geqslant 3$ so called strange attractors on $T^3(\mu)$ might be expected [2].

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