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On a Recent Paper of Amrein, Georgescu and Jauch

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Abstract. Condition (D) of a recent paper of Amrein, Georgescu and Jauch is shown to be a consequence of their conditions (θ) and (A).

In a recent paper [1] on quantum scattering theory W. O. Amrein, V. Georgescu and J. M. Jauch were forced to introduce an objectionable assumption. They conjectured that their assumption, which they called condition (D), is in fact a consequence of their conditions (θ) and (A). The purpose of this paper is to prove that their conjecture is true.

Suppose, therefore, that $\mathscr{H}$ denotes a Hilbert space and that the following assumptions are true.

(θ) H_0 and $H = H_0 + V$ are self-adjoint operators on a common dense domain

$$D_H = D_{H_0} \equiv \mathscr{D}$$

and V is symmetric on a domain $D_V \supset \mathscr{D}$.

(A) For all $\psi \epsilon \mathscr{H}$ the strong limits

$$\underset{t \to \mp\infty}{\text{s-lim}}\, V_t^* U_t \psi = \psi_\pm \equiv \Omega_\pm \psi$$

exist where $V_t = e^{-iHt}$, $U_t = e^{-iH_0 t}$.

From these assumptions two preliminary results follow.

Lemma 1. *If conditions (θ) and (A) are satisfied, the equation*

$$\Omega_+ \psi = \underset{\epsilon \downarrow 0}{\text{s-lim}}\, \Omega_\epsilon \psi,$$

where

$$\Omega_\epsilon \psi \equiv \epsilon \int_{-\infty}^{0} dt\, e^{\epsilon t}\, V_t^*\, U_t\, \psi,$$

is true for every $\psi \epsilon \mathscr{H}$.

Proof. This is proved in Theorem 0 of [1]. Q.E.D.

Lemma 2. *If condition* (θ) *is satisfied, then for every* $z \epsilon C$, $\operatorname{Im} z \neq 0$, *the operators* $H(z-H)^{-1}$, $H(z-H_0)^{-1}$, $H_0(z-H)^{-1}$ and $H_0(z-H_0)^{-1}$ *are defined and bounded on* $\mathscr{H}$.

Proof. This is an easy consequence of Lemma 1 of [1]. Q.E.D.

Using these two results the following theorem can be proved.

Theorem 1. *If conditions* (θ) *and* (A) *are satisfied, then for every* $\epsilon > 0$ *the operators* Ω_+ *and* Ω_ϵ *map* $\mathscr{D}$ *into* $\mathscr{D}$. *Moreover, there exist non-negative constants* α, β *and* γ *such that the inequality*

$$\|V(\Omega_+ - \Omega_\epsilon)\psi\| \leqslant \alpha \|H_0 \psi\| + (\beta + \epsilon\gamma)\|\psi\| \tag{1}$$

is true for all $\psi \epsilon \mathscr{D}$. *Finally, the equation*

$$\operatorname*{s-lim}_{\epsilon \downarrow 0} V(\Omega_+ - \Omega_\epsilon)\psi = 0 \tag{2}$$

is true for all $\psi \epsilon \mathscr{D}$.

Proof. Let ψ be an arbitrary, but fixed, vector in $\mathscr{D}$.

Because H_0 and $U_t = e^{-iH_0 t}$ commute on $\mathscr{D}$, the equation

$$\Omega_\epsilon H_0 \psi \equiv \epsilon \int_{-\infty}^{0} dt\, e^{\epsilon t} V_t^* U_t H_0 \psi = \epsilon \int_{-\infty}^{0} dt\, e^{\epsilon t} V_t^* H_0 U_t \psi \tag{3}$$

holds for all $\epsilon > 0$. Existence of the Bochner integral in equation (3) is implied (Theorem 3.7.4 of [2]) by the strong continuity and boundedness of the operator $V_t^* U_t$.

It is also true that $\Omega_\epsilon \psi \epsilon \mathscr{D}$ and that the equation

$$H\Omega_\epsilon \psi = \epsilon \int_{-\infty}^{0} dt\, e^{\epsilon t} V_t^* H U_t \psi \tag{4}$$

holds for all $\epsilon > 0$. To prove this one observes that $U_t \psi \epsilon \mathscr{D}$ and hence that the vector $V_t^* H U_t \psi = H V_t^* U_t \psi$ is well defined. If z is a fixed complex number with nonzero imaginary part, this vector can be written in the form

$$V_t^* H U_t \psi = V_t^* B(z)(z - H_0) U_t \psi = V_t B(z) U_t (z - H_0)\psi,$$

where $B(z) \equiv H(z-H_0)^{-1}$. The operator $B(z)$ is, by Lemma 2, defined and bounded on all of $\mathscr{H}$. It follows that the operator $V_t^* B(z) U_t$ is continuous and bounded on $\mathscr{H}$ and hence (Theorem 3.7.4 of [2]) that the Bochner integral in equation (4) exists. Theorem 3.7.12 of [2] and the fact that H is closed now imply that $\Omega_\epsilon \psi \epsilon \mathscr{D}$ and that equation (4) is true.

Equations (3) and (4) imply that the equation

$$(H\Omega_\epsilon - \Omega_\epsilon H_0) = \epsilon \int_{-\infty}^{0} dt\, e^{\epsilon t} V_t^* (H - H_0) U_t \psi \tag{5}$$

is true for all $\epsilon > 0$. A consequence of Lemma 2 of [1] is that this integral can be evaluated by partial integration, with the result that

$$(H\Omega_\epsilon - \Omega_\epsilon H_0)\,\psi = i\epsilon(\Omega_\epsilon - I)\,\psi,$$

where I is the identity. It now follows from the boundedness of Ω_ϵ, $\|\Omega_\epsilon\psi\| \leqslant \|\psi\|$, that the inequality

$$\|(H\Omega_\epsilon - \Omega_\epsilon H_0)\,\psi\| < 2\epsilon\|\psi\| \tag{6}$$

holds for all $\epsilon > 0$.

Lemma 1 implies that the vectors $\Omega_\epsilon H_0\psi$ converge to $\Omega_+ H_0\psi$ as $\epsilon \to 0$. Inequality (6) therefore implies that $H\Omega_\epsilon\psi$ converges to the same vector $\Omega_+ H_0\psi$ as $\epsilon \to 0$. Because (by Lemma 1) the vectors $\Omega_\epsilon\psi$ converge to $\Omega_+\psi$, it now follows from the fact that H is a closed operator that $\Omega_+\psi\epsilon\mathscr{D}$ and that the intertwining relation $(H\Omega_+ - \Omega_+ H_0)\,\psi = 0$ is true.

Because $(\Omega_+ - \Omega_\epsilon)\,\psi$ is now known to lie in $\mathscr{D}$ for every $\epsilon > 0$, one can speak of the vector $V(\Omega_+ - \Omega_\epsilon)\,\psi = (H - H_0)(\Omega_+ - \Omega_\epsilon)\,\psi$. If ζ is a fixed complex number with nonzero imaginary part, this vector can be written in the form

$$V(\Omega_+ - \Omega_\epsilon)\,\psi = C(\zeta)\,(\zeta - H)(\Omega_+ - \Omega_\epsilon)\,\psi.$$

The operator $C(\zeta) \equiv V(\zeta - H)^{-1} = (H - H_0)(\zeta - H)^{-1}$ is, by Lemma 2, defined and bounded on $\mathscr{H}$. Hence, there is a positive constant B, independent of ϵ, such that the inequality

$$\|V(\Omega_+ - \Omega_\epsilon)\,\psi\| \leqslant B\|(\zeta - H)(\Omega_+ - \Omega_\epsilon)\,\psi\| \tag{7}$$

is true for all $\epsilon > 0$.

The intertwining property implies that

$$(\zeta - H)(\Omega_+ - \Omega_\epsilon)\,\psi = (\Omega_+ - \Omega_\epsilon)(\zeta - H_0)\,\psi + (H\Omega_\epsilon - \Omega_\epsilon H_0)\,\psi$$

and hence that (using inequalities (6) and (7))

$$\|V(\Omega_+ - \Omega_\epsilon)\,\psi\| \leqslant B\{\|(\Omega_+ - \Omega_\epsilon)(\zeta - H_0)\,\psi\| + 2\epsilon\|\psi\|\}. \tag{8}$$

The fact that for all $\epsilon > 0$ the vectors $\Omega_+\psi$ and $\Omega_\epsilon\psi$ lie in $\mathscr{D}$ has been established. Inequality (8), together with the previously used inequality $\|\Omega_\epsilon\psi\| \leqslant \|\psi\|$, implies inequality (1) with $\alpha = 2B$, $\beta = 2B|\zeta|$ and $\gamma = 2B$. Equation (2) is implied by inequality (8) and Lemma 1. The theorem has thus been proved for a fixed ψ. Since ψ was arbitrary, the proof is complete. Q.E.D.

The final result of this paper is the following (cf. Theorem 9(a) of [1]).

Theorem 2. *If the conditions* (θ) *and* (A) *are satisfied, then for all* $\epsilon_1 > 0$ *the equation*

$$\operatorname*{s-lim}_{\epsilon_2 \downarrow 0} \int_0^\infty dt\, e^{-\epsilon_1 t}[U_t^* V(\Omega_+ - \Omega_{\epsilon_2})\,U_t + U_t V(\Omega_+ - \Omega_{\epsilon_2})\,U_t^*]\,\psi = 0 \tag{9}$$

is true for every $\psi\epsilon\mathscr{D}$.

Proof. Inequality (1) of Theorem 1 implies that the integrand in equation (9) is continuous and bounded in norm by the function $2e^{-\epsilon_1 t}[\alpha\|H_0\psi\| + (\beta + \epsilon_2\gamma)\|\psi\|]$. This function is Lebesque integrable for all $\epsilon_1, \epsilon_2 > 0$, with the consequence that the Bochner integral in equation (9) exists (Theorem 3.7.4 of [2]) for all $\epsilon_1, \epsilon_2 > 0$. Equation (2) of Theorem 1 implies further that for every $t \epsilon [0, \infty)$ the strong limit as $\epsilon_2 \to 0$ of the integrand is zero. The Lebesque-dominated convergence theorem for Bochner integrals (Theorem 3.7.9 of [2]) therefore applies, with the result that equation (9) is true for all $\epsilon_1 > 0$ and all $\psi \epsilon \mathscr{D}$. Q.E.D.

Equation (9) is precisely the condition (D) of [1]. The conjecture of Amrein, Georgescu and Jauch that conditions (θ) and (A) imply condition (D) is thus proved.

A generalization of [1] and this paper to multichannel scattering is relatively straightforward. A detailed report on these calculations, as well as some new results, appears in [3].

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