Zeitschrift: Helvetica Physica Acta
Band: 45 (1972)
Heft: 4

Artikel: Scattering systems with finite total cross-section

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DOI: https://doi.org/10.5169/seals-114397

Helvetica Physica Acta
Vol. 45, 1972. Birkhäuser Verlag Basel

Scattering Systems with Finite Total Cross-Section

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Abstract. We give a sufficient condition for the finiteness of the total cross-section in a simple scattering system. Various corollaries of our main theorem are also discussed.

1. Introduction

In the scattering theory developed by Jauch [1] the point of departure is a physically motivated asymptotic condition from which the existence and properties of the scattering operator can be derived.

However, it has so far not been possible to prove that the total scattering cross-section, or equivalently that the forward scattering amplitude is finite. The best known physically important system where this is not the case is the Coulomb system. This system does not satisfy the asymptotic condition of Ref. [1]. On the other hand there is no system which satisfies the asymptotic condition but has infinite total cross-section. It seems therefore natural to conjecture that a finite total cross-section is a consequence of the asymptotic condition.

We have not been able to establish such a theorem. In this paper we prove a theorem which assures a finite total cross-section with a slightly stronger hypothesis than the asymptotic condition. This condition is satisfied by any short-range potential for which the S-matrix exists and for a large class of more general interactions.

The total cross-section for a spherically symmetrical spin-less system is expressed in terms of the phase shifts δ_l and it is proportional to the sum $\sum_l (2l+1) \sin^2 \delta_l$ extended over all the different angular momentum states $l = 0, 1, 2 \ldots$.

The finiteness of this sum is assured if the operator $T_\lambda = S_\lambda - I_\lambda$ which is the operator $T = S - I$ taken on the energy shell λ, is a trace-class operator, since the above-mentioned sum is simply the imaginary part of the trace of T_λ. Thus we shall need a theorem which assures that T_λ is a trace-class operator. The content of this paper is the establishment and proof of such a theorem.

Closely related results of this nature were first obtained by Krein, Birman and Entina [3, 4]. These important results seem to have been overlooked by many workers in this field partly no doubt because there seems to be no complete proof of their results in the literature that was available to us. The theorem that we prove here is slightly more general than their results but with less detailed conclusions. The method of proof is based on a lemma which we have proved in a previous publication with Misra [2].

2. Notations and Statement of the Problem

Throughout this paper we shall be dealing with a simple scattering system in the sense of Jauch [1]. It is defined by two self-adjoint operators H and $\mathring{H}$ representing respectively the total and the free Hamiltonian for the particle. The associated unitary one-parameter groups are denoted by

$$V_t = e^{-iHt} \quad \text{and} \quad U_t = e^{-i\mathring{H}t}, \quad (-\infty < t < \infty).$$

They satisfy the asymptotic condition

$$\underset{t\to\pm\infty}{s\text{-limit}}\, V_t^* U_t = \Omega_\mp.$$

The operators $\Omega_\pm$ are isometries with identical range and the unitary operator

$$S = \Omega_-^* \Omega_+$$

is the scattering operator.

We denote by

$$R_z = (H - Z)^{-1} \quad \text{and} \quad \mathring{R}_z = (\mathring{H} - z)^{-1}$$

the resolvents. They are bounded operators for z in the respective resolvent sets $z \in \rho(H)$ and $z \in \rho(\mathring{H})$. These sets include all z with $\operatorname{Im} z \neq 0$.

The problem that we propose to solve is to find a convenient sufficient condition on the operators H and $\mathring{H}$ such that the operator $T_\lambda = S_\lambda - I_\lambda$ is of trace-class.

These S_λ-operators are defined as follows: Since S commutes with $\mathring{H}$ we may represent S in the spectral representation of $\mathring{H}$ as an operator-valued function on the spectrum Λ of $\mathring{H}$. We denote by $\lambda \in \Lambda$ a point on the spectrum and write for S in this representation

$$S = \{S_\lambda\}$$

where S_λ are unitary operators in a Hilbert space $\mathscr{H}_\lambda$ for almost all λ.

The remaining parts of this paper are based on the following

Theorem 0. *Let $\mathring{H}$ be self-adjoint with absolutely continuous spectrum Λ. Let Γ be any trace-class operator in the Hilbert space $\mathscr{H}$. Then there exists a dense set $\mathscr{D} \subseteq \mathscr{H}$ such that for all $f,g \in \mathscr{D}$*

$$G[f,g] = \int_{-\infty}^{+\infty} (U_t f, \Gamma U_t g)\, dt$$

exists and defines a sesquilinear form on $\mathscr{D} \times \mathscr{D}$ (i.e. linear in g and antilinear in f).

If $f_\lambda, g_\lambda \in \mathscr{H}_\lambda$ are the components of f and g respectively in the direct integral representation of $\mathscr{H}$, then

$$G[f,g] = \int_\Lambda (f_\lambda, G_\lambda g_\lambda)_\lambda\, d\lambda$$

where G_λ is an essentially unique family of trace-class operators in $\mathscr{H}_\lambda$ for almost all λ and

$$\int_\Lambda \mathrm{Tr}_\lambda G_\lambda \, d\lambda = 2\pi \, \mathrm{Tr}\, \Gamma,$$

$$\int_\Lambda \|G_\lambda\|_{1,\lambda} \, d\lambda \leqslant 2\pi \|\Gamma\|_1$$

In the last three expressions we have written $(\cdot,\cdot)_\lambda$ for the scalar product, Tr_λ for the trace in the Hilbert space $\mathscr{H}_\lambda$, and $\|\cdot\|_{1,\lambda}$ for trace norm in $\mathscr{H}_\lambda$. This theorem is proved in Ref. [2] to which we refer the reader for details.

3. The Main Theorem

Throughout the rest of this paper we shall assume that V, defined by $H = \mathring{H} + V$, is a symmetric operator with domain $\mathscr{D}(V) \supseteq \mathscr{D}(\mathring{H}) = \mathscr{D}(H)$ where we have denoted by $\mathscr{D}(A)$ the domain of definition of the operator A.

Theorem 1. *Let $R_z^n \, V \, R_z^m$ belong to the trace-class operators in $\mathscr{H}$, where n and m are integers $\geqslant 0$. Then*

S_λ is of the form, $S_\lambda = I_\lambda + T_\lambda$

where I_λ is the identity operator in $\mathscr{H}_\lambda$ and T_λ is trace-class in $\mathscr{H}_\lambda$.

Proof. According to the definition of S we can write

$$\begin{aligned} S - I &= \Omega_-^* (\Omega_+ - \Omega_-) \\ &= \Omega_-^* \left[\text{s-}\lim_{t\to-\infty} V_t^* U_t - \text{s-}\lim_{t\to+\infty} V_t^* U_t \right] \\ &= -\Omega_-^* \int_{-\infty}^{+\infty} \frac{d}{dt} V_t^* U_t \, dt, \end{aligned}$$

the integral being taken in the strong Bochner sense.

It follows that

$$\begin{aligned} S - I &= -i \int_{-\infty}^{+\infty} \Omega_-^* V_t^* V U_t \, dt \\ &= -i \int_{-\infty}^{+\infty} U_t^* \Omega_-^* V U_t \, dt. \end{aligned}$$

In the last step we have used the interwining relation $V_t \Omega = \Omega U_t$. Since $[S, \mathring{H}] = 0$ and $R_z \Omega = \Omega \mathring{R}_z$ we obtain

$$\begin{aligned} \mathring{R}_z^n (S - I) \mathring{R}_z^m &= \mathring{R}_z^{n+m} (S - I) \\ &= -i \int_{-\infty}^{+\infty} U_t^* [\Omega_-^* R_z^n V \mathring{R}_z^m] U_t \, dt. \end{aligned}$$

Denoting by Γ the trace-class operator $-i[\Omega_-^* R_z^n V \mathring{R}_z^m]$, we conclude that

$$\mathring{R}_z^{n+m}(S-I) = \int_{-\infty}^{+\infty} U_t^* \Gamma U_t \, dt.$$

Applying theorem 0 to the above identity we obtain for all pairs $f,g \in \mathscr{H}$

$$(\mathring{R}_z^{n+m}(S-I)f,g) = \int_{\Lambda} \frac{(f_\lambda, T_\lambda g_\lambda)_\lambda}{(\lambda - z)^{n+m}} d\lambda = \int_{-\infty}^{+\infty} (U_t f, \Gamma U_t g) \, dt$$

where T_λ is an essentially unique family of trace-class operators in $\mathscr{H}_\lambda$ for almost all $\lambda \in \Lambda$.

Furthermore

$$\int_{\Lambda} \frac{\mathrm{Tr}_\lambda T_\lambda}{(\lambda - z)^{n+m}} d\lambda = 2\pi \,\mathrm{Tr}\, \Gamma = -2\pi i \,\mathrm{Tr}\,(\Omega_-^* R_z^n V \mathring{R}_z^m)$$

and

$$\int_{\Lambda} \frac{\|T_\lambda\|_{1,\lambda}}{|\lambda - z|^{n+m}} d\lambda \leqslant 2\pi \|\Gamma\|_1 < \infty.$$

4. Applications of the Main Theorem

Corollary 0. *Let $V \in \mathscr{L}_1(\mathscr{H})$ (=Trace-class operators in $\mathscr{H}$), then $T_\lambda \equiv S_\lambda - I_\lambda \in \mathscr{L}_1(\mathscr{H}_\lambda)$ for almost all λ. This follows from Theorem 1 by choosing $m = n = 0$.*

Corollary 1. *Let $D_z = R_z - \mathring{R}_z \in \mathscr{L}_1(\mathscr{H})$ for some $z \in \rho(H) \cap \rho(\mathring{H})$, then $T_\lambda \equiv S_\lambda - I_\lambda \in \mathscr{L}_1(\mathscr{H}_\lambda)$ for almost all λ.*

Proof. Choose $n = m = 1$ and use the identity

$$D_z = R_z - \mathring{R}_z = -R_z V \mathring{R}_z$$

and Theorem 1 gives the result.

Note that from the hypothesis follows also that $D_z \in \mathscr{L}_1(\mathscr{H})$ for all $z \in \rho(H) \cap \rho(\mathring{H})$.

Corollary 2. *Let $|V|^{1/2} R_z^n \in \mathscr{L}_2(\mathscr{H})$ and $|V|^{1/2} \mathring{R}_z^n \in \mathscr{L}_2(\mathscr{H})$ for some $n > 0$ then $T_\lambda \in \mathscr{L}_1(\mathscr{H}_\lambda)$ for almost all λ.*

We have used the notation $\mathscr{L}_2(\mathscr{H})$ for the Hilbert-Schmidt operators in $\mathscr{H}$ and $|V|^{1/2} = (V^* V)^{1/4}$.

Proof. We use the decomposition of Birman and Entina [3] valid for any $f \in \mathscr{D}(V) \supseteq \mathscr{D}(\mathring{\mathscr{H}})$:

$$Vf = |V|^{1/2} U |V|^{1/2} f,$$

where U is a certain self-adjoint contraction operator.

$(|V|^{1/2} R_z^n)^*$ is the closure of the densely defined bounded operator $R_{\bar{z}}^n |V|^{1/2}$ and since

$$|V|^{1/2} R_{\bar{z}}^n = |V|^{1/2} R_z^n - (z - \bar{z}) |V|^{1/2} R_z^n R_{\bar{z}} \in \mathscr{L}_2(\mathscr{H})$$

we find that $R_z^n |V|^{1/2} \in \mathscr{L}_2(\mathscr{H})$.

It follows that

$$R_{\bar{z}}^n V \mathring{R}_z^n = (R_{\bar{z}}^n |V|^{1/2}) U(|V|^{1/2} \mathring{R}_z^n)$$

and the result follows from theorem 1 with $n = m$.

Corollary 3. *Let* $R_z^n - \mathring{R}_z^n \in \mathscr{L}_1(\mathscr{H})$ *for some* $z \in \rho(H) \cap \rho(\mathring{H})$, *then* $T_\lambda \in \mathscr{L}_1(\mathscr{H}_\lambda)$ *for almost all* λ.

Proof. By repeated use of the resolvent identity we find

$$\begin{aligned} R_z^n - \mathring{R}_z^n &= R_z^{n-1}(R_z - \mathring{R}_z^n) + R_z^{n-2}(R_z - \mathring{R}_z)\mathring{R}_z + \cdots + (R_z - \mathring{R}_z)R_z^{n-1} \\ &= -[R_z^n V \mathring{R}_z + R_z^{n-1} V \mathring{R}_z^2 + \cdots + R_z V \mathring{R}_z^n]. \end{aligned}$$

Now

$$(\mathring{R}_z^{n+1}[S - I]f, g) = -i \int_{-\infty}^{+\infty} (U_t f, [\Omega_-^* R_z^{n-k} V \mathring{R}_z^{k+1}] U_t g)\, dt$$

for $k = 0, 1, \ldots, n-1$.

Therefore

$$n(\mathring{R}_z^{n+1}[S - I]f, g) = -i \int_{-\infty}^{+\infty} (U_t f, \{\Omega_-^* [R_z^n V \mathring{R}_z + R_z^{n+1} V \mathring{R}_z^2 + \cdots + R_z V \mathring{R}_z^n]\} U_t g)\, dt$$

$$= +i \int_{-\infty}^{+\infty} (U_t f, [\Omega_-^* (R_z^n - \mathring{R}_z^n)] U_t g)\, dt$$

and the conclusion follows from Theorem 1.

Remarks. Hypotheses of Corollary 0 and 2 were used by Birman and Entina [3] to obtain the above results, and hypotheses of Corollaries 1 and 3 in a slightly different form were used by Krein and Birman [4]. In both papers the authors had more detailed conclusions. We obtain a slightly weaker result with an elementary method of proof. Recently Kuroda [5] has obtained some general results in this direction.

If we now further assume spherical symmetry in the problem, then S_λ becomes reducible in the space $L^2(S_2)$, the space of square-integrable functions on the unit-sphere in 3-dimensions. In fact S_λ is given by $\{\mathscr{S}_l(\lambda) = e^{zi\delta_l(\lambda)}\}_{l=0}^{\infty}$ where $\delta_l(\lambda)$ is a real, measurable function of λ for $l = 0, 1, 2, \ldots$. Then the trace-class nature of the operator $T_\lambda = S_\lambda - I_\lambda$ is assured by the absolute convergence of the infinite series:

$$\sum_{l=0}^{\infty} (2l + 1)(e^{2i\delta_l(\lambda)} - 1) \quad \text{and} \quad \|S_\lambda - I_\lambda\|_{1,\lambda} = \sum_{l=0}^{\infty} (2l + 1)\,|e^{2i\delta_l(\lambda)} - 1|.$$

Recalling partial wave analysis of scattering problems, the scattering amplitude $f(\theta, \lambda)$ and the total scattering cross-section $\sigma_{tot}(\lambda)$ are given as

$$f(\theta,\lambda) = \frac{1}{2i\sqrt{\lambda}} \sum_l (2l+1)\,(e^{2i\delta_l} - 1)\, P_l\,(\cos\theta)$$

and

$$\sigma_{tot}\,(\lambda) = \frac{4\pi}{\lambda} \sum_l (2l+1)\,\sin^2\delta_l\,(2).$$

Since δ_l's are all real,

$$\sigma_{tot}\,(\lambda) \leqslant \frac{4\pi}{\lambda} \sum_l (2l+1)\,|\sin\delta_l|$$

$$= \frac{2\pi}{\lambda} \|S_\lambda - I_\lambda\|_{1,\lambda} < \infty, \quad \text{for a.a.}\lambda.$$

Similarly, using $|P_l(\cos\theta)| \leqslant 1$, we obtain

$$|f(\theta,\lambda)| \leqslant \frac{1}{2\sqrt{\lambda}} \|S_\lambda - I_\lambda\|_{1,\lambda} < \infty, \quad \text{for a.a.}\lambda.$$

Thus we can conclude that the above-mentioned conditions on the potential are sufficient to insure the finiteness of the scattering amplitude and total cross-section for a.a.λ. We can easily see that a weaker condition, viz. $S_\lambda - I_\lambda \in \mathscr{L}_2(\mathscr{H}_\lambda)$, the Hilbert-Schmidt class of operators, will suffice for this purpose. In fact, we have

$$\sigma_{tot}\,(\lambda) = \frac{\pi}{\lambda} \|S_\lambda - I_\lambda\|_{2,\lambda}^2$$

We now recall the integral condition, viz.

$$\int_0^\infty \frac{\|S_\lambda - I_\lambda\|_{1,\lambda}}{\lambda^2+1}\, d\lambda < \infty,$$

where we have assumed

$$\Lambda \equiv \text{sp}(\mathring{H}) = [0,\infty)$$

and $n = m = 1$. Then

$$\int_0^\infty \frac{\lambda^{1/2}|f(\theta,\lambda)|}{\lambda^2+1}\, d\lambda < \infty$$

and

$$\int_0^\infty \frac{\lambda\sigma_{tot}\,(\lambda)}{\lambda^2+1}\, d\lambda < \infty.$$

Since the function $\lambda/\lambda^2 + 1$ is not integrable at infinity, it follows essentially from the definition of Lebesgue integral that either of the two following possibilities must occur.

Either $\sigma_{tot}(\lambda)$ does not go to any limit at all (i.e. to any positive real number including ∞) as $\lambda \to \infty$. Or, $\sigma_{tot}(\lambda)$ converges to zero as $\lambda \to \infty$.

Loosely speaking, the total cross-section either oscillates rapidly or decays to zero as energy gets larger and larger. For short-range potentials, one expects physically the second kind of behaviour for the total cross-section. See, for example, Newton [6].

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