On a covariant expression of energy-momentum in the relativistic theory of gravitation

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On a Covariant Expression of Energy-Momentum in the Relativistic Theory of Gravitation

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Abstract. The relativistic linear formalism developed by W. Scherrer leads to a covariant expression of gravitational energy-momentum. In the case of a mass distribution with spherical symmetry (Schwarzschild's *exterior* solution), this tensor gives a total gravitational energy equal to $-Mc^2$. How can the minus sign be interpreted? A detailed analysis of Schwarzschild's *interior* solution shows that the gravitational energy has to be considered as a generalization of the classical potential energy, which is *negative* in our attractive case.

1. Introduction

The linear formalism (LT) developed and applied to General Relativity by Scherrer since 1954, leads to a covariant expression of energy-momentum [1]. The field equations of this theory in fact allow us to set conservation theorems of the form

$$\frac{\partial \mathfrak{T}_{\lambda,}{}^{,\mu}}{\partial x^{\mu}} = 0 \quad (\lambda, \mu = 0, 1, 2, 3), \tag{1.1}$$

where $\mathfrak{T}_{\lambda,}{}^{,\mu} = g_{(+1)} T_{\lambda,}{}^{,\mu}, g_{(+1)} = \det(g^{\lambda,}{}_{,\mu})$ (for the definition of the notations employed in this work, the reader should refer to Scherrer's original paper). Recall to mind that the indices placed *before* the comma correspond to the Lorentz transformations, and the indices placed *after* the comma correspond to the general coordinate transformations. On the other hand, the $g^{\lambda,}{}_{,\mu}$ of the LT are connected with the $g_{\mu\nu}$ of the quadratic formalism (QT) by the relations

$$g_{(\mu\nu)} = e_{\alpha\beta} g^{\alpha,}{}_{,\mu} g^{\beta,}{}_{,\nu} \tag{1.2}$$

where the $e_{\alpha\beta}$ are the Eisenhart symbols: $e_{00} = 1$, $e_{ii} = -1$, $e_{\alpha\neq\beta} = 0$. In opposition to what happens in the QT, the $T_{\lambda,}{}^{,\mu}$ form a 'true' tensor, which can be identified to the energy-momentum tensor of the system. The tensorial nature of the $T_{\lambda,}{}^{,\mu}$ allows us in particular to localize the energy-momentum, which is impossible in the QT.

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Applying this formalism to the case of a mass with spherical symmetry, Scherrer showed that the gravitational energy of the system is equal to Mc^2 [2]. In the case of the plane gravitational waves (PGW), however, we obtain an energy density apparently negative definite [2], [3], as well as a momentum density opposed to the propagation direction of the waves. In a later paper, we shall show quite generally that the tensor $T_{\lambda,}{}^{,\mu}$ must in fact be multiplied by

$$-\frac{1}{\kappa_E}, \; \kappa_E = \frac{8\pi\kappa_N}{c^4},$$

κ_N being the Newtonian gravitation constant. The energy and momentum of the PGW then have the correct sign.

In return, the formula $E = Mc^2$, obtained by Scherrer from Schwarzschild's exterior solution, becomes now $E = -Mc^2$. The purpose of the present work is to show, with the help of a 'concrete' example, that, far from being surprising, the minus sign which appears in this formula is on the contrary necessary to the self-consistence of the formalism. To show this, we shall consider Schwarzschild's interior solution.

2. Schwarzschild's Interior Solution in the LT

Scherrer recently showed [4] that, for a sphere of perfect incompressible fluid with constant density ρ, the LT provides for the same line element as the QT, i.e. in polar coordinates, for the potentials different from the Euclidian $g_{\mu\nu}$

$$g_{00} = [f(r)]^2, \quad g_{11} = -[h(r)]^2, \tag{2.1}$$

$$f(r) = A - \sqrt{1 - \left(\frac{r}{R}\right)^2}, \quad h(r) = \frac{1}{\sqrt{1 - \left(\frac{r}{R}\right)^2}},$$

where

$$A = \frac{3}{2}\sqrt{1 - \left(\frac{R_\odot}{R}\right)^2}, \quad R^2 = \frac{3}{\kappa_E \rho c^2},$$

$R_\odot$ = coordinate radius of the sphere. On the other hand, we find for the pressure

$$p = \frac{1}{\kappa_E R^2} \, \frac{\frac{3}{2}\sqrt{1 - \left(\frac{r}{R}\right)^2} - A}{A - \frac{1}{2}\sqrt{1 - \left(\frac{r}{R}\right)^2}}. \tag{2.2}$$

3. Energy

The total energy of the system (sphere + gravitation field) decomposes as follows:

a) the 'material' energy E_m of the sphere, defined by the phenomenological tensor $\Theta^{,\alpha\beta}$ of the field equations;

b) the gravitational energy of the internal field E_{gi};

c) the gravitational energy of the external field E_{ge}.

We can thus write the total energy

$$E_{tot} = \int \left(\Theta_{0,}{}^{,0} - \frac{1}{\kappa_E} T_{0,}{}^{,0} \right) \underset{(+1)}{g} \, d^3 x \tag{3.1}$$

the integral extending to the whole three-dimensional space (we point out that the field shows no singularity in our case).

We give below the expressions of the various quantities included in the integral (3.1)

$$\Theta_{0,}{}^{,0} = g_{0,}{}^{,\alpha} \Theta_{,\alpha}{}^{,0} = \frac{1}{f} \rho c^2, \tag{3.2}$$

$$T_{0,}{}^{,0} = \frac{(h-1)^2}{r^2 f h^2} \quad \text{(Scherrer [5])}. \tag{3.3}$$

In the internal case, f and h are given by the expressions (2.1). In the external case, we have

$$f = \frac{1}{h} = \left[1 - \frac{2a}{r} \right]^{1/2} \tag{3.4}$$

where $2a$ represents the gravitation radius. On the other hand

$$\underset{(+1)}{g} = f h r^2 \sin\vartheta. \tag{3.5}$$

In introducing the expressions (3.2)–(3.5) into the integral (3.1), we finally obtain the total energy. According to the decomposition of E_{tot} mentioned in the beginning of 3, we have successively

$$E_m = \int_{\text{sphere}} \Theta_{0,}{}^{,0} \underset{(+1)}{g} \, d^3 x = \int_{\text{sphere}} \rho c^2 h r^2 \sin\vartheta \, dr \, d\vartheta \, d\phi$$

$$= \frac{6\pi R}{\kappa_E} \left[\arcsin \frac{R_\odot}{R} - \frac{R_\odot}{R} \sqrt{1 - \left(\frac{R_\odot}{R} \right)^2} \right], \tag{3.6a}$$

$$E_{gi} = -\frac{1}{\kappa_E} \int_{\text{sphere}} T_{0,}{}^{,0} \underset{(+1)}{g} \, d^3 x = -\frac{1}{\kappa_E} \int_{\text{sphere}} \frac{(h-1)^2}{h} \sin\vartheta \, dr \, d\vartheta \, d\phi$$

$$= -\frac{6\pi R}{\kappa_E} \left[\arcsin \frac{R_\odot}{R} - \frac{4}{3} \frac{R_\odot}{R} + \frac{R_\odot}{3R} \sqrt{1 - \left(\frac{R_\odot}{R} \right)^2} \right], \tag{3.6b}$$

$$E_{ge} = -\frac{1}{\kappa_E} \int_{r > R_\odot} T_{0,}{}^{,0} \underset{(+1)}{g} \, d^3 x = -\frac{16\pi a}{\kappa_E} \left[-\frac{1}{2} + \frac{1}{1 + \sqrt{1 - \dfrac{2a}{R_\odot}}} \right], \tag{3.6c}$$

where $R^2 = 3/\kappa_E \rho c^2$, $R_\odot$ = coordinate radius of the sphere, $2a = R_g$ = gravitation radius. Here we are not so much interested in the *exact* expression of the total energy, which we obtain simply by adding the results (3.6a–c), but in the total energy of the Newtonian case ($R_\odot/R \ll 1$). In developing the expressions (3.6a–c) in power series of $(R_\odot/R)$, we find

$$E_m \cong M_N c^2 + \frac{3}{5}\frac{\kappa_N M_N^2}{R_\odot}, \tag{3.7a}$$

$$E_{gi} \cong -\frac{1}{10}\frac{\kappa_N M_N^2}{R_\odot}, \tag{3.7b}$$

$$E_{ge} \cong -\frac{1}{2}\frac{\kappa_N M_N^2}{R_\odot} \quad \text{with} \quad M_N = \frac{4\pi R_\odot^3}{3}\rho. \tag{3.7c}$$

In this approximation, the total energy is sensibly equal to

$$E_{\text{tot}} \cong M_N c^2 \tag{3.8}$$

which is evidently a very satisfactory result. However we obtain a more interesting interpretation of the above calculations in introducing the relativistic mass $M_\odot$, defined by the relation

$$M_\odot = \int_{\text{sphere}} \rho \gamma \, d^3 x \tag{3.9}$$

$\gamma d^3 x$ being the Riemannian volume element of the sphere. By means of some elementary calculations, this definition allows us to put the total energy into the form

$$E_{\text{tot}} \cong M_\odot c^2 - \frac{3}{5}\frac{\kappa_N M_N^2}{R_\odot}\,[2] \tag{3.10}$$

The second term is just the classical potential energy of the system, which must be negative (attractive case).

Since this term is the contribution of gravitational energy, the discussed example shows clearly that the choice of the factor $-1/\kappa_E$ is perfectly correct. In the Newtonian approximation, this factor effectively leads to the correct sign of classical potential energy. In short we can say that, in the relativistic theory, the gravitational energy appears as a generalization of the classical potential energy. Under these circumstances, it is quite natural to obtain $-M_N c^2$ (instead of $+M_N c^2$) as total energy corresponding to the only exterior Schwarzschild's solution.

4. Another Approximation Method for the PGW

The question of the energy-momentum sign being thus resolved in a most general way, the problem of the PGW presents no more contradictions. It is interesting to compare the solution given by the LT to the corresponding results obtained within the QT.

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[2]) This result has already been obtained by Tolman in 1930 [6] with the help of quasi-Galilean coordinates. The originality of our method lies in its absolutely covariant character.

In trying to resolve the field equations in first approximation in a quasi-Lorentzian coordinate system, we write

$$g_{\mu\nu} \cong e_{\mu\nu} + h_{\mu\nu} \quad (|h_{\mu\nu}| \ll 1). \tag{4.1}$$

Introducing (4.1) into the $R_{\mu\nu}$

$$R_{\mu\nu} = \frac{\partial \Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} \tag{4.2}$$

we can see that $R_{\mu\nu}$ decomposes into a term of first order $r_{\mu\nu}$ (given by the $\partial\Gamma/\partial x$) and a term of second order $s_{\mu\nu}$ (from the $\Gamma\Gamma$). We shall similarly write $R = r + s$, so that the vacuum field equations become in this approximation

$$r_{\mu\nu} - \tfrac{1}{2}e_{\mu\nu} r = -s_{\mu\nu} + \tfrac{1}{2}e_{\mu\nu} s + \tfrac{1}{2}e_{\mu\nu} s. \tag{4.3}$$

The left members of these equations are simply the $S_{\mu\nu} \equiv (R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R)$ in the first approximation. The right members are of second order. In comparison with the general field equations

$$S_{\mu\nu} = -\kappa_E \Theta_{\mu\nu} \tag{4.4}$$

we identify these right members (divided by $-\kappa_E$) with the components $\Theta_{\mu\nu}$ of the gravitational energy-momentum tensor in the considered approximation.

On the other hand we shall eliminate the terms containing $\partial\Gamma/\partial x$ from $\Theta_{\mu\nu}$, because we wish to avoid that the energy-momentum depends on the second derivatives of the $g_{\mu\nu}$. Thus we can write

$$\Theta_{\mu\nu} = \frac{1}{\kappa_E}[s_{\mu\nu} - \tfrac{1}{2}e_{\mu\nu} s] \tag{4.5}$$

that is

$$\Theta_{\mu\nu} = \frac{1}{\kappa_E}[(\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma}) - \tfrac{1}{2}e_{\mu\nu}e^{\alpha\beta}(\Gamma^{\rho}_{\alpha\sigma}\Gamma^{\sigma}_{\beta\rho} - \Gamma^{\rho}_{\alpha\beta}\Gamma^{\sigma}_{\rho\sigma})]. \tag{4.6}$$

Applied to the case in which, for example, only the component g_{23} oscillates, the formulas (4.6) give, for the energy ($u_{\mathbf{grav}}$) and momentum density ($j^{1}_{\mathbf{grav}}$)

$$u_{\mathbf{grav}} = \frac{1}{2\kappa_E}h'^2, \quad j^{1}_{\mathbf{grav}} = -\frac{1}{2\kappa_E}h'\dot{h}, \tag{4.7}$$

where

$$g_{23} \cong h[k(x^{\circ} - x^{1})], \quad h' = \frac{dh}{dx^{1}}, \quad \dot{h} = \frac{dh}{dx^{\circ}} \cdot$$

These results agree with those of the LT and those of the QT obtained by other means.

5. Conclusion

The preceding considerations on the sign of gravitational energy-momentum allows us to eliminate immediately the difficulty about the gravitational waves mentioned in 1. The energy and momentum densities become both positive definite for waves propagating in the positive direction of the x^1-axis (for example). Scherrer's theory then presents no internal contradiction, and the 'true' energy-momentum tensor which it allows to be defined constitutes probably one of its most beautiful successes. Therefore this formalism represents a natural extension of Einstein's quadratic formalism.

BIBLIOGRAPHY

[1] W. SCHERRER, *Grundlagen zu einer linearen Feldtheorie*, Z. Phys. *138*, **16** (**1954**), and later papers.
[2] W. SCHERRER, *Bericht über eine einheitliche Feldtheorie*, Z. Phys. *174*, **351** (**1963**).
[3] J. CHEVALIER, Ph.D. Thesis (University of Bern, 1970).
[4] W. SCHERRER, *Der phänomenologische Energietensor in Rahmen der linearen Feldtheorie*, Helv. phys. Acta *44*, **530** (**1971**).
[5] W. SCHERRER, *Darstellung der Gravitationsenergie*, Helv. phys. Acta *37*, **317** (**1964**).
[6] TOLMAN, Phys. Rev. *35*, 875 (1930).