# Statistical description of elementary processes. II, Quantum field theory

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# Statistical Description of Elementary Processes
# II. Quantum Field Theory

by **L. P. Horwitz**[2])

University of Geneva

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*Abstract.* In a preceding paper, the consequences of the assumption that the measurements set up to characterize the quantum state of certain systems do not form (non-trivially) a complete set was investigated in the framework of the one-particle Hilbert space. In this paper, systems of identical particles, and the structure of the associated quantum fields, are discussed. The general form of the $n$-body density matrices characterizing incompletely measured states is given, and a special class of observables which 'carry their own incoherence' is constructed. As an illustration, a free charged Klein–Gordon field is constructed; it is shown that the field is non-local if the energy momentum is a non-trivial function of unmeasured variables. Coherent states are discussed, and it is shown that fields with some similarity to those of the Veneziano operator theory appear as a special case in which the spectrum of unmeasured observables corresponds to the four-fold tensor indices of space-time.

## 1. Introduction

In a previous paper [1], a basis for the statistical description of elementary processes was formulated in terms of incomplete measurement. It was assumed that the measurements set up to characterize the quantum state of certain systems do not form (non-trivially) a complete set. The consequences of this assumption for the structure of quantum mechanical states and their evolution in time was investigated.

In the present work, we consider systems of many identical subsystems (particles), and the structure of the associated quantum fields is investigated.

A many-body system is characterized by observables that may depend upon correlations between subsystems as well as the properties of the individual subsystems. In the extension of the work of I to many-body systems, we shall assume that all correlations can be measured with arbitrary precision within the restriction imposed on the one-body measurements, i.e., that the set of one-body observables actually measured, or used in the preparation of a state, do not form a complete set of one-body observables. The closed algebra of generalized incomplete measurements of the second kind is used, following Schwinger's procedure [2], to construct the quantum fields which annihilate and create subsystems in states which correspond to the overcomplete sets discussed in I. These fields have a number of components equal to the dimensionality of the spectrum of the unobserved subset of observables (which may be infinite).

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[2]) Present address: Dept. of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel.

In the third section, the general form of the $n$-body density matrices (correlation functions) is given, and a special class of observables which 'carry their own incoherence' is constructed. It is shown that the one-particle pure states created by our field variables correspond to mixed states in the one-particle Hilbert space (the argument can be extended to any subspace of the Fock space).

In the fourth section, an investigation of the space-time properties of these fields is carried out for the example of a free Klein–Gordon scalar particle (for which $k^\mu$ and the charge $Q$ constitute a complete set of observables). It is assumed that $k^\mu$ is a function of another set of observables, of which a subset is not measured. It is shown that the assumption of locality implies that the total energy momentum cannot be a non-trivial operator on the unmeasured factor of the direct product basis for the one-particle Hilbert space. Dropping the assumption of locality, an example can be constructed of a field for which the first commutator with the energy momentum operator does not induce a simple derivative, but for which the second commutator induces the local d'Alembertian. These non-local fields still provide, moreover, a representation for the translation group (with respect to a second set of space-time parameters). The conserved currents associated with these fields are briefly examined.

In the last section, coherent states constructed with these fields are examined. It is remarked that fields with some similarity to those of the Veneziano operator theory [3] appear as a special case in which the spectrum of unmeasured variables corresponds to the four-fold tensor indices of space time.

## 2. N-Body Measurements and Field Variables

We shall follow the method of Schwinger [2] in developing the quantum field theoretical extension of the one-body theory discussed in I because it is associated, at every step, with interpretation of the measuring process.

As in I, we shall assume that the state of the $i$th of $N$ identical subsystems (particles) can be characterized by the values of a complete set of observables $\{\alpha_i, \beta_i\}$, of which the subset $\alpha_i$ is not measured by the apparatus used. The one-dimensional projection operator $M(a'_i)$, corresponding to a filtering process carried out on the $i$th particle selecting the values $a'$ of an arbitrary complete set of observables $A$, corresponds to the density matrix for a pure state of the $i$th subsystem. Since the subset of variables $\alpha_i$ is assumed unmeasured, the 'maximal' measurement ('infinite temperature' will be assumed, for simplicity) is given by

$$M(a'_i)\,\mathfrak{p}_i = \frac{1}{N_\alpha} \sum_{\alpha'\alpha''} \bar{M}(\alpha'_i\,\alpha''_i)\, M(a'_i)\, \bar{M}(\alpha''_i, \alpha'_i), \tag{2.1}$$

where $\mathfrak{p}_i$ is a projection operator in the Liouville space associated with the one-body Hilbert space of the $i$th subsystem. The operators $\bar{M}(\alpha'_i, \alpha''_i)$ are defined by

$$\bar{M}\,(\alpha'_i, \alpha''_i) = \sum_{\beta'} |\alpha'_i\,\beta'_i\rangle\langle\alpha''_i\,\beta'_i|. \tag{2.2}$$

The spectra of identical observables associated with each of the particles will be assumed ordered in the same way, and therefore the index $i$ need not be indicated on the summation range in (2.2). In an $N$-body system, there are $n$-body observables for $n = 2, 3, \dots N$, as well as $n = 1$, and we may ask about the role of these operators in specifying a state.

The possibility of constructing a complete one-body measurement is implicit in the usual approach to the many-body problem ([2], [4]); in this way the Hilbert space is spanned by the (appropriately symmetrized) direct product of one-particle vectors. As an interesting special case of the application of the approach described in I, the structure of a theory for which no one-particle measurements could be constructed could be studied. The maximal measurements would then be two-body (or three-body, etc.) measurements, and the interpretation of the resulting annihilation-creation operators would be changed (as in the theory of superconductivity [5]). In the following, we shall restrict ourselves to the simplest situation, and assume that *the only limitation on the measurements one can construct is that a subset of a complete set of one-particle observables is not subject to measurement.*

We therefore define the maximal one-body measurement for an $N$-body system as

$$\mathbf{M}(a') = \sum_i M(a_i')\, \mathfrak{p}_i. \tag{2.3}$$

The algebra of $N$-body measurements cannot close on simple products because cross-terms arising from the products of sums of the type (2.3) are not $N$-body measurements. Using antisymmetric multiplication, the cross-terms can be eliminated [2]:

$$\begin{aligned}[\mathbf{M}(a'), \mathbf{M}(a'')] &= \sum_i [M(a_i')\, \mathfrak{p}_i, M(a_i'')\, \mathfrak{p}_i] \\ &= \frac{1}{N_\alpha^2} \sum_{i,\alpha',\ldots} [\bar{M}(\alpha_i', \alpha_i'')\, M(a_i')\, \bar{M}(\alpha_i'', \alpha_i'), \bar{M}(\alpha_i''', \alpha_i^{\mathrm{IV}})\, M(a_i'')\, \bar{M}(a_i^{\mathrm{IV}}, \alpha_i''')] \\ &= \frac{1}{N_{\alpha^2}} \sum_{i,\alpha',\ldots} \{\langle a'|\bar{M}(\alpha'', \alpha^{\mathrm{IV}})|a''\rangle\, \bar{M}(\alpha_i', \alpha_i'')\, M(a_i', a_i'')\, \bar{M}(\alpha_i^{\mathrm{IV}}, \alpha_i') \\ &\quad - \langle a''|\bar{M}(\alpha^{\mathrm{IV}}, \alpha''')|a'\rangle\, \bar{M}(\alpha_i''', \alpha_i^{\mathrm{IV}})\, M(a_i'', a_i')\, \bar{M}(\alpha_i'', \alpha_i''')\},\end{aligned} \tag{2.4}$$

where we have used the fact that scalar products of corresponding vectors are the same in the Hilbert space of every subsystem. We therefore define the generalized incomplete $N$-body measurement of the second kind as

$$\begin{aligned}\mathbf{M}_{\alpha'\alpha''}(a', a'') &= \sum_{i,\alpha'''} \bar{M}(\alpha_i''', \alpha_i')\, M(a_i', a_i'')\, \bar{M}(\alpha_i'', \alpha_i''') \\ &= \sum_i M_{\alpha_i'\alpha_i''}(a_i', a_i''),\end{aligned} \tag{2.5}$$

in the notation of equation (4.5) of I. The set of operators defined in this way belongs to a closed commutator algebra which contains $\mathbf{M}(a')$. The commutators have the form

$$\begin{aligned}&[\mathbf{M}_{\alpha'\alpha''}(a', a''), \mathbf{M}_{\alpha'''\alpha^{\mathrm{IV}}}(a''', a^{\mathrm{IV}})] \\ &\quad = \langle a''|\bar{M}(\alpha'', \alpha''')|a'''\rangle\, \mathbf{M}_{\alpha'\alpha^{\mathrm{IV}}}(a', a^{\mathrm{IV}}) - \langle a^{\mathrm{IV}}|\bar{M}(\alpha^{\mathrm{IV}}, \alpha')|a'\rangle\, \mathbf{M}_{\alpha'''\alpha''}(a''', a'').\end{aligned} \tag{2.6}$$

Following Schwinger [2], we construct a representation of this algebra in terms of fields $\{\psi_{\alpha'}(a')\}$ with

$$[\psi_{\alpha'}(a'), \psi_{\alpha''}^\dagger(a'')]_\pm = \langle a'|\bar{M}(\alpha', \alpha'')|a''\rangle, \tag{2.7}$$

where we have assumed commutation or anti-commutation relations corresponding to Bose–Einstein or Fermi–Dirac statistics[3]).

In terms of these fields

$$\mathbf{M}_{\alpha'\alpha''}(a', a'') = \psi^{\dagger}_{\alpha'}(a')\,\psi_{\alpha''}(a''), \tag{2.8}$$

and

$$\mathbf{M}(a') = \frac{1}{N_\alpha} \sum_{\alpha'} \psi^{\dagger}_{\alpha'}(a')\,\psi_{\alpha'}(a'). \tag{2.9}$$

Since

$$\sum_{a'} \mathbf{M}(a') = N, \tag{2.10}$$

where $N$ is the number of subsystems, we have

$$N = \frac{1}{N_\alpha} \sum_{a',\alpha'} \psi^{\dagger}_{\alpha'}(a')\,\psi_{\alpha'}(a'). \tag{2.11}$$

To make explicit the structure of these field variables, we write (2.5) in a different way:

$$\begin{aligned}\mathbf{M}_{\alpha'\alpha''}(a', a'') &= \sum_{i,\beta',\beta'',\alpha''} |\alpha'''_i, \beta'_i\rangle \langle \alpha'_i \beta'_i | M(a'_i a''_i) | \alpha''_i \beta''_i\rangle \langle \alpha'''_i \beta''_i | \\ &= \sum_{\beta'\beta''} \langle \alpha' \beta' | a' \rangle \langle a'' | \alpha'' \beta'' \rangle \sum_i \bar{M}(\beta'_i, \beta''_i),\end{aligned} \tag{2.12}$$

where, as in (4.19) of I,

$$\bar{\mathbf{M}}(\beta', \beta'') = \sum_{i,\alpha'} M(\alpha'_i \beta'_i, \alpha'_i \beta''_i). \tag{2.13}$$

The $N$-body measurements (2.13) satisfy an algebra identical in structure to that of the corresponding algebra in the case of complete measurement, i.e.,

$$[\bar{\mathbf{M}}(\beta', \beta''), \bar{\mathbf{M}}(\beta''', \beta^{\mathrm{IV}})] = \delta(\beta'', \beta''')\,\bar{\mathbf{M}}(\beta', \beta^{\mathrm{IV}}) - \delta(\beta^{\mathrm{IV}}, \beta')\,\bar{\mathbf{M}}(\beta''', \beta''), \tag{2.14}$$

and hence we may construct a representation in terms of fields $\psi(\beta')$ with

$$[\psi(\beta'), \psi(\beta'')^{\dagger}]_{\pm} = \delta(\beta', \beta''), \tag{2.15}$$

again assuming commutation or anticommutation relations.

In terms of these fields,

$$\bar{\mathbf{M}}(\beta', \beta'') = \psi(\beta')^{\dagger}\,\psi(\beta'') \tag{2.16}$$

and therefore (2.12) can be written as

$$\mathbf{M}_{\alpha'\alpha''}(a', a'') = \sum_{\beta'\beta''} \langle a' | \alpha' \beta' \rangle^{*}\, \psi(\beta')^{\dagger}\, \psi(\beta'') \langle a'' | \alpha'' \beta'' \rangle. \tag{2.17}$$

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[3]) Note that $[\psi_{\alpha'}(a'), \psi_{\alpha'}(a')^{\dagger}]_{\pm} = \Sigma_{\beta'} |\langle a' | \alpha' \beta' \rangle|^2 \geqslant 0$.

Comparing (2.8) and (2.17), we conclude (within an arbitrary phase) that

$$\psi_{\alpha'}(a') = \sum_{\beta'} \langle a'|\alpha'\beta'\rangle \psi(\beta'); \tag{2.18}$$

the (anti-)commutation relations (2.7) are clearly satisfied. As for the vectors $\{|\beta')\}$ discussed in I, the fields $\psi(\beta')$ are *algebraic* fields, with $N_\alpha$ components. It follows in fact, from (2.13) that

$$\bar{\mathbf{M}}(\beta', \beta'') = \sum_{\alpha'} \psi_0(\alpha'\beta')^\dagger \psi_0(\alpha', \beta''), \tag{2.19}$$

where $\psi_0(\alpha', \beta')$ are the usual 'pure state' fields satisfying (in an arbitrary representation) the canonical relations

$$[\psi_0(a'), \psi_0(a'')^\dagger]_\pm = \delta(a', a''). \tag{2.20}$$

Comparing (2.16) and (2.20), one concludes that

$$(\psi(\beta'))_{\alpha'} = \psi_0(\alpha', \beta') \tag{2.21}$$

$$\begin{aligned}(\psi_{\alpha'}(a'))_{\alpha''} &= \sum_{\beta'} \langle a'|\alpha'\beta'\rangle (\psi(\beta'))_{\alpha''} \\ &= \sum_{a''} \langle a'|M(\alpha', \alpha'')|a''\rangle \psi_0(a'').\end{aligned} \tag{2.22}$$

The result (2.22), involving the reproducing kernel of the overcomplete set of states discussed in I, is analogous to (4.26) of I, except that the roles of $\alpha', \alpha''$ are interchanged (the fields $\psi$ transform like dual vectors).

Equations (2.8) and (2.16) contain implicit summations over $\alpha''$. The relation (2.15) is to be understood as

$$[(\psi(\beta'))_{\alpha'}, (\psi(\beta'')^\dagger)_{\alpha''}]_\pm = \delta(\alpha', \alpha'')\, \delta(\beta', \beta''),$$

i.e., as a matrix equation, it is diagonal in the $\{\alpha'\}$ space.

The number operator, expressed in terms of the $\psi(\beta')$, is

$$\begin{aligned}N &= \sum_{\beta'} \psi(\beta')^\dagger \psi(\beta') \\ &= \sum_{\alpha'\beta'} \psi_0(\alpha'\beta')^\dagger \psi_0(\alpha', \beta').\end{aligned} \tag{2.23}$$

From (2.18), one finds

$$\psi(\beta') = \sum_{a'} \langle \alpha'\beta'|a'\rangle \psi_{\alpha'}(a'), \tag{2.24}$$

and we note that

$$\sum_{a'} \langle \alpha''\beta'|a'\rangle \psi_{\alpha'}(a') = 0 \qquad (\alpha' \neq \alpha''). \tag{2.25}$$

Hence, from (2.23),

$$N = \sum_{a'a''\beta'} \psi_{\alpha'}(a')^\dagger \langle a'|\alpha'\beta'\rangle\langle\alpha''\beta'|a''\rangle\psi_{\alpha''}(a'')$$

$$= \sum_{a'a''} \psi_{\alpha'}(a')^\dagger\langle a'|\bar{M}(\alpha',\alpha'')|a''\rangle\psi_{\alpha''}(a'')$$

$$= \sum_{a'} \psi_{\alpha'}(a')^\dagger\psi_{\alpha'}(a'), \tag{2.26}$$

where the last equality follows from (2.22) and the properties of the reproducing kernel. It is independent of $\alpha'$, and is therefore consistent with (2.11) (the implicit summation index in (2.26) corresponds to the 'layers' discussed in I, and these contributions occur in the same way in (2.11) and (2.26)).

We remark that the vacuum of $\psi(\beta')$ (or $\psi_{\alpha'}(a')$) is the same as the vacuum for $\psi_0(\alpha',\beta')$. Defining the vacuum state $|0\rangle$ as the solution of

$$\psi(\beta')|0\rangle = 0 \tag{2.27}$$

for all $\beta'$, it follows that

$$\|\psi(\beta')|0\rangle\|^2 = \langle 0|\psi(\beta')^\dagger\psi(\beta')0\rangle$$

$$= \sum_{\alpha'}\langle 0|\psi_0(\alpha'\beta')^\dagger\psi_0(\alpha',\beta')|0\rangle$$

$$= \sum_{\alpha'}\|\psi_0(\alpha'\beta')|0\rangle\|^2 = 0,$$

and hence

$$\psi_0(\alpha'\beta'|0\rangle = 0$$

for all $\alpha',\beta'$.

The interpretation of $\psi_{\alpha'}(a')$ and $\psi(\beta')$ as annihilation operators (and their adjoints as creation operators) is obtained, as for $\psi_0(\alpha',\beta')$, in terms of their commutators with $N$:

$$[N,\psi(\beta')^\dagger] = \psi(\beta')^\dagger \tag{2.28}$$

and

$$[N,\psi_{\alpha'}(a')^\dagger] = \sum_{a''}\psi_{\alpha''}(a'')^\dagger\langle a''|\bar{M}(\alpha'',\alpha')|a'\rangle$$

$$= \psi_{\alpha'}(a')^\dagger. \tag{2.29}$$

The number operator is invariant with respect to the choice of description for the particles, but the physical interpretation of the Fock space elements created by these fields is quite different. $\psi_{\alpha'}(a')^\dagger$ creates a particle in a state that is isomorphic to the one-body Hilbert space vector $|a'(\alpha'))$ discussed in I.

## 3. Observables

With the help of the fields $\psi_0(a')$, it is possible to construct $1, 2, 3 \ldots n, \ldots, N$-body operators in the usual 'second quantized' form. Schwinger's procedure [2] for the one-body operator, for example, is to recognize that an operator defined for the $i$th subsystem, in its appropriate Hilbert space, is given by

$$f_i^{(1)} = \sum_{a'a''} \langle a_i' | f_i^{(1)} | a_i'' \rangle M(a_i' a_i''). \tag{3.1}$$

The sum of operators of the form (3.1) then corresponds to a one-body operator in the Hilbert space for an $N$-body system, i.e., using the fact that matrix elements for identical particles do not depend on $i$,

$$\begin{aligned} F^{(1)} &= \sum_{a'a''} \langle a' | f^{(1)} | a'' \rangle \sum_i M(a_i' a_i'') \\ &= \sum_{a'a''} \psi_0(a')^\dagger \langle a' | f^{(1)} | a'' \rangle \psi_0(a''). \end{aligned} \tag{3.2}$$

For the description of processes subject only to incomplete measurement, however, the density matrix for the $i$th subsystem is given (in place of (3.1)) by

$$\check{\rho}_i^{(1)} = \sum_{a'a''} \langle a_i' | \rho_i^{(1)} | a_i'' \rangle (M(a_i', a_i'')\, \mathfrak{p}_i), \tag{3.3}$$

and therefore the one-body density operator (in place of (3.2)) is

$$\check{\rho}^{(1)} = \frac{1}{N_\alpha} \sum_{a'a'',\alpha'} \psi_{\alpha'}(a')^\dagger \langle a' | \rho^{(1)} | a'' \rangle \psi_{\alpha'}(a''). \tag{3.4}$$

For the two-body density matrix (correlation function), in the two-particle Hilbert space for the $i$th and $j$th ($i \neq j$) subsystems, we have (taking into account the assumption characterizing our model which was discussed in connection with (2.3))

$$\check{\rho}_{ij}^{(2)} = \frac{1}{2!} \sum_{\substack{a'a''a'''a^{\mathrm{IV}} \\ i\neq j}} \langle a_i' a_j''' | \rho_{ij}^{(2)} | a_i'' a_j^{\mathrm{IV}} \rangle (M(a_i' a_i'')\, \mathfrak{p}_i)(M(a_j''' a_j^{\mathrm{IV}}\, \mathfrak{p}_j). \tag{3.5}$$

In the $N$-body Hilbert space, the two-body density operator is then

$$\begin{aligned} \check{\rho}^{(2)} &= \frac{1}{2!} \sum_{a'a''a'''a^{\mathrm{IV}}} \langle a' a''' | \rho^{(2)} | a'' a^{\mathrm{IV}} \rangle \Big\{ \sum_{ij} M(a_i' a_i'')\, \mathfrak{p}_i M(a_j''' a_j^{\mathrm{IV}})\, \mathfrak{p}_j \\ &\quad - \sum_i (M(a_i' a_i'')\, \mathfrak{p}_i)(M(a_i''', a_i^{\mathrm{IV}})\, \mathfrak{p}_i) \Big\} \\ &= \frac{1}{2!} \frac{1}{N_\alpha^2} \sum_{a'a''a'''a^{\mathrm{IV}}} \langle a' a''' | \rho^{(2)} | a'' a^{\mathrm{IV}} \rangle \Big\{ \sum_{\alpha'\alpha''} \psi_{\alpha'}^\dagger(a') \psi_{\alpha'}(a'') \psi_{\alpha''}^\dagger(a''') \psi_{\alpha''}(a^{\mathrm{IV}}) \\ &\quad - \langle a'' | \bar{M}(\alpha' \alpha'') | a''' \rangle \psi_{\alpha'}^\dagger(a') \psi_{\alpha''}(a^{\mathrm{IV}}) \Big\}. \end{aligned} \tag{3.6}$$

Using the (anti-)commutation relation (2.7) in the first term of (3.6), we obtain

$$\check{\rho}^{(2)} = \frac{1}{2!} \frac{1}{N_\alpha^2} \sum_{\substack{a'a''a'''a^{\mathrm{IV}} \\ \alpha'\alpha''}} \psi_{\alpha'}^\dagger(a') \psi_{\alpha''}^\dagger(a''') \langle a' a''' | \rho^{(2)} | a'' a^{\mathrm{IV}} \rangle .\, \psi_{\alpha''}(a^{\mathrm{IV}}) \psi_{\alpha'}(a''). \tag{3.7}$$

This result may be easily generalized to the higher order correlation functions. It is evident from (3.4) and (3.7) that the replacement of $\psi_0(a')$ by $\psi_{\alpha'}(a')$ (with associated sums) in the usual form of the $n$-body correlation functions, accounts for the incoherence which accompanies incomplete measurement with respect to a complete set of one-particle observables.

The density operators $\check{\rho}^{(1)}$, $\check{\rho}^{(2)}$, ... may be used in the usual way to calculate average values of observables such as $F^{(1)}$, $F^{(2)}$, ...

For example, using (2.22) one obtains

$$\begin{aligned} \mathrm{Tr}_1(F^{(1)}\check{\rho}^{(1)}) &= \sum_{a',\ldots a^{\mathrm{V}},\alpha'} \langle 0|\psi_0(a^{\mathrm{V}})\,\psi_0^\dagger(a')\langle a'|f^{(1)}|a''\rangle\,\psi_0(a'')\,\psi_{\alpha'}^\dagger(a''') \\ &\quad \cdot\langle a'''|\rho^{(1)}|a^{\mathrm{IV}}\rangle\,\psi_{\alpha'}(a^{\mathrm{IV}})\,\psi_0^\dagger(a^{\mathrm{V}})|0\rangle \\ &= \sum_{a',\alpha',\alpha''} \langle a'|f^{(1)}(\overline{M}(\alpha'',\alpha')\,\rho^{(1)}\,\overline{M}(\alpha',\alpha''))|a'\rangle. \end{aligned}$$

There is a special class of $n$-body observables, however, which carry their own incoherence.

In the discussion of induced dispersion given in I (section 5), it was found that, for the single particle theory, an observable $A$ has non-vanishing dispersion in every contracted state unless an element of its spectral family commutes with all of the $\overline{M}(\alpha',\alpha'')$. In the demonstration, the quantities $\mathrm{Tr}(A(\rho\mathfrak{p}))$ and $\mathrm{Tr}(A^2(\rho\mathfrak{p}))$ were considered. Alternatively, the operators $A\mathfrak{p}$ and $A^2\mathfrak{p}$ could have been used with arbitrary uncontracted states $\rho$ (note that $A^2\mathfrak{p}\neq(A\mathfrak{p})^2$. The operator $A\mathfrak{p}$ 'carries its own incoherence' from the point of view which considers an observable as a linear superposition of measurements (in this case, incomplete), since

$$\check{A}=A\mathfrak{p}=\sum a'(M(a')\,\mathfrak{p}). \tag{3.8}$$

Following the procedure for which (3.4) was obtained for the one-body density operator, we find

$$\check{F}^{(1)}=\frac{1}{N_\alpha}\sum_{\alpha'a'a''}\psi_{\alpha'}^\dagger(a')\langle a'|f^{(1)}|a''\rangle\,\psi_{\alpha'}(a'') \tag{3.9}$$

for the one-body operator which carries its own incoherence.

The commutative kernel (section 4 of I) of the operator (3.8) is

$$\begin{aligned} K_{\alpha'\alpha''}(\check{A}) &= \sum_{\alpha''',\alpha^{\mathrm{IV}},\alpha^{\mathrm{V}}} \overline{M}(\alpha''',\alpha')\,\overline{M}(\alpha^{\mathrm{IV}},\alpha^{\mathrm{V}})\,A\overline{M}(\alpha^{\mathrm{V}}\,\alpha^{\mathrm{IV}})\,\overline{M}(\alpha'',\alpha''') \\ &= \delta(\alpha',\alpha'')\,\check{A}, \end{aligned}$$

i.e., $\check{A}$ is its own commutative kernel. But this property is also true of products of contracted operators.

Let $\check{f},\check{f}_2$ be of the form

$$\check{f}_i=\frac{1}{N_\alpha}\sum_{a'a''\alpha'\alpha''}\langle a'|f_i|a''\rangle\,\overline{M}(\alpha',\alpha'')\,M(a',a'')\,\overline{M}(\alpha'',\alpha'). \tag{3.10}$$

Then

$$\check{f}_1\check{f}_2 = \frac{1}{N_\alpha^2}\sum_{a'a''\alpha'\alpha''} \langle a'|f_1\bar{M}(\alpha',\alpha'')f_2|a''\rangle M_{\alpha'\alpha''}(a',a'') \tag{3.11}$$

has the property

$$K_{\alpha'\alpha''}(\check{f}_1\check{f}_2) = \delta(\alpha',\alpha'')\check{f}_1\check{f}_2. \tag{3.12}$$

The most general form of an operator which is its own commutative kernel (and hence carries its own incoherence) is therefore

$$\check{f}^{(1)} = \sum_{a'a''\alpha'\alpha''} \langle a'|f^{(1)}(\alpha',\alpha'')|a''\rangle M_{\alpha'\alpha''}(a',a''). \tag{3.13}$$

(3.13) does not constitute a real generalization of (3.8) in an algebraic sense, but provides a formal convenience for retaining reference to the original factors in an expression such as (3.11) (the $M_{\alpha'\alpha''}(a',a'')$ are not linearly independent). A chain of multiplications would lead to the same form for (3.13), but the corresponding operator in the matrix element in (3.11) would be, for example, for the case of three factors,

$$\sum_{\alpha''} f_1\bar{M}(\alpha',\alpha'')f_2\bar{M}(\alpha'',\alpha''')f_3$$

In fact, (3.13) can be written

$$\check{f}^{(1)} = \sum_{\alpha'\alpha''\alpha'''} \bar{M}(\alpha''',\alpha')f^{(1)}(\alpha',\alpha'')\bar{M}(\alpha'',\alpha''')$$

$$= \sum_{\alpha'''\beta'\beta''} |\alpha'''\beta'\rangle f_{\beta'\beta''}\langle\alpha'''\beta''|,$$

and the same is true for operators of the type (3.8).

The many-body sum of operators of the form (3.13) results, instead of (3.9), in

$$\check{F}^{(1)} = \sum_{\alpha'\alpha'',a'a''} \psi^\dagger_{\alpha'}(a')\langle a'|f^{(1)}(\alpha',\alpha'')|a''\rangle\psi_{\alpha''}(a''), \tag{3.14}$$

and we remark that

$$\langle 0|\psi_0(a')\check{F}^{(1)}\psi_0^\dagger(a'')|0\rangle = \langle a'|\check{f}^{(1)}|a''\rangle,$$

and

$$\check{F}^{(1)} = \sum_{a'a''} \psi_0^\dagger(a')\langle a'|\check{f}^{(1)}|a''\rangle\psi_0(a''), \tag{3.15}$$

relating $\check{F}^{(1)}$ of (3.14) and $\check{f}^{(1)}$ of (3.13).

The commutator of two one-body observables of the form (3.9) is

$$[\check{F}_1^{(1)},\check{F}_2^{(1)}] = \frac{1}{N_\alpha^2}\sum_{\substack{\alpha'\alpha''\\a'a''}} \psi^\dagger_{\alpha'}(a')\langle a'|\{f_1^{(1)}\bar{M}(\alpha',\alpha'')f_2^{(1)}$$

$$-f_2^{(1)}\bar{M}(\alpha',\alpha'')f_1^{(1)}\}|a''\rangle\psi_{\alpha''}(a''), \tag{3.16}$$

while the commutator of two operators of the form (3.14) is

$$[\check{F}_1^{(1)}, \check{F}_2^{(1)}] = \sum_{a',a'',\alpha' \ldots} \psi^\dagger_{\alpha'}(a') \langle a' | \{ f_1^{(1)}(\alpha', \alpha'') \, \bar{M}(\alpha'', \alpha''') f_2^{(1)}(\alpha''', \alpha^{\mathrm{IV}})$$

$$- f_2^{(1)}(\alpha', \alpha'') \, \bar{M}(\alpha'', \alpha''') f_1^{(1)}(\alpha''', \alpha^{\mathrm{IV}}) \} | a'' \rangle \, \psi_{\alpha^{\mathrm{IV}}}(a''). \tag{3.17}$$

With respect to the matrix elements occurring in (3.9) and (3.14), (3.16) appears to introduce something new (relative to (3.9)), while (3.17) is of the same form as (3.14). Hence the commutator algebra of operators of type (3.14) formally closes.

We remark that the internal indices of (3.17) couple through the reproducing kernel $\langle a' | \bar{M}(\alpha', \alpha'') | a'' \rangle$, which appears as a vertex function. The appearance of reproducing kernels in the two-body operators which carry their own incoherence is of particular interest, and we therefore construct these operators as well.

Using $f^{(2)}(\alpha' \alpha''', \alpha'' \alpha^{\mathrm{IV}})$ to represent the appropriate operator products occurring in the closure of the algebra of two-body operators, we may write

$$\check{F}^{(2)} = \frac{1}{2!} \sum_{\substack{a', \ldots \\ \alpha', \ldots}} \langle a' \, a''' | f^{(2)}(\alpha' \alpha''', \alpha'' \, \alpha^{\mathrm{IV}}) | a'' \, a^{\mathrm{IV}} \rangle \sum_{i \neq j} M_{\alpha'_i \alpha''_i}(a'_i \, a''_i) M_{\alpha'''_j \alpha^{\mathrm{IV}}_j}(a'''_j \, a^{\mathrm{IV}}_j)$$

$$= \frac{1}{2!} \sum_{\substack{a', \ldots \\ \alpha', \ldots}} \langle a' \, a''' | f^{(2)}(\alpha' \, \alpha''', \alpha'' \, \alpha^{\mathrm{IV}}) | a'' \, a^{\mathrm{IV}} \rangle \Big\{ \sum_{ij} M_{\alpha'_i \alpha''_i}(a'_i, a''_i) \, M_{\alpha'''_j \alpha^{\mathrm{IV}}_j}(a'''_j, a^{\mathrm{IV}}_j)$$

$$- \sum_i \langle a'' | \bar{M}(\alpha'' \, \alpha''') | a''' \rangle \, M_{\alpha'_i \alpha^{\mathrm{IV}}_i}(a'_i, a^{\mathrm{IV}}_i) \Big\}.$$

Following the same procedure used in obtaining (3.7), we find

$$\check{F}^{(1)} = \frac{1}{2!} \sum_{\substack{a', \ldots \\ \alpha', \ldots}} \psi^\dagger_{\alpha'}(a') \, \psi^\dagger_{\alpha'''}(a''') \langle a' \, a''' | f^{(2)}(\alpha' \, \alpha''', \alpha'' \, \alpha^{\mathrm{IV}}) | a'' \, a^{\mathrm{IV}} \rangle \, \psi_{\alpha^{\mathrm{IV}}}(a^{\mathrm{IV}}) \, \psi_{\alpha''}(a''). \tag{3.18}$$

The operator $f^{(2)}(\alpha' \, \alpha''', \alpha'', \alpha^{\mathrm{IV}})$ contains, in general, direct products of pairs of reproducing kernels which have the appearance of scattering amplitudes coupling dynamical four-point functions. The matrix elements of an operator of the type $\check{F}_2$ in states of the form $\psi^\dagger_{\alpha'}(a') \, \psi^\dagger_{\alpha''}(a'') | 0 \rangle$ contains four-point reproducing kernels in initial and final channels as well. We shall not discuss these properties of the two-body operator further here, but conclude this section with some remarks on one-body operators.

The vector valued field operator $\psi^\dagger_{\alpha'}(a')$, as we have seen, creates a family (with respect to the vector index $\alpha''$) of one-particle states on the vacuum that is a family of pure states. This family is associated with a mixed state in the corresponding one-particle Hilbert space. The density operator associated with this family is

$$\rho_{\alpha'}(a') = \psi^\dagger_{\alpha'}(a') \, \psi_{\alpha'}(a'), \tag{3.19}$$

a special case of (3.14). The matrix elements of the corresponding operator in the one-particle Hilbert space, according to (3.15), are given by

$$\langle a'' | \rho^{(1)}_{\alpha'}(a') | a''' \rangle = \langle 0 | \psi_0(a'') \, \rho_{\alpha'}(a') \, \psi^\dagger_0(a''') | 0 \rangle$$

$$= \langle a'' | M_{\alpha' \alpha'}(a', a') | a''' \rangle, \tag{3.20}$$

so that

$$\rho^{(1)}_{\alpha'}(a') = M_{\alpha'\alpha'}(a', a') = \sum_{\beta'\beta''} \langle\alpha'\beta'|M(a')|\alpha'\beta''\rangle \bar{M}(\beta',\beta''),$$

where (cf. section 4 of I)

$$\bar{M}(\beta',\beta'') = \sum_{\alpha''} |\alpha''\beta'\rangle\langle\alpha''\beta''|.$$

Hence $\rho^{(1)}_{\alpha'}(a')$ corresponds to a mixed state.

A comparison of (3.14) and (3.15) makes it clear that, in the construction of operators that carry their own incoherence, either a replacement of field variables $\psi_0$ by $\psi_{\alpha'}$, or a replacement of the operator in the one-particle matrix element $f^{(1)}$ by $\check{f}^{(1)}$ is effective, and results in equivalent representations. The use of the new field variables $\psi_{\alpha'}$ has some intuitive advantage, as in (3.19), and permits the direct transcription of one-body (or two-body, etc.) observables to Fock space operators without carrying out a contraction first. The one-body (or two-body, etc.) matrix elements then retain their dynamical interpretation, and the kinematical constraints imposed by the measuring process are reflected in the structure of the fields.

## 4. An Illustration in Space Time

Let us consider an example of a free charged particle, for which $k^\mu$ and the charge $Q$ constitute a complete set of observables. We shall assume that $\{k^\mu, Q\}$ is, however, a function of another complete set $\{\alpha, \beta, Q\}$ for which only $\beta$ and $Q$ are precisely observed. According to (2.29),

$$[\psi_{+\alpha'}(k), N_+] = \psi_{+\alpha'}(k)$$

$$[\psi_{-\alpha'}(k), N_-] = \psi_{-\alpha'}(k),$$

and therefore $\psi_{\pm\alpha'}(k)$ have the usual interpretation as annihilation operators.

Since $\psi_{\pm\alpha'}(k)$ are vector-valued fields (possibly infinite component, if the set $\alpha$ has infinite spectrum), it is possible to assume more complicated wave equations [6]; our discussion of the one-particle wave equations given in section 4 of I (4.30), for example) also suggest more general equations. We shall assume, however, for simplicity, that there is a family of charged scalar fields $\varphi_{\alpha'}(x)$ which satisfies a Klein–Gordon equation, so that

$$\varphi_{\alpha'}(x) = \int \frac{d^3k}{\sqrt{2k_0(2\pi)^3}} (e^{-ikx} a_{+\alpha'}(k) + e^{ikx} a^\dagger_{-\alpha'}(k)), \tag{4.1}$$

where $k_0 = +\sqrt{k^2 + m^2}$, with a universal mass $m$, and we have replaced $\psi_{\pm\alpha'}(k)$ by $a_{\pm\alpha'}(k)$ to conform with the usual notation for a scalar charged field. If the Lagrangian density is taken to be

$$\mathfrak{L} = \sum_{\alpha'} (\partial_\mu \varphi^\dagger_{\alpha'} \partial^\mu \varphi_{\alpha'} - m^2 \varphi^\dagger_{\alpha'} \varphi_{\alpha'}), \tag{4.2}$$

then the 'canonical momenta' are formally

$$\pi_{\alpha'}(x) = \varphi^\dagger_{\alpha'}(x). \tag{4.3}$$

We assume commutation relations corresponding to (2.7) in the form

$$[a_{\pm\alpha'}(k), a_{\pm\alpha''}(k')^\dagger] = \langle k|\bar{M}(\alpha', \alpha'')|k'\rangle, \tag{4.4}$$

independently of the charge.

We shall first show that *the assumption that $\varphi_{\alpha'}(x)$ is a local field implies that the energy momentum operator is trivial in the $\alpha$-factor of the one-particle Hilbert space, and that the representation of the translation operator for the fields* (4.1) *as an operator which carries its own incoherence (in terms of the fields $a_{\pm\alpha'}(k)$ is equivalent to the usual representation*[4]).

The assumption of locality implies the existence of an operator $\hat{p}^\mu$ for which

$$i\frac{\partial \varphi_{\alpha'}(x)}{\partial x_\mu} = [\varphi_{\alpha'}(x), \hat{p}^\mu], \tag{4.5}$$

and hence (from (4.1)) that

$$[a_\pm(k), \hat{p}^\mu] = k^\mu a_{\pm\alpha'}(k). \tag{4.6}$$

Since

$$a_{\pm\alpha'}(k) = \sum_{\beta'} \langle k|\alpha'\beta'\rangle a_\pm(\beta'),$$

(4.6) is equivalent to

$$\sum_{\beta'} \langle k|\alpha'\beta'\rangle [a_\pm(\beta'), \hat{p}^\mu] = k^\mu \sum_{\beta'} \langle k|\alpha'\beta'\rangle a_\pm(\beta'),$$

from which we obtain (by multiplying by $\langle\alpha''\beta''|k\rangle$ and integrating over $d^3k$)

$$\delta(\alpha', \alpha'')[a_\pm(\beta''), \hat{p}^\mu] = \sum_{\beta'} \int d^3k \langle\alpha''\beta''|k\rangle k^\mu \langle k|\alpha'\beta'\rangle a_\pm(\beta'). \tag{4.7}$$

It follows that

$$\int d^3k \langle\alpha''\beta''|k\rangle k^\mu \langle k|\alpha'\beta'\rangle \tag{4.8}$$

is diagonal in $\alpha', \alpha''$ and independent of $\alpha'$. $\hat{p}^\mu$ may therefore be represented by

$$\hat{p}^\mu = \sum_{\beta'\beta'', Q=\pm} \int d^3k\, a_Q(\beta'')^\dagger \langle\alpha'\beta''|k\rangle k^\mu \langle k|\alpha'\beta'\rangle a_Q(\beta')$$

$$= \int d^3k \sum_{Q=\pm} a^\dagger_{Q\alpha'}(k)\, k^\mu a_{Q\alpha'}(k), \tag{4.9}$$

independently of $\alpha'$. To verify this result, one may calculate the commutator of $a_{\pm\alpha''}(k)$ with (4.9):

$$[a_{\pm\alpha''}(k), \hat{p}^\mu] = \int d^3k' \langle k|\bar{M}(\alpha'', \alpha')|k'\rangle k'^\mu a_{\pm\alpha'}(k'). \tag{4.10}$$

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[4]) It is interesting to compare some of the results which follow with those of Poncet [7].

This appears to be in contradiction with (4.6), but it is, in fact, equivalent. To make this equivalence explicit, we recover (4.7) by multiplying by $\langle\alpha''\beta'|k\rangle$ and integrating, i.e.,

$$\int d^3k\,\langle\alpha''\beta'|k\rangle[a_{\pm\alpha''}(k),\hat{p}^\mu]=[a_\pm(\beta'),\hat{p}^\mu]$$
$$=\int d^3k'\langle\alpha'\beta'|k'\rangle k'^\mu a_{\pm\alpha'}(k'). \tag{4.11}$$

Using the diagonal property of the right-hand side, we can immediately write (4.7). The equivalence with (4.6) then follows by multiplying (4.7) by $\langle k'|\alpha''\beta''\rangle$ and summing over both $\alpha''$, $\beta''$ as follows:

$$[a_{\pm\alpha'}(k'),\hat{p}^\mu]=\sum_{\alpha''\beta''\beta'}\int d^3k\,\langle k'|\alpha''\beta''\rangle\langle\alpha''\beta''|k\rangle k^\mu\langle k|\alpha'\beta'\rangle a_\pm(\beta')$$
$$=k'^\mu a_{\pm\alpha'}(k').$$

We have therefore proved the very simple relation

$$\int d^3k'\,\langle k|\bar{M}(\alpha'',\alpha')|k'\rangle k'^\mu a_{\pm\alpha'}(k')=k^\mu a_{\pm\alpha''}(k). \tag{4.12}$$

This result is a consequence of our assumption that the field $\varphi_{\alpha'}(x)$ is local. If $\hat{p}^\mu$ were to act non-trivially in the $\alpha$-factor, the fields defined by (4.1) could not be local, i.e., they could not satisfy (4.5).

It is now easy to show that

$$\hat{p}^\mu=p^\mu=\int d^3k\sum_{Q=\pm}a^\dagger_{Q_0}(k)\,k^\mu a_{Q_0}(k), \tag{4.13}$$

the usual representation of the translation operator for the charged scalar field. Since (from 2.22)),

$$(a_{\pm\alpha'}(k))_{\alpha''}=\int d^3k'\langle k|\bar{M}(\alpha',\alpha'')|k'\rangle a_{\pm 0}(k'), \tag{4.14}$$

we may write (4.9) in the form

$$\hat{p}^\mu=\sum_{\alpha'',Q}\int d^3k d^3k'\,a_{Q_0}(k')^\dagger\langle k'|\bar{M}(\alpha'',\alpha')|k\rangle k^\mu(a_{Q\alpha'}(k))_{\alpha''};$$

with the help of (4.12) and the fact that

$$\sum_{\alpha''}(a_{Q\alpha''}(k))_{\alpha''}=a_{Q_0}(k), \tag{4.15}$$

(4.13) follows immediately. ■

We have demonstrated that locality of the fields $\varphi_{\alpha'}(x)$ implies that $p^\mu$ is a function of the $\beta$ (measured) variables alone, since in this case the operator (4.8) is trivial in the $\alpha$-factor. Suppose, on the other hand, that we do *not* assume that the fields (4.1) are local.

One may always construct local fields using $a_{\pm 0}(k)$ (on the same mass shell $k^2=m^2$, according to (4.15)), and for these the form (4.13) for the translation operator is correct.

To see the effect of a translation on the fields (4.1), we therefore calculate the commutators

$$[(a_{\pm\alpha'}(k))_{\alpha''}, p^\mu] = \int d^3k' \langle k|\bar{M}(\alpha', \alpha'')|k'\rangle k'^\mu a_{\pm 0}(k') \tag{4.16}$$

$$[[(a_{\pm\alpha'}(k))_{\alpha''}, p^\mu], p_\mu] = \int d^3k' \langle k|\bar{M}(\alpha', \alpha'')|k'\rangle k'^2 a_{\pm 0}(k')$$

$$= m^2 (a_{\pm\alpha'}(k))_{\alpha''} \tag{4.17}$$

Hence

$$[[\varphi_{\alpha'}(x), p^\mu], p_\mu] = m^2 \varphi_{\alpha'}(x) = -\partial^\mu \partial_\mu \varphi_{\alpha'}(x)$$

even if we do not assume locality (but retain the Klein–Gordon equation restricting $a_{\pm\alpha'}(k)$ to a mass-shell).

Since

$$a_{\pm 0}(k) = \sum_{\alpha''} \int d^3k' \langle k|\bar{M}(\alpha'', \alpha')|k'\rangle (a_{\pm\alpha'}(k')_{\alpha''}, \tag{4.18}$$

(4.16) can be written

$$[(a_{\pm\alpha'}(k))_{\alpha''}, p^\mu] = \sum_{\alpha'''} \int d^3k' \langle k|\bar{M}(\alpha', \alpha'') p^{\mu(1)} \bar{M}(\alpha''', \alpha^{\mathrm{IV}})|k'\rangle \,.\, (a_{\pm\alpha^{\mathrm{IV}}}(k'))_{\alpha'''} \tag{4.19}$$

for arbitrary $\alpha^{\mathrm{IV}}$. In (4.19),

$$p^{\mu(1)} = \int d^3k |k\rangle k^\mu \langle k|$$

$$= \sum_{\alpha'\alpha''} K_{\alpha'\alpha''}(p^{\mu(1)}) \bar{M}(\alpha', \alpha'')$$

where (see (4.7) of I) $K_{\alpha'\alpha''}(p^{\mu(1)})$ is the commutative kernel of $p^{\mu(1)}$. Hence

$$[(a_{\pm\alpha'}(k))_{\alpha''}, p^\mu] = \sum_{\alpha'''} \int d^3k' \langle k|K_{\alpha''\alpha'''}(p^{\mu(1)}) \bar{M}(\alpha', \alpha^{\mathrm{IV}})|k'\rangle (a_{\pm\alpha^{\mathrm{IV}}}(k'))_{\alpha'''}$$

$$= \sum_{\alpha'''} \int d^3k' \langle k|K_{\alpha''\alpha'''}(p^{\mu(1)})|k'\rangle (a_{\pm\alpha'}(k'))_{\alpha'''}. \tag{4.20}$$

The commutative kernel $K_{\alpha'',\alpha'''}$ is proportional to the unit operator in the $\alpha$-factor; since the observable $k$ is a function of both $\alpha$ and $\beta$, there is a spreading in momentum space, as illustrated in (4.20), under translations, which also couples different components[5]) ($\alpha'''$) of the fields $a_{\pm\alpha'}(k')$.

Defining

$$(\mathscr{D}(k, k'; A))_{\alpha'\alpha''} = \langle k|K_{\alpha'\alpha''}(A)|k'\rangle, \tag{4.21}$$

---

[5]) Note that in the one-particle quantum theory discussed in I, the general equations of motion for a state $|a'(\alpha')\rangle$ involved a sum over $\alpha'$. In the second quantized theory, the roles of label and component are reversed, as remarked in connection with (2.22).

(4.20) can be written

$$[a_{\pm\alpha'}(k), p^{\mu}] = \int d^3 k\, \mathscr{D}(k, k'; p^{\mu(1)})\, a_{\pm\alpha'}(k'). \tag{4.22}$$

Furthermore, defining

$$\mathscr{D}(x, x'; A) = \frac{1}{(2\pi)^3} \int \frac{d^3 k}{\sqrt{2k_0}} \frac{d^3 k'}{\sqrt{2k'_0}} \mathscr{D}(k, k'; A)\, e^{-ikx}\, e^{ik'x'}, \tag{4.23}$$

it follows from (4.22) that

$$[\varphi_{\alpha'}^{(+)}(x), p^{\mu}] = i \int d^3 x'\, \mathscr{D}(x, x'; p^{\mu(1)})\, \overleftrightarrow{\partial'_0}\, \varphi_{\alpha'}^{(+)}(x'), \tag{4.24}$$

where $\varphi_{\alpha'}^{(+)}(x)$ is the positive frequency part of $\varphi_{\alpha'}(x)$. Conjugating (4.22), one obtains for the negative frequency part

$$[\varphi_{\alpha'}^{(-)}(x), p^{\mu}] = -i \int d^3 x'\, \varphi_{\alpha'}^{(-)}(x')\, \overleftrightarrow{\partial'_0}\, \mathscr{D}(x', x; p^{\mu(1)}). \tag{4.25}$$

The discussion of broken symmetries given in I can be followed here as well; starting with (4.21) (for $A = p^{\mu(1)}$), the wave vector can be separated out, leaving a commutator.

The contribution of the wave vector leads to the expected derivative in (4.24) and (4.25), and the commutator contributions provide the mixing among the components of $\varphi_{\alpha'}(x)$ as well as a spreading over points in space-time.

Although the relation (4.24) is non-local, we remark that the positive (or negative) frequency part of the field $\varphi_{\alpha'}(x)$ still provides a representation for the translation group. The function $\mathscr{D}(x, x'; A)$ is linear in $A$, and according to (4.21) and (4.23),

$$\int d^3x'\, \mathscr{D}(x, x'; A)\, \overleftrightarrow{\partial'_0}\, \mathscr{D}(x', x''; B) = \mathscr{D}(x, x''; AB). \tag{4.26}$$

Hence (4.24) implies (since $\mathscr{D}(k, k'; 1) = \delta^3(k - k^1)$)

$$\varphi_{\alpha'}^{(+)}(x, \epsilon) = e^{ip\epsilon}\, \varphi_{\alpha'}^{(+)}(x)\, e^{-ip\epsilon} = i \int d^3 x'\, \mathscr{D}(x, x'; e^{ip^{(1)}\epsilon})\, \overleftrightarrow{\partial'_0}\, \varphi_{\alpha'}^{(+)}(x'), \tag{4.27}$$

so that

$$\varphi_{\alpha'}^{(+)}(x, \epsilon + \delta) = e^{ip(\epsilon+\delta)}\, \varphi_{\alpha'}^{(+)}(x)\, e^{-ip(\epsilon+\delta)}$$

$$= i \int d^3 x'\, d^3 x''\, \mathscr{D}(x, x'; e^{ip^{(1)}\epsilon})\, \overleftrightarrow{\partial'_0}\, \mathscr{D}(x', x''; e^{ip^{(1)}\delta})\, \overleftrightarrow{\partial''_0}\, \varphi^{(+)}(x''),$$

with which we can define the representations of this (4 parameter) group.

In a way analogous to the definition (4.23), we define the coordinate representation of the reproducing kernel as

$$\mathscr{K}_{\alpha'\alpha''}^{(+)}(x, x') = \frac{1}{(2\pi)^3} \int \frac{d^3 k}{\sqrt{2h_0}} \frac{d^3 k'}{\sqrt{2k'_0}} \langle k | \bar{M}(\alpha', \alpha'') | k' \rangle\, e^{-ikx}\, e^{ik'x'}. \tag{4.28}$$

Then

$$[\varphi_{\alpha'}(x), \varphi_{\alpha''}^{\dagger}(x')] = i\mathscr{K}_{\alpha'\alpha''}(x, x'), \tag{4.29}$$

where $\mathscr{H} = 2\mathrm{Im}\mathscr{H}^{(+)}$, and the unit operator in the vector indices (components) of the fields is suppressed. The Feynman propagator, i.e., the vacuum expectation value of the time-ordered product, is

$$\langle 0|T(\varphi_{\alpha'}(x)\,\varphi^{\dagger}_{\alpha''}(x'))|0\rangle = \theta(t-t')\,\mathscr{H}_{\alpha'\alpha''}(x,x') + \theta(t'-t)\,\mathscr{H}_{\alpha'\alpha''}(x,x')^{*}. \tag{4.30}$$

We remark that, since $A = \sum K_{\alpha'\alpha''}(A)\,\bar{M}(\alpha',\alpha'')$ [(4.7) of I],

$$\sum_{\alpha'\alpha''} \int d^3x''\,\mathscr{D}(x,x'';A)_{\alpha'\alpha''}\,i\overleftrightarrow{\partial_0''}\,\mathscr{H}^{(+)}_{\alpha'\alpha''}(x'',x')$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2k_0}}\,\frac{d^3k'}{\sqrt{2k_0'}}\,e^{-ikx}\,e^{ik'x'}\,\langle k|A|k'\rangle. \tag{4.31}$$

For $A = p^{\mu(1)}$, (4.31) is $i\,\partial^{\mu}\Delta_{+}(x-x')$, and for $A = 1$, it is just

$$\sum_{\alpha'} \mathscr{H}^{(+)}_{\alpha'\alpha'}(x,x') = \Delta_{+}(x-x'),$$

where

$$\Delta_{+}(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2k_0}\,e^{-ikx}$$

In terms of the local field

$$\varphi_0(x) = \int \frac{d^3k}{\sqrt{2k_0(2\pi)^3}}\,(e^{-ikx}\,a_{+0}(k) + e^{-ikx}\,a^{\dagger}_{-0}(k)),$$

we may express the field (4.1) as

$$(\varphi_{\alpha'}(x))_{\alpha''} = \int d^3x'\{\mathscr{H}_{\alpha',\alpha''}(x,x')\,i\overleftrightarrow{\partial_0'}\,\varphi_0(x') + \varphi_0(x')\,i\overleftrightarrow{\partial_0'}\,\mathscr{H}_{\alpha'',\alpha'}(x',x)\}. \tag{4.32}$$

The reproducing kernels, in coordinate representation, appear in (4.32) as smoothing or 'test' functions. The appearance of the labels of overcomplete families of states in the context of field theory in this manner has been mentioned by Klauder [8].

In the remaining paragraphs of this section, we shall discuss a class of currents and charges associated with the fields $\varphi_{\alpha'}(x)$. Let

$$j^{\mu}_{\alpha'}(x'\,\mathbf{\Gamma}) = i[\varphi^{\dagger}_{\alpha'}(x)\,\mathbf{\Gamma}\,\partial^{\mu}\,\varphi_{\alpha'}(x) - \partial^{\mu}\,\varphi^{\dagger}_{\alpha'}(x)\,\mathbf{\Gamma}\varphi_{\alpha'}(x)], \tag{4.33}$$

where $\mathbf{\Gamma}$ is a matrix in the suppressed indices $\alpha'', \alpha'''$. It follows from the Klein–Gordon equation that $\partial_{\mu} j^{\mu}_{\alpha'}(x,\mathbf{\Gamma}) = 0$, and therefore that the charges

$$Q^{\Gamma}_{\alpha'} = \int d^3x\,j^{0}_{\alpha'}(x,\mathbf{\Gamma}) = \int d^3x\,\varphi^{\dagger}_{\alpha'}(x)\,\mathbf{\Gamma}\,i\overleftrightarrow{\partial_0}\,\varphi_{\alpha'}(x) \tag{4.34}$$

are constant. In terms of the fields $a_{\pm\alpha'}(k)$,

$$Q^{\Gamma}_{\alpha'} = \int d^3k[a^{\dagger}_{+\alpha'}(k)\,\mathbf{\Gamma}a_{+\alpha'}(k) - a^{\dagger}_{-\alpha'}(k)\,\mathbf{\Gamma}a_{-\alpha'}(k)], \tag{4.35}$$

within a possibly infinite constant. With the help of (4.14), (4.35) can be written

$$Q_{\alpha'}^{\Gamma} = \int d^3k\, d^3k' [a_{+0}^{\dagger}(k) \langle k|\Gamma|k'\rangle\, a_{+0}(k') - a_{-0}^{\dagger}(k) \langle k|\Gamma|k'\rangle\, a_{-0}(k')], \tag{4.36}$$

independently of $\alpha'$, and

$$\Gamma = \sum_{\alpha'\alpha''} \mathbf{\Gamma}_{\alpha'\alpha''} \bar{M}(\alpha', \alpha'') \tag{4.37}$$

is an operator non-trivial only in the $\alpha$-factor. Although the charges $Q_{\alpha'}^{\Gamma}$, are independent of $\alpha'$, the currents (4.33) are not. With the relations (4.29), we find that

$$\begin{aligned} &[j_{\alpha'}^{0}(x, \mathbf{\Gamma}), j_{\alpha''}^{0}(x', \mathbf{\Gamma}')] \\ &= -i\{\varphi_{\alpha'}^{\dagger}(x)\, \mathbf{\Gamma}\, \overleftrightarrow{\partial_0}\, \mathscr{H}_{\alpha'\alpha''}(x, x')\, \overleftrightarrow{\partial_0'}\, \mathbf{\Gamma}'\, \varphi_{\alpha''}(x') \\ &\quad - \varphi_{\alpha''}^{\dagger}(x')\, \mathbf{\Gamma}'\, \overleftrightarrow{\partial_0'}\, \mathscr{H}_{\alpha''\alpha'}(x', x)\, \overleftrightarrow{\partial_0}\, \mathbf{\Gamma}\varphi_{\alpha'}(x)\}. \end{aligned} \tag{4.38}$$

One can require that $\mathscr{H}_{\alpha'\alpha''}(x,x')$ vanish for space-like displacements, but the bilocal algebra [9] (first derivatives in time of the bilocal currents in the closure of the algebra occur here since we are working with solutions of the Klein–Gordon equation) in light-like and null regions is determined, for the free fields, by the structure of ($\mathbf{\Gamma}$ and) the reproducing kernels.

## 5. Coherent States

To discuss the properties of coherent states, we consider the neutral boson field

$$\begin{aligned} \varphi_{\alpha'}(x) &= \int \frac{d^3k}{\sqrt{(2\pi)^3\, 2k_0}} \{e^{-ikx} a_{\alpha'}(k) + e^{ikx} a_{\alpha'}(k)^{\dagger}\} \\ &= a_{\alpha'}(x) + a_{\alpha'}(x)^{\dagger}. \end{aligned} \tag{5.1}$$

The coherent states for this field, defined by the simplest eigenvalue condition for $a_{\alpha'}(x)$, are just, as will be shown, the usual ones constructed with the help of the fields (2.21) ($a_0(x)$). In these states, one-particle observables (in second-quantized form) have the same expectation value as that of the corresponding uncontracted operators in the one-particle Hilbert space. It is possible, however, to construct coherent states using the field operators (2.22), for which the expectation value of one-particle observables corresponds to that of contracted operators, at finite or infinite 'temperature' for the thermal bath provided by unmeasured degrees of freedom.

From

$$|x\rangle = \int \frac{d^3k}{\sqrt{2k_0(2\pi)^3}}\, e^{ikx} |\vec{k}\rangle, \tag{5.2}$$

we may define (in the notation of I)

$$|x(\alpha')) = \int \frac{d^3k}{\sqrt{2k_0(2\pi)^3}}\, e^{ikx} |\vec{k}(\alpha')). \tag{5.3}$$

The wave function for a boson of the type (5.1), in the one-particle Hilbert space, is then

$$\varphi^{f}_{\alpha'}(x) = \int \frac{d^3 k}{\sqrt{2k_0(2\pi)^3}} \{e^{-ikx} f_{\alpha'}(k) + e^{ikx} f^{*}_{\alpha'}(k)\}, \tag{5.4}$$

where

$$f_{\alpha'}(k) = (k(\alpha')|f\rangle,$$

i.e.,

$$(f_{\alpha'}(k))_{\alpha''} = \langle k|\bar{M}(\alpha', \alpha'')|f\rangle. \tag{5.5}$$

In terms of the states (5.3),

$$\varphi^{f}_{\alpha'}(x) = (x(\alpha')|f\rangle + \langle f|x(\alpha')),$$

or

$$(\varphi^{f}_{\alpha'}(x))_{\alpha''} = 2\,\mathrm{Re}\,\langle x|\bar{M}(\alpha', \alpha'')|f\rangle. \tag{5.6}$$

In analogy with the electromagnetic case (see, for example, [10]), we ask for a state $|f\rangle_c$ such that

$${}_c\langle f|\varphi_{\alpha'}(x)|f\rangle_c = \varphi^{f}_{\alpha'}(x),$$

or

$$a_{\alpha'}(x)|f\rangle_c = (x(\alpha')|f\rangle . |f\rangle_c. \tag{5.7}$$

Since (from (2.22))

$$(a_{\alpha'}(x))_{\alpha''} = \sum_{a'} \langle x|\bar{M}(\alpha', \alpha'')|a'\rangle\, a_0(a'), \tag{5.8}$$

(5.7) will be satisfied if

$$a_0(a')|f\rangle_c = \langle a'|f\rangle . |f\rangle_c. \tag{5.9}$$

It therefore follows that $|f\rangle_c$ is of the usual form,

$$|f\rangle_c = e^{-\frac{1}{2}\|f\|^2} e^{a_0^{\dagger}(f)}|0\rangle, \tag{5.10}$$

where

$$a_0(f) = \sum_{a'} \langle f|a'\rangle\, a_0(a'). \tag{5.11}$$

The operator

$$\begin{aligned}(a_{\alpha'}(f))_{\alpha''} &= \sum_{a'} \langle f|\bar{M}(\alpha', \alpha'')|a'\rangle\, a_0(a') \\ &= \sum_{a'} {}_{\alpha''}(f(\alpha')|a'\rangle\, a_0(a')\end{aligned} \tag{5.12}$$

corresponds, however, to a special class of $|f\rangle$, i.e., those projected into $|f(\alpha'))$, where ((4.26) of I)

$$|f(\alpha'))_{\alpha''} = \bar{M}(\alpha'', \alpha')|f\rangle. \tag{5.13}$$

The coherent state (5.10), constructed for this special class of vectors, has the form

$$|f(x')_{\alpha''}\rangle_c = e^{-\frac{1}{2}\langle f|\bar{M}(\alpha')|f\rangle}\, e^{\sum_{a'} a_0^\dagger(a')\langle a'|\bar{M}(\alpha'',\alpha')|f\rangle}\, |0\rangle. \tag{5.14}$$

A one-particle observable of the form

$$\mathcal{O} = \sum_{a',a''} a_0^\dagger(a')\langle a'|\mathcal{O}^{(1)}|a''\rangle\, a_0(a'') \tag{5.15}$$

has the expectation value, in the state (5.10),

$${}_c\langle f|\mathcal{O}|f\rangle_c = \langle f|\mathcal{O}^{(1)}|f\rangle, \tag{5.16}$$

and, therefore, for the state (5.14),

$${}_c\langle f(\alpha')_{\alpha''}|\mathcal{O}|f(\alpha')_{\alpha''}\rangle_c = \langle f|\bar{M}(\alpha',\alpha'')\,\mathcal{O}^{(1)}\,\bar{M}(\alpha'',\alpha')|f\rangle. \tag{5.17}$$

Summing over $\alpha', \alpha''$ we find that

$$\begin{aligned}\sum_{\alpha',\alpha''} {}_c\langle f(\alpha')_{\alpha''}|\mathcal{O}|f(\alpha')_{\alpha''}\rangle_c &= \langle f|(\mathcal{O}^{(1)}\,\mathfrak{p})|f\rangle \\ &= \sum_{\alpha'} \mathrm{tr}(f(\alpha')|\mathcal{O}^{(1)}|f(\alpha')) \\ &= \mathrm{Tr}(\mathcal{O}^{(1)}\,\check{\rho}),\end{aligned} \tag{5.18}$$

where $\check{\rho}$ is the minimally mixed state $(|f\rangle\langle f|)\,\mathfrak{p}$.

According to section 4 of I, finite 'temperature' is accounted for by the presence of a density matrix (corresponding to the canonical ensemble) in the $\alpha$ subalgebra as a factor inside the $\alpha$-trace in the third member of (5.18).

Let $\kappa_{\alpha'\alpha''}$ be a matrix in the $\alpha$-subalgebra, and replace $a_{\alpha'}(f)$ in (5.12) by $\kappa a_{\alpha'}(f)$. Then (5.14) becomes

$$|(f(\alpha')\kappa)_{\alpha''}\rangle_c = e^{-\frac{1}{2}(\kappa\kappa^\dagger)_{\alpha''\alpha''}\langle f|\bar{M}(\alpha')|f\rangle}\, e^{\sum_{a',\alpha'''} a_0^\dagger(a')\kappa^*_{\alpha''\alpha'''}\langle a'|\bar{M}|\alpha''',\alpha')|f\rangle}|0\rangle \tag{5.19}$$

and the expectation value of $\mathcal{O}$ is (summing over $\alpha', \alpha''$)

$$\begin{aligned}&\sum_{\alpha',\alpha''} {}_c\langle (f(\alpha')\,\kappa)_{\alpha''}|\mathcal{O}|(f(\alpha')\,\kappa)_{\alpha''}\rangle_c \\ &= \sum_{\alpha',\alpha'',\alpha'''} \kappa_{\alpha''\alpha'''}\,{}_{\alpha'''}(f(\alpha')|\mathcal{O}^{(1)}|f(\alpha'))_{\alpha^{\mathrm{IV}}}\,\kappa^*_{\alpha''\alpha^{\mathrm{IV}}} \\ &= \sum_{\alpha'} \mathrm{tr}((f(\alpha')|\mathcal{O}^{(1)}|f(\alpha'))(\kappa^\dagger\kappa)).\end{aligned} \tag{5.20}$$

Taking for $\kappa$ the positive square root of $\rho_\alpha(T)$, (5.20) corresponds to the finite 'temperature' version of (5.18).

It was remarked in I that the states which are irreducible representations of a physical symmetry group retain their transformation properties under contraction if the group is a direct product of groups acting on the $\alpha$ (unmeasured variables) factor and the $\beta$ (measured variables) factor, i.e., if $g = g_\alpha \otimes g_\beta$. Under these conditions, the $\kappa$ appearing in (5.20) transforms like a second-rank tensor. We may also construct a coherent state using vector parameters, corresponding to a linear mapping of the algebraic vector space into a vector space of the usual type, selecting a direction within the manifold associated with each algebraic vector. Let $\{p_{\alpha'}\} = p$ be such a vector, and let us replace $a_{\alpha'}(f)$ in (5.12) by

$$p \cdot a_{\alpha'}(f) = \sum_{\alpha''} p_{\alpha''} (a_{\alpha'}(f))_{\alpha''}. \tag{5.21}$$

Then

$$|f(\alpha') \cdot p\rangle_c = e^{-\frac{1}{2}|p|^2 \langle f|\bar{M}(\alpha')|f\rangle} \, e^{\sum_{a'} a_0^\dagger(a')\langle a'|f(\alpha')) \cdot p^*} |0\rangle, \tag{5.22}$$

where $|p|^2 = \sum_{\alpha'} |p_{\alpha'}|^2$. For this state,

$$a_0(a') |f(\alpha') \cdot p\rangle_c = \langle a'|f(\alpha')) \cdot p^* \, |f(\alpha') \cdot p\rangle_c, \tag{5.23}$$

and

$$\sum_{\alpha'} {}_c\langle f(\alpha') \cdot p|\mathcal{O}|f(\alpha') \cdot p\rangle_c = \sum_{\alpha'} p \cdot (f(\alpha')|\mathcal{O}^{(1)}|f(\alpha')) \cdot p^*$$

$$= \sum_{\alpha'', \alpha'''} \langle f|K_{\alpha'', \alpha'''}(\mathcal{O}^{(1)})|f\rangle \, p_{\alpha''} \, p^*_{\alpha'''}. \tag{5.24}$$

The Lorentz group is a good symmetry of the type under consideration. As a particularly interesting example, let us consider $\{\alpha'\}$ to be in one-to-one correspondence with the Lorentz tensor indices $\mu(0, 1, 2, 3)$; since this representation is not unitary, we must take care to distinguish covariant and contravariant indices. Then, (2.22) is of the form

$$(a_\mu(a'))_\nu = \sum_{a''} \langle a'|\bar{M}_{\mu\nu}|a''\rangle \, a_0(a''). \tag{5.25}$$

For Lorentz covarience of the commutation relations, we must assume the relation

$$\bar{M}_{\mu\nu} \bar{M}_{\nu'\mu'} = g_{\nu\nu'} \bar{M}_{\mu\mu'}, \tag{5.26}$$

directly reflecting the non-unitary nature of the finite dimensional representation. With (5.26), we find that

$$[(a_\mu(a'))_\nu, (a^\dagger_{\mu'}(a''))_{\nu'}] = g_{\nu\nu'} \langle a'|\bar{M}_{\mu\mu'}|a''\rangle. \tag{5.27}$$

With the choice $(+, -, -, -)$ for the metric tensor, negative norm states are generated by $(a^\dagger_\mu(a'))_\nu$ for $\nu = 1, 2, 3$, and positive norm states for $\nu = 0$, independently of $\mu$.

The fields $a_\mu(a')$ are not identical to the Veneziano field operators [3] since they transform like second-rank tensors, but any even or odd angular momentum can be generated by products. The operator generating the coherent state (5.22) from the vacuum is analogous to the one entering bilinearly in the vertex function of the Veneziano theory ($p_{\alpha'}$ is to be replaced by $p_{\mu}$ in this case).

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