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On the Decay of an Unstable Particle[1])

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Abstract. It is shown that the absence of regeneration of the unstable particle from the decay products is inconsistent with a Hamiltonian bounded below. Consequences of some decay laws are also derived.

1. Introduction

In this paper, we study the problem of decay of an unstable particle in the usual quantum mechanical formalism, following the works of Williams [1] and Horwitz et al. [2]. There have been many investigations of the problem (Refs. [3]–[7]) prior to those of the above-mentioned authors, but they initiated a mathematically rigorous formulation of the problem. This study is essentially a continuation of their investigations. Here we do not attempt to give a physically acceptable detailed model of an 'unstable particle', but we start from an essentially model-independent set-up and derive some consequences a decaying system forces on the theory. For an interesting discussion of various models and their relative merits, the reader is referred to the above mentioned authors.

We shall start by assuming that an unstable particle can be described by vectors in a Hilbert space $\mathscr{H}_u$. We shall also assume that $\mathscr{H}_u$ can be embedded in a larger Hilbert space $\mathscr{H}$ such that $\mathscr{H} = \mathscr{H}_u \oplus \mathscr{H}_D$ where $\mathscr{H}_D$ is the Hilbert space of the decay products. The last assumption is natural to interpret the decay of unstable particle vectors in $\mathscr{H}_u$, as being due to a loss probability from $\mathscr{H}_u$ into the Hilbert space of decay products, $\mathscr{H}_D$.

Let $V(t)$ be the unitary evolution of the total system (i.e. unstable particle and the decay products), generated by a self-adjoint Hamiltonian H in $\mathscr{H}$. It is natural to assume the unitarity of the evolution operator because once we have included all the decay products in the system, the total system is isolated from the rest of the Universe as far as the decay phenomenon is concerned. Then the probability that a particle in the initial state $\Psi \in \mathscr{H}_u$ remain in the same state after time $t \geqslant 0$ is given by

$$p_\Psi(t) = |(\Psi, V(t)\Psi)|^2.$$

This quantity $p_\Psi(t)$ is expected to converge to zero as $t \to \infty$ for every $\Psi \in \mathscr{H}_u$, if it were to describe a decay. Exact decay law, i.e. the rate of convergence of $p_\Psi(t)$ to zero, has a striking influence on the theory, as we shall see in the sequel. Most experiments

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tend to show a very rapid rate of decay, viz. the exponential one. It is convenient to introduce an operator $Z(t)$, called contracted evolution defined for $t \geqslant 0$ as follows

$$Z(t) = PV(t)\,P, \quad \text{where} \quad P\mathscr{H} = \mathscr{H}_u.$$

$Z(t)$ is essentially a collection of all matrix elements of the type $(\Phi, V(t)\Psi)$; $\Phi, \Psi \in \mathscr{H}_u$. So it describes the decay law of any vector in $\mathscr{H}_u$ into other. As a natural generalization of exponential decay law for one dimension, one is tempted to think in terms of a semigroup law for $Z(t)$ for positive times. In fact, this is what was done in [1] and [2]. We shall discuss this more fully, give simpler derivation of similar results and also show that a relaxation of semigroup law for $Z(t)$ does not alter the conclusions.

2. Semigroup Law for $Z(t)$ and Regeneration

We assume here that $Z(t) = PV(t)\,P$ obeys a semigroup law for positive time, i.e.

$$Z(t_1)\,Z(t_2) = Z(t_1 + t_2); \quad t_1, t_2 \geqslant 0. \tag{1}$$

We write (1) more explicitly, viz.

$$PV(t_1)\,PV(t_2)\,P = PV(t_1 + t_2)\,P.$$

Denoting by $\bar{P}$, the projection onto the subspace orthogonal to $\mathscr{H}_u$, the subspace of decay products, we obtain

$$PV(t_1)\,\bar{P}V(t_2)\,P = PV(t_2)\,\bar{P}V(t_1)\,P = 0$$

for all $t_1, t_2 \geqslant 0$.

This equation means that there is no regeneration of the vectors in $\mathscr{H}_u$, in subsequent evolution from the vectors in $\mathscr{H}_D$. In other words, a semigroup law for the contracted evolution $Z(t)$ for positive times implies that the decay products at any positive time cannot regenerate the unstable particle. This point has been emphasized by Fonda and Ghirardi [8].

As a next step towards generalization, one can assume that regeneration is not absent for all positive times but rather it continues for an arbitrary finite time T_r, called the regeneration time. Mathematically, this is symbolized by

$$PV(t_1)\,\bar{P}V(t_2)\,P = 0$$

for all $t_2 \geqslant 0$ and for $t_1 > T_r > 0$. This implies what we term an 'approximate semigroup law' for the contracted evolution $Z(t)$, viz.

$$Z(t_1)\,Z(t_2) = Z(t_1 + t_2); \quad t_1 > T_r > 0, t_2 \geqslant 0. \tag{2}$$

Now we will study the consequences of (1) and (2). Case (1) has been studied in detail by the authors in [1] and [2]. We give simpler proofs and apply the same technique to the treatment of (2).

Theorem 2.1. *Assume* (1). *Then H, the generator of $V(t)$, has the whole real line as its spectrum.*

The proof is essentially that of Williams [1].

Proof. Under (1) we have already noted that

$$PV(t_1)\,\bar{P}V(t_2)\,P = 0 \quad \text{for all}\; t_1, t_2 \geqslant 0.$$

Let us consider the following expression

$$\begin{aligned}&(\bar{P}V(-t_1)P)^* V(t_2)(\bar{P}V(-t_1)P) \quad \text{with } t_1, t_2 \geqslant 0\\ &= PV(t_1)\,\bar{P}V(t_2)\,\bar{P}V(-t_1)\,P\\ &= PV(t_1)\,\bar{P}V(t_2)\,V(-t_1)\,P - PV(t_1)\,\bar{P}V(t_2)\,PV(-t_1)\,P,\end{aligned}$$

where we have used the fact that $\bar{P} = I - P$. The second term in the above is zero because of (1) and the expression reduces to

$$PV(t_1)\,\bar{P}V(t_2 - t_1)\,P = 0,$$

if we choose $t_2 > t_1 \geqslant 0$. Similarly,

$$\begin{aligned}&(\bar{P}V(-t_1)P)^* V(-t_2)(\bar{P}V(-t_1)P); \quad t_1, t_2 \geqslant 0\\ &= PV(t_1)\,\bar{P}V(-t_2)\,\bar{P}V(-t_1)\,P\\ &= PV(t_1)\,V(-t_2)\,\bar{P}V(-t_1)\,P - PV(t_1)\,PV(-t_2)\,\bar{P}V(-t_1)\,P\\ &= (PV(t_1)\,\bar{P}V(t_2 - t_1)P)^* - PV(t_1)\,(PV(t_1)\,\bar{P}V(t_2)P)^*\\ &= 0\end{aligned}$$

by the same choice $t_2 > t_1 \geqslant 0$. Thus for any vector $\Psi \in \mathscr{H}$,

$$(\bar{P}V(-t_1)\,P\Psi, V(t)\,\bar{P}V(-t_1)P\Psi) = 0 \quad \text{if } |t| > t_1 \geqslant 0.$$

In terms of the spectral family $\{E_\lambda\}$ of H, the generator of the unitary group $V(t)$, the above can be written as

$$\int e^{it\lambda}\, d\|E_\lambda\,\bar{P}V(-t_1)P\Psi\|^2 = 0 \quad \text{for } |t| > t_1 \geqslant 0.$$

Then, by Lemma 6 of Appendix, the spectral measure has the whole real line for its support and we have the desired result.

Remark. In the literature, there is a lot of confusion about the statement and application of Paley and Wiener's theorem [9]. It is worth mentioning that this theorem relates to the Fourier transform of L^2-functions and hence not immediately applicable to the problem at hand.

Theorem 2.2. *Assume (II). Then spectrum of H is the whole real line.*

Proof. Since in this case

$$PV(t_1)\,\bar{P}V(t_2)P = 0 \quad \text{for } t_2 \geqslant 0 \text{ and } t_1 > T_r > 0$$

we can follow the identical construction as in Theorem 2.1 and conclude that for any vector $\Psi \in \mathscr{H}$,

$$(\bar{P}V(-t_1)P\Psi, V(t)\bar{P}V(-t_1)P\Psi) = 0 \quad \text{if } |t| > t_1 > T_r > 0.$$

Hence similar argument as in previous theorem leads to the stated conclusion.

Thus we observe that any absence of regeneration even after a finite but arbitrary time leads to a Hamiltonian necessarily having unphysical spectrum, viz. it is not bounded below. In order to avoid this, one must allow regeneration to continue for all positive times, i.e. one must have

$$PV(t_1)\bar{P}V(t_2)P \neq 0 \quad \text{for all } t_1 \geqslant 0, t_2 \geqslant 0.$$

3. Sz. Nagy's Theorem on Extension and its Consequences

We state Sz. Nagy's theorem here without proof. For the proof, the reader is referred to Reference [10].

Theorem 3.1.

*Let G be a *semigroup and let $\{Z(g)\}$ be a family of bounded linear operators in $\mathscr{L}$, satisfying the following:*

(i) $Z(e) = I, \quad Z(g^*) = Z(g)^*$

(ii) $\sum_{i,j=1}^{n} (\Phi_i, Z(g_i g_j^*)\Phi_j) \geqslant 0$ *for finite set* $\{i\}$ *and* $\{j\}$

(iii) $\sum_{i,j} (\Phi_i, Z(g_i^* h^* h g_j)\Phi_j) \leqslant C_h \sum_{i,j} (\Phi_i, Z(g_i^* g_j)\Phi_i)$

with a constant $C_h > 0$, where the same set of $\{i\}$ and $\{j\}$ are chosen as in (ii).

Then $\exists$ a triple $\{\mathscr{K}, S(g), Q\}$ such that

$$Z(g) = QS(g)Q \; \forall g \in G$$

$$\mathscr{L} = Q\mathscr{K}$$

and the Hilbert space $\mathscr{K}$ is minimum in the sense that

$$\overline{\bigcup_{g \in G} S(g)\, Q\mathscr{K}} = \mathscr{K}$$

i.e. $\mathscr{L}$ is generating under the action of $S(g)$, the unitary representation of G in $\mathscr{K}$. Also the structure $\{\mathscr{K}, S(g), Q\}$ is unique up to an isomorphism of the structure.

It is easy to check that with $G \equiv \mathbb{R}^1$, the additive group on the reals; $Z(t) = PV(t)P$ in $\mathscr{L} = P\mathscr{H}$ for $t \geqslant 0$ and the additional identification that

$$Z(-t) = PV(-t)P = Z(t)^* \quad \text{for } t \geqslant 0,$$

$Z(t)$ satisfies all the hypotheses of Sz. Nagy's Theorem. Then there exists $\mathscr{K} \supseteq P\mathscr{H}$, a projector Q in $\mathscr{K}$ and a unitary representation of $\mathbb{R}^1$ in $\mathscr{K}$, $S(t)$ such that

$$Q\mathscr{K} = P\mathscr{H},$$

$$Z(t) = QS(t)Q$$

and $\mathscr{K}$ is minimal, i.e.

$$\mathscr{K} = \overline{\bigcup_{t \in \mathbb{R}^1} S(t)\, Q\mathscr{K}}$$

Notice that $\mathscr{H}$, a priori need not be minimal with respect to $V(t)$ and P. But then we can work with the structure $\{\mathscr{K}, S(t), Q\}$ which is by Sz. Nagy's theorem structurally isomorphic to all other minimal structures. Also, it seems that minimal structure is the only one which has any right to describe the process under consideration.

Otherwise, there will be vectors in the Hilbert space which are not obtainable from the unstable particle states by the given evolution and hence cannot describe decay product states either. Henceforth by $\{\mathscr{H}, V(t), P\}$ we shall mean the minimal structure implying

$$\overline{\bigcup_{t\in\mathbb{R}^1} V(t)\, P\mathscr{H}} = \mathscr{H}$$

4. Decay Laws and the Spectrum of H

The probability amplitude that the unstable particle with an initial state $\Phi \in P\mathscr{H}$ will be in a state Ψ after time $t \geqslant 0$ is given by $(\Psi, V(t)\Phi)$. In a decay process that we attempt to describe here, one expects this quantity or rather its modulus square $|(\Psi, V(t)\Phi)|^2$ to decay to zero as $t \to \infty$. This is achieved economically by asking for weak convergence of $Z(t)$ to zero as $t \to \infty$.

By a decay law, we mean finding a positive function $\phi(t)$ such that for every pair of vectors

$$|(\Psi, Z(t)\Phi)| = 0(\phi(t))$$

or, equivalently, $\left|\frac{(\Psi, Z(t)\Phi)}{\phi(t)}\right| \leqslant$ a constant, for large $t > 0$. In this definition, the function $\phi(t)$ can in general depend on the vectors Φ and Ψ.

Remark. From the definition $Z(-t) = Z(t)^*$ it is clear that there is complete symmetry between positive and negative times as far as the decay law is concerned.

Theorem 4.1. *Let $Z(t)$ converge weakly to zero as $t \to \infty$. Then the spectrum of H is continuous.*

Proof of this theorem has been given by Horwitz et al. [1]. We give a simpler proof using the theorem on means given in Appendix.

Proof.

$$(\Psi, Z(t)\Psi) = \int e^{it\lambda}\, d\|E_\lambda P\Psi\|^2$$

converges to zero as $t \to \infty$, then by Lemma 1 of the Appendix, the spectral measure $\|E_\lambda P\Psi\|^2$ is continuous and hence $P\mathscr{H} \subseteq \mathscr{H}_c$, where by $\mathscr{H}_c$, we denote the continuous subspace with respect to H, as in Kato [11]. Since the structure $\{\mathscr{H}, V(t), P\}$ is assumed to be minimal,

$$\mathscr{H} = \overline{\bigcup_{t\in\mathbb{R}^1} V(t)\, P\mathscr{H}} \subseteq \mathscr{H}_c \subseteq \mathscr{H}$$

and thus $\mathscr{H} = \mathscr{H}_c$, proving the theorem.

Theorem 4.2 *Let*

$$\int_{-\infty}^{\infty} |(\Psi, Z(t)\Psi)|\,dt < \infty$$

for every $\Psi \in \mathscr{H}$. *Then* $Z(t)$ *converges weakly to zero as* $t \to \infty$ *and the spectrum of* H *is absolutely continuous.*

Proof.

$$(\Psi, Z(t)\Psi) = \int e^{it\lambda}\, d\|E_\lambda P\Psi\|^2.$$

From Lemma 4 in the Appendix we conclude that $(\Psi, Z(t)\Psi)$ converges to zero as $t \to \infty$ for every $\Psi \in \mathscr{H}$ and hence by polarisation identity $Z(t)$ converges weakly. By the same Lemma, we conclude that the spectral measure $\|E_\lambda P\Psi\|^2$ is absolutely continuous and as before, using the minimality of the structure $\{\mathscr{H}, V(t), P\}$ we arrive at the result that $\mathscr{H} = \mathscr{H}_{\text{a.c.}}$, the absolutely continuous subspace with respect to H.

Corollary. *If the decay law is given by* $\phi(t) = 1/t^{1+\epsilon}$ *for* $\epsilon > 0$, *then the spectrum of* H *is absolutely continuous.*

Proof. The result follows immediately on application of the Corollary to Lemma 4 in the Appendix.

Theorem 4.3. *Let the decay law be exponential, viz.* $\phi(t) = e^{-\beta|t|}$; $\beta > 0$. *Then the spectrum of* H *is the whole real line and is absolutely continuous.*

Proof. Let Ψ be any vector in $\mathscr{H}$. Then

$$(\Psi, Z(t)\Psi) = \int e^{it\lambda}\, d\|E_\lambda P\Psi\|^2$$

and hence, by Lemma 5 in Appendix, the spectral measure $\|E_\lambda P\Psi\|^2$ must have whole real line as its support and is an absolutely continuous function. This means that $P\mathscr{H} \subseteq \mathscr{H}_{\text{a.c.}}$. Since the structure $\{\mathscr{H}, V(t), P\}$ is assumed to be minimal, as before

$$\mathscr{H} = \overline{\bigcup_{t \in \mathbb{R}^1} V(t)\, P\mathscr{H}} \subseteq \mathscr{H}_{\text{a.c.}} \subseteq \mathscr{H}$$

and thus $\mathscr{H} = \mathscr{H}_{\text{a.c.}}$, showing absolute continuity of the whole spectrum of H.

Appendix

Here we prove a few useful results regarding the Fourier transforms of a Stieltjes measure. Let $\sigma(\lambda)$ be a Stieltjes measure on R^1 $(-\infty < \lambda < \infty)$ and $f(t)$ be its Fourier transform, i.e.

$$f(t) = \int e^{it\lambda}\, d\sigma(\lambda); \quad -\infty < t < \infty.$$

It is clear that $f(t)$ is a bounded continuous function on $\mathbb{R}^1$. For the properties of such measures and their Fourier transforms, the reader is referred to Bochner [12].

We state two theorems on the inverse Fourier transform whose proofs can be found in Ref. [12].

Theorem 1. *Let $\sigma(\lambda)$ and $f(t)$ be as stated before, then*

$$\sigma(\lambda)-\sigma(0)=\lim_{\omega\to\infty}\frac{1}{2\pi}\int_{-\omega}^{\omega} f(t)\,\frac{e^{-it\lambda}-1}{-it}\,dt.$$

We define the mean value m of a continuous bounded function on $\mathbb{R}^1$ as follows:

$$m\{f(t)\}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)\,dt.$$

Note that for such functions the mean value always exists, i.e. $m\{f(t)\}<\infty$.

Theorem 2. *Let $f(t)$ be the Fourier transform of a Stieltjes measure $\sigma(\lambda)$. Then the following relation holds for all $\lambda(-\infty<\lambda<\infty)$:*

$$m\{f(t)\,e^{-it\lambda}\}=\sigma(\lambda+0)-\sigma(\lambda-0).$$

The right-hand side expression is sometimes called the 'jump' of the function σ at the point λ.

Now we prove a few lemmas which have been used in this investigation.

Lemma 1. *Let $f(t)$ converge to zero as $t\to\infty$. Then $\sigma(\lambda)$ is continuous in λ, i.e. it has no 'jumps'.*

Proof. Since $f(t)$ converges to zero as $t\to\infty$, so does $f(t)\,e^{-it\lambda}$ for all real λ, uniformly in λ; i.e.

$$|f(t)\,e^{-it\lambda}|=|f(t)|<\epsilon,\quad \text{when } |t|>T_0(\epsilon)$$

$$\left|\int_{-T}^{T} f(t)\,e^{-it\lambda}\,dt\right|\leqslant\int_{-T_0}^{T_0}|f(t)|\,dt+\int_{T_0}^{T}|f(t)|\,dt+\int_{-T}^{-T_0}|f(t)|\,dt$$

$$\leqslant 2C(T_0)\,T_0+2\epsilon(T-T_0).$$

Then

$$\left|\frac{1}{2T}\int_{-T}^{T} f(t)\,e^{it\lambda}\,dt\right|\leqslant C(T_0)\frac{T_0}{T}+\epsilon\left(1-\frac{T_0}{T}\right).$$

Letting $T\to\infty$ with ϵ fixed, we obtain

$$m\{f(t)\,e^{-it\lambda}\}\leqslant\epsilon.$$

But since ϵ was arbitrary to start with,

$$m\{f(t)\,e^{-it\lambda}\}=0\quad\text{for all }\lambda,$$

thus giving the required result by virtue of Theorem 2.

A weak converse of the above can be proven.

Lemma 2. *Let $\sigma(\lambda)$ be continuous in λ, then there exists at least one sequence $\{t_j\}$ tending to infinity such that $f(t_j)$ converges to zero.*

For the proof of this result, the reader is referred to Lax and Phillips [13].

The following Lemma is the Riemann-Lebesgue Lemma which we produce for the sake of completeness. The proof of this can be found in most text books, in particular (12).

Lemma 3. *Let $\sigma(\lambda)$ be an absolutely continuous function. Then $f(t)$ converges to zero as $|t| \to \infty$.*

Remark. For definition of absolutely continuous functions and its relation to the absolute continuity of the measure generated by it, the reader is referred to Rudin [14].

It is to be noted that the Riemann-Lebesgue Lemma does not give any idea of the rate of decay of $f(t)$ as $t \to \infty$. The next Lemma is essentially the converse of Riemann-Lebesgue Lemma, but we need some restriction on the decay rate.

Lemma 4. *Let $f(t)$ be the Fourier transform of a positive Stieltjes measure $\sigma(\lambda)$ and let $f(t)$ be absolutely integrable in $\mathbb{R}^1$. Then $\sigma(\lambda)$ is an absolutely continuous function and hence the Stieltjes measure associated to it is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^1$.*

Proof. By Theorem 1,

$$\sigma(\lambda) - \sigma(0) = \lim_{\omega \to \infty} \frac{1}{2\pi} \int_{-\omega}^{\omega} f(t) \frac{e^{it\lambda} - 1}{-it} \, dt.$$

Since $f(t)$ is continuous, bounded and also integrable in $\mathbb{R}^1$ and $(e^{-it\lambda} - 1)/(-it)$ is continuous, bounded everywhere, the limit exists as a Lebesgue integral, i.e.

$$\sigma(\lambda) - \sigma(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \frac{e^{-it\lambda} - 1}{-it} \, dt.$$

Now we compute the derivation of the continuous function $\sigma(\lambda)$.

$$\frac{\sigma(\lambda) - \sigma(\lambda')}{\lambda - \lambda'} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \frac{e^{-it\lambda} - e^{-it\lambda'}}{-it(\lambda - \lambda')} \, dt,$$

$$\left| \frac{\sigma(\lambda) - \sigma(\lambda')}{\lambda - \lambda'} - \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \, e^{-it\lambda} \, dt \right| \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(t)| \left| \frac{e^{-it(\lambda - \lambda')} - 1}{-it(\lambda - \lambda')} - 1 \right| dt.$$

The function

$$\frac{e^{-it(\lambda - \lambda')} - 1}{-it(\lambda - \lambda')} - 1$$

goes to zero for fixed t pointwise in λ as $\lambda' \to \lambda$ and also

$$\left|\frac{e^{-it(\lambda-\lambda')}-1}{-it(\lambda-\lambda')}-1\right| \leqslant \text{Constant} \quad \text{for} |\lambda-\lambda'| \text{ small.}$$

Hence by Lebesgue dominated convergence theorem, we have that the continuous Stieltjes function $\sigma(\lambda)$ is differentiable *everywhere* in $-\infty < \lambda < \infty$ and the derivative being,

$$\frac{d\sigma(\lambda)}{d\lambda} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-it\lambda}\, dt$$

the right-hand side making sense because $f(t) \in L^1$. It is easy to see that the real function $d\sigma(\lambda)/d\lambda$ is locally integrable and one verifies that

$$\sigma(\lambda)-\sigma(\rho) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,\frac{e^{-it\lambda}-e^{-it\rho}}{-dt}\, dt$$

$$= \int_{\rho}^{\lambda} \frac{d\sigma(\mu)}{d\mu}\, d\mu.$$

Hence by a theorem of Dieudonné [15], the Stieltjes (positive) measure μ_σ associated with the function σ by the rule $\mu_\sigma(a, b) = \sigma(b) - \sigma(a)$, is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^1$.

Remark. Since the measure generated by σ is finite, the derivative $d\sigma(\lambda)/d\lambda$ is not only locally integrable, but also integrable. Therefore, by Riemann-Lebesgue Lemma, it follows that $f(t) \to 0$ as $t \to \infty$. In other words, the above Lemma actually proves also the following: if a function $f(t)$ is of positive type in the sense of Bochner [12] and is absolutely integrable, then $f(t) \to 0$ as $t \to \infty$. Note that positivity of the measure is used though it is not essential. Since we have in mind the spectral measure of a self-adjoint operator in Hilbert space, this is enough for our applications.

Corollary. *Let $f(t)$ be a Fourier transform of a positive Stieltjes measure $\sigma(\lambda)$, and let*

$$f(t) = 0\left(\frac{1}{t^{1+\epsilon}}\right); \ \epsilon > 0.$$

Then σ is absolutely continuous.

Proof.

$$f(t) = 0\left(\frac{1}{t^{1+\epsilon}}\right)$$

implies that $f(t)$ is integrable in $\mathbb{R}^1$ and also clearly $f(t) \to 0$ as $t \to \infty$. By the above Lemma then the result follows.

Lemma 5. *Let* $f(t) = 0(e^{-\beta|t|})$. *Then* σ *is absolutely continuous and has whole* $\mathbb{R}^1$ *as its support.*

Proof. Since $f(t) = 0\ (e^{-\beta|t|})$ for $|t| \to \infty$, $f \in L^1$ and hence, by our previous result, σ is absolutely continuous. Using the inversion formula, viz.

$$\sigma(\lambda) - \sigma(0) = (2\pi)^{-1} \int_{-\infty}^{\infty} f(t) \frac{e^{-it\lambda} - 1}{-it} dt$$

we notice that this can be extended to a function of complex variable $z = \lambda + i\mu$ by the relation

$$\sigma(z) - \sigma(0) = (2\pi)^{-1} \int_{-\infty}^{\infty} f(t) \frac{e^{-itz} - 1}{-it} dt.$$

The above integral is well defined for z in the open strip $-\beta < \mu < \beta$ and defines a function $\sigma(z)$ analytic in the strip whose boundary value for $\mu \to 0$ is the original Stieltjes measure $\sigma(\lambda)$. Hence the support of $\sigma(\lambda)$ has to be the whole real line.

Lemma 6. *Let* $f(t) = 0$ *for* $|t| > B$. *Then* σ *is absolutely continuous and has whole* $\mathbb{R}^1$ *as its support.*

Proof. Defining $\sigma(z)$ as above, we conclude that $\sigma(z)$ is an entire function whose boundary value for $\mu \to 0$ is $\sigma(\lambda)$ and hence the result.

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