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On the Uniqueness of the Energy Density in the Infinite Volume Limit for Quantum Field Models

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Abstract. We isolate two properties of the vacuum energy E_V (for volume V) that are sufficient to ensure the existence and uniqueness of $\lim_{V\to\infty} E_V/V$. The first property has been recently verified by Glimm and Jaffe for the $P(\varphi)_2$ quantum field model. The second property is shown to hold in a simplified $P(\varphi)_2$ model where the free field energy H_0 is replaced by the number operator N.

The construction of quantum field models has progressed rapidly in the last few years [3]. Such models are obtained by starting from cut-off Hamiltonians $H(g)$ with a space cut-off g, and by taking the infinite volume limit, i.e. the limit $g \to 1$.

A function g is called a space cut-off if $g \in L^2(\mathbb{R})$, $0 \leqslant g(x) \leqslant 1$ for $x \in \mathbb{R}$, and g is of compact support. (Sometimes one requires additional smoothness properties for g.) We note that $g_a(\cdot) \equiv g(\cdot - a)$ is also a space cut-off for all $a \in \mathbb{R}$. The operators $H(g)$ are given by

$$H(g) = H_0 + \lambda H_I(g) + \text{counterterms}, \tag{1}$$

where H_0 is the free Hamiltonian and $H_I(g)$ is the interaction. The parameter λ is the coupling constant and is supposed to be positive. In the $P(\varphi)_2$ models $H(g)$ is given by

$$H_I(g) = \int : P(\varphi(x)) : g(x)\, dx, \qquad \text{counterterms} = 0, \tag{1'}$$

where $P(\xi)$ is a polynomial of even degree with real coefficients, the coefficient of the leading term is one. The symbol : : denotes Wick ordering with respect to the free vacuum. In this case the operators $H(g)$ are known to be self-adjoint, semi-bounded linear operators in the Fock space $\mathscr{F}$ of a free boson field $\varphi(x)$ of mass $m > 0$.

Although a great deal is known about the theory in the infinite volume limit [1, 2], nothing is known about uniqueness. The simplest object to study is the ground state energy $E(g)$, the minimum eigenvalue of $H(g)$. Our purpose is to study its behavior when g tends to 1. (The convergence of $E(g)$/vol. supp. g should yield convergence of ω_g as $g \to 1$.) The following theorem is general and model independent.

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We introduce some notation. For a given space cut-off g let V_g = support of g, $|V_g|$ = measure of V_g, and

$$V'_g = \{x \in \mathbb{R};\, g(x) = 1\}, \qquad V''_g = \{x \in \mathbb{R};\, 0 < g(x) < 1\}.$$

Also for a given sequence $\{g_n\}_{n \in Z^+}$ of space cut-offs, we write $V_n = V_{g_n}$, etc. Let $\mathscr{C}$ denote the convex set of all space cut-off functions and let $E(\cdot)$ be any real valued function on $\mathscr{C}$ which is translation invariant, i.e. $E(g_a) = E(g)$ for all $a \in \mathbb{R}$ and $g \in \mathscr{C}$.

Definition: Property P. *We say $E(\cdot)$ has property P if there exists $c > 0$ such that for all $g, h \in \mathscr{C}$ with $g + h \in \mathscr{C}$*

$$|E(g+h) - E(h)| \leqslant c(|V_g| + 1).$$

Note that (P) implies the linear bound on $|E(g)|$, namely

$$|E(g)| \leqslant c(|V_g| + 1). \tag{2}$$

Definition: Property S. *We say that $E(\cdot)$ has property S if there exists $\Delta_0 > 0$ and a monotonically decreasing function $\rho: [\Delta_0, \infty) \to \mathbb{R}^+$, with $\lim_{x\to\infty} \rho(x) = 0$ such that*

$$\sum_{i \in I} E(g_i) \leqslant E\left(\sum_{i \in I} g_i\right) + \rho(\Delta)\left(\sum_{i \in I} |V_i| + 1\right),$$

for all finite families $\{g_i\}_{i \in I}$ in $\mathscr{C}$ with $\sum_{i \in I} g_i \in \mathscr{C}$ where each V_i is an interval and $\Delta = \inf \operatorname{dist}(V_i, V_j) > \Delta_0$.

Remarks. If $E(g)$ is the ground state energy for the $P(\varphi)_2$ Hamiltonian $H(g)$, given by (1) and (1′) then property P is known to hold, see Glimm and Jaffe [2]. Property S has not yet been established in that case. However, we will show below that S holds for the ground state energy of a modified Hamiltonian

$$H(g) = N + \lambda \int : P(\varphi(x)) : dx, \tag{3}$$

where N is the number operator. We note that in this case $E(g)$ is a simple eigenvalue of $H(g)$, satisfying the estimate (2) for some $c > 0$. More generally $E(g)$ has property P [2].

For the discussion of the infinite volume limit we now make precise the way we will let the space cut-off g tend to 1.

Definition. *A sequence $\{g_n\}_{n \in Z^+}$ of space cut-offs tends to 1 if*

a) V'_n is an interval $[\alpha_n, \beta_n]$ for all $n \in Z^+$, and

b) $|V_n| \to \infty$ and $|V''_n| \cdot |V_n|^{-1} \to 0$ as $n \to \infty$.

Condition a) may be weakened, but for simplicity we will work with this definition. The main result of this paper is the

Theorem. *Let $E(\cdot)$ be any translation invariant real valued function on the set $\mathscr{C}$ of all cut-offs g which has properties P and S. Then for any sequence $\{g_n\}_{n\in Z^+}$ in $\mathscr{C}$, tending to* 1, *the limit*

$$e = \lim_{n\to\infty} E(g_n)|V_n|^{-1}$$

exists and is independent of the special choice of the sequence.

If $E(g)$ is the ground state energy for a Hamiltonian $H(g)$, then the theorem gives sufficient conditions for the uniqueness of the energy density in the infinite volume limit. In particular e is unique if $H(g)$ is given by (3).

Corollary. *Let $e = e(\lambda)$ denote the infinite volume energy density for the ground state obtained from a $P(\varphi)_2$ Hamiltonian or from a Hamiltonian given by* (3), *and assume uniqueness. Then $e(\lambda)$ is a convex function of the coupling constant for non-negative values of λ.*

The corollary implies that $e(\lambda)$ is continuous in λ. It would be interesting to know for the $P(\varphi)_2$ model, where $e(\lambda) \leqslant 0$, whether $e(\lambda) < 0$. At least we may conclude that if $e(\lambda) < 0$ for some $\lambda_0 > 0$, then $e(\lambda) < 0$ for all $\lambda \geqslant \lambda_0$.

To prove the corollary it is of course sufficient to show that $E(g) = E(\lambda, g)$ is a convex function of λ for any space cut-off g. However, this follows from

$$\begin{aligned} &E(\alpha\lambda_1 + (1-\alpha)\lambda_2, g) \\ &\quad = (\Omega(\alpha\lambda_1 + (1-\alpha)\lambda_2, g), \{\alpha H(\lambda_1, g) + (1-\alpha) H(\lambda_2, g)\}\, \Omega(\alpha\lambda_1 + (1-\alpha)\lambda_2, g)) \\ &\quad \geqslant \alpha E(\lambda_1, g) + (1-\alpha) E(\lambda_2, g), \end{aligned}$$

where $\Omega(\alpha\lambda_1 + (1-\alpha)\lambda_2, g)$ is the ground state of the Hamiltonian

$$H(\alpha\lambda_1 + (1-\alpha)\lambda_2, g) = H_0 + (\alpha\lambda_1 + (1-\alpha)\lambda_2) H_I(g), \qquad \text{or}$$

$$H(\alpha\lambda_1 + (1-\alpha)\lambda_2, g) = N + (\alpha\lambda_1 + (1-\alpha)\lambda_2) H_I(g); \quad 0 \leqslant \alpha \leqslant 1.$$

Proof of the theorem: We use arguments familiar from discussions of the thermodynamic limit of the free energy density of continuous systems. In statistical mechanics the perturbation property P is replaced by a monotonicity property, and the subadditivity property S follows from a temperedness condition on the potential, see e.g. [4], chapter III and literature quoted there.

We start from a special sequence of space cut-offs. Denote by $\xi_i(x)$ the characteristic function of the interval $[-i, +i]$, $i \in Z^+$. Then the linear bound (2) gives the existence of a subsequence ξ_{i_k}, $1 \leqslant i_k < i_{k+1}$, such that $\lim_{k\to\infty} E(\xi_{i_k})\,(2i_k)^{-1} = e$ for some $e \in [-c, c]$. Let us denote ξ_{i_k} by χ_k and $[-i_k, i_k]$ by W_k. For any $\epsilon > 0$ there is a $K(\epsilon)$, such that for all $k \geqslant K(\epsilon)$, c, Δ_0 and $\rho(x)$ as in (P) and (S),

$$|E(\chi_k)|W_k|^{-1} - e| < \epsilon, \qquad 2c|W_k|^{-1/2} < \epsilon,$$

$$2\rho(|W_k|^{1/2}) < \epsilon, \qquad |W_k|^{1/2} \geqslant \Delta_0. \tag{5}$$

We abbreviate $\chi_{K(\epsilon)}$ and $W_{K(\epsilon)}$ by χ_ϵ and W_ϵ respectively. Now we take any sequence of space cut-offs $g_n(\cdot)$ which tends to 1. We assert that

$$E(g_n)|V_n|^{-1} \to e \qquad \text{as } n \to \infty. \tag{6}$$

This of course implies the theorem. The proof of (6) involves two steps. The first step is to bound $E(g_n)|V_n|^{-1}$ from below by

$$E(g_n)|V_n|^{-1} - e \geqslant -7\epsilon. \tag{7}$$

First choose $N(\epsilon)$ so large that for all $n > N(\epsilon)$

$$|V_n'| > |W_\epsilon|, 2c(|V_n''| + 1)|V_n|^{-1} < \epsilon, \qquad 2c|W_\epsilon|\,|V_n|^{-1} < \epsilon. \tag{8}$$

Such an $N(\epsilon)$ always exists since g_n tends to 1. Now we fill V_n' with translates W_ϵ^i of W_ϵ such that they are at a mutual distance of at least $|W_\epsilon|^{1/2}$. More precisely we set

$$V_n' = \left(\bigcup_{i=1}^{r} W_\epsilon^i\right) \cup \left(\bigcup_{i=1}^{r-1} \Delta_\epsilon^i\right) \cup R_1, \tag{9}$$

where Δ_ϵ^i is the interval separating W_ϵ^i from W_ϵ^{i+1} $(i = 1, 2, \ldots r - 1)$ and is supposed to be of length $|W_\epsilon|^{1/2}$. R_1 is the 'remainder' and with a suitable choice of r we obtain

$$0 \leqslant |R_1| \leqslant |W_\epsilon| + |W_\epsilon|^{1/2}. \tag{10}$$

Corresponding to (9) there is a decomposition of g_n

$$g_n(x) = \sum_{i=1}^{r} \chi_\epsilon^i(x) + h(x),$$

where χ_ϵ^i is the characteristic function of W_ϵ^i and $h(x) = g(x)$ on $(\bigcup_{i=1}^{r-1} \Delta_\epsilon^i) \cup R_1 \cup V_n''$ and zero otherwise. Note that due to the definition of Δ_ϵ and due to (10), the measure of the support of h is smaller than $r|W_\epsilon|^{1/2} + |W_\epsilon| + |V_n''|$. Thus we get using property P

$$\begin{aligned} E(g_n)|V_n|^{-1} &\geqslant E\left(\sum_1^r \chi_\epsilon^i\right)|V_n|^{-1} - c(r|W_\epsilon|^{1/2} + |W_\epsilon| + |V_n''| + 1)|V_n|^{-1} \\ &\geqslant E\left(\sum_1^r \chi_\epsilon^i\right)(r|W_\epsilon|)^{-1} - \left|E\left(\sum_1^r \chi_\epsilon^i\right)\right|[(r|W_\epsilon|)^{-1} - |V_n|^{-1}] \\ &\quad - c(r|W_\epsilon|^{1/2} + |W_\epsilon| + |V_n''| + 1)|V_n|^{-1}. \end{aligned} \tag{11}$$

Furthermore, using property P together with (5), (8) and (9) we get

$$\begin{aligned} \left|E\left(\sum_1^r \chi_\epsilon^i\right)\right|[r|W_\epsilon|^{-1} - |V_n|^{-1}] &\leqslant 2cr|W_\epsilon|(|V_n| - r|W_\epsilon|)(r|W_\epsilon|\,|V_n|)^{-1} \\ &\leqslant 2c(r|W_\epsilon|^{1/2} + |W_\epsilon| + |V_n''|)|V_n|^{-1} \\ &\leqslant 3\epsilon. \end{aligned} \tag{12}$$

Inserting (12) into (11) and using property S, (5) and the translation invariance $E(\cdot)$ we get

$$E(g_n)|V_n|^{-1} \geqslant E(\chi_\epsilon)|W_\epsilon|^{-1} - 6\epsilon \geqslant e - 7\epsilon,$$

which implies (7).

The second step is to bound $E(g_n)|V_n|^{-1}$ from above by

$$8\epsilon \geqslant E(g_n)|V_n|^{-1} - e. \tag{13}$$

Take any $n > N(\epsilon)$ and keep it fixed. Then pick a $k > K(\epsilon)$ (see (5)), such that

$$W_k = \left(\bigcup_{i=1}^{l} V_n^i\right) \cup \left(\bigcup_{i=1}^{l-1} \Delta_n^i\right) \cup R_2, \tag{14}$$

with $l > 2c\epsilon^{-1}$. V_n^i are intervals of length $|V_n|$, V_n^i being separated from V_n^{i+1} by the interval Δ_n^i of length $|V_n|^{1/2}$. The remainder R_2 is estimated by $0 \leqslant |R_2| \leqslant |V_n| + |V_n|^{1/2}$. The decomposition (14) and the same sequence of arguements as above leads to the following chain of inequalities (χ_n^i denotes the characteristic function of V_n^i):

$$\begin{aligned} E(\chi_k)|W_k|^{-1} &\geqslant E\left(\sum_1^l \chi_n^i\right)|W_k|^{-1} - 2c(l|V_n|^{1/2} + |V_n|)(l|V_n|)^{-1} \\ &\geqslant E\left(\sum_1^l \chi_n^i\right)(l|V_n|)^{-1} - \left|E\left(\sum_1^l \chi_n^i\right)\right|[(l|V_n|)^{-1} - |W_k|^{-1}] \\ &\quad - 2c(l|V_n|^{1/2} + |V_n|)(l|V_n)^{-1} \\ &\geqslant E(\chi_n^i)|V_n|^{-1} - 6\epsilon \\ &\geqslant E(g_n)|V_n|^{-1} - c(|V_n''| + 1) - 6\epsilon \\ &\geqslant E(g_n)|V_n|^{-1} - 7\epsilon, \end{aligned}$$

and (13) follows after another application of (5).

Since ϵ is arbitrary, inequalities (7) and (13) prove the relation (6) and hence the theorem.

The remaining part of this work will be devoted to the proof that the ground state energy for the Hamiltonian given by (3) satisfies property S. More precisely we have the

Proposition. *The ground state energy $E(g)$ for the Hamiltonian $H(g)$ defined by* (3) *satisfies S with $\Delta_0 = 3$ and*

$$\rho(\Delta) = c_1 \exp\left(-\frac{m}{4}\Delta\right)$$

for some $c_1 < \infty$, where m is the mass of the free boson.

For the proof we use certain localization projection operators in the one particle space which have been introduced by *B*. Simon [5]. We recall the definition: Let the one particle space be described by $\mathscr{K} = L^2(\mathbb{R})$. For any interval $J = [\alpha, \beta]$ of $\mathbb{R}$ with $-\infty < \alpha < \beta < \infty$, let $\mathscr{K}_J$ be the closure of the linear subspace consisting of all $f \in \mathscr{S}(\mathbb{R})$, such that $\operatorname{supp} \mu_x^{1/2} f$ is a compact subset of J. The operator $\mu_x^{1/2}$ is multiplication by $\mu^{1/2}(p) = (m^2 + p^2)^{1/4}$ in the Fourier transform space, i.e. $\widetilde{\mu_x^{1/2} f}(p) = \mu^{1/2}(p)\tilde{f}(p)$, where $\sim$ denotes the Fourier transform. Let P_J be the orthogonal projection onto $\mathscr{K}_J$ and let $\mathscr{K}_J^{\perp}$ be the orthogonal complement of $\mathscr{K}_J$. Denote by $\mathscr{F}$, $\mathscr{F}_J$, $\mathscr{F}_{J\perp}$ the Fock spaces built with the one particle spaces $\mathscr{K}$, $\mathscr{K}_J$ and $\mathscr{K}_J^{\perp}$ respectively. Then $\mathscr{F} = \mathscr{F}_J \times \mathscr{F}_{J\perp}$ and $H_I(g) = H_I(g)|\mathscr{F}_J \times 1$ if $\operatorname{supp} g \subset J$. Likewise $d\Gamma(P_J) = N|\mathscr{F}_J \times 1$, where $d\Gamma(\cdot)$ denotes the second quantization of a one particle operator. Note that $N = d\Gamma(1)$. We improve Theorem III.1 in [5] to yield the

Lemma. *There exists* $c_2 < \infty$ *such that for any finite set* $\{J_i\}_{i\in I}$ *of intervals with* $\Delta = \inf_{i\neq j} \operatorname{dist}(J_i, J_j) \geqslant 3$

$$\|\sum_{i\in I} P_{j_i}\| \leqslant 1 + c_2 \exp\left(-\frac{m}{2}\Delta\right).$$

The proof of the proposition is now straightforward. Let $\{g_i\}_{i\in I}$ be as in (S) and set $P_i = P_{V_{g_i}}$. Then

$$E(g_i) = \inf \operatorname{spec} H(g_i) \leqslant \inf \operatorname{spec} (d\Gamma(P_i) + \lambda H_I(g_i)) \leqslant d\Gamma(P_i) + \lambda H_I(g_i).$$

Since biquantization preserves positivity, summing yields

$$\sum_{i\in I} E(g_i) \leqslant d\Gamma\left(\mathbb{1}\left(1 + c_2 \exp\left(-\frac{m}{2}\Delta\right)\right)\right) + \lambda H_I\left(\sum_{i\in I} g_i\right)$$

$$= H\left(\sum_{i\in I} g_i\right) + c_2 \exp\left(-\frac{m}{2}\Delta\right) N.$$

In particular if we take the expectation values in the ground state $\Omega \sum g_i$ of $H(\sum_{i\in I} g_i)$, then

$$\sum_{i\in I} E(g_i) \leqslant E\left(\sum_{i\in I} g_i\right) + c_2 \exp\left(-\frac{m}{2}\Delta\right) (\Omega_{\Sigma g_i}, N\Omega_{\Sigma g_i}). \tag{15}$$

To estimate the error term in (15), we write [1]

$$(\Omega_{\Sigma g_i}, N\Omega_{\Sigma g_i}) = (\Omega_{\Sigma g_i}, 2H(\textstyle\sum g_i) - H(2\sum g_i)\, \Omega_{\Sigma g_i})$$

$$\leqslant c'\left(\sum_{i\in I} |V_{g_i}| + 1\right).$$

The last inequality follows from (2) and the prime on c' indicates that we have taken (2) for the coupling constant $\lambda' = 2\lambda$. This proves the proposition.

Proof of the lemma: For convenience we assume that I is the set $\{1, 2, \ldots |I|\}$ and that the interval J_{i+1} lies to the right of J_i. We complete the set $\{J_i\}_{i\in I}$ to an infinite set of intervals $\{J_i\}_{i\in Z}$, such that we still have $\Delta = \inf_{i\neq j} \operatorname{dist}(J_i, J_j) \geqslant 3$, with J_{i+1} to the right of J_i. Then

$$\Delta_{ij} = \operatorname{dist}(J_i, J_j) \geqslant |i-j|\Delta.$$

It has been shown by B. Simon ([5], Theorem A.2) that if $\{P_i\}_{i\in Z}$ is a family of projections on a Hilbert space and if $d_{ij} = \|P_i P_j\|$ is the matrix of a bounded operator D on $l_2(Z)$, then $\|\sum_{i=-n}^{n'} P_i\| \leqslant \|D\|$ for all n, n'. We set $P_i = P_{J_i}$ and prove the lemma by estimating first $d_{ij} = \|P_i P_j\|$ and then $\|D\|$. We assert that there is a constant c_3, independent of J_i or J_j, such that

$$d_{ij} = \|P_i P_j\| \leqslant c_3 \exp\left(-\frac{m}{2}|i-j|\Delta\right). \tag{16}$$

Inequality (16) of course follows if we can prove that for all $\varphi, \psi \in \mathscr{S}(\mathbb{R})$

$$|(\varphi, P_i P_j \psi)| \leqslant c_3 \exp\left(-\frac{m}{2}|i-j|\Delta\right)\|\varphi\|\,\|\psi\|. \tag{17}$$

In order to use the support properties of $\mu_x^{1/2} P_i \varphi$, we pick C_0^∞ functions ρ_i which are equal to 1 on J_i, 0 at points whose distance from J_i is larger than 1 and $0 \leqslant \rho_i(x) \leqslant 1$ for all $x \in \mathbb{R}$. We denote the support of ρ_i by $\hat{J}_i$. Furthermore we require that $\rho_i'(x)$ is of the form $\xi(-x+\alpha_i) - \xi(x-\beta_i)$ for all i, where α_i and β_i are the endpoints of the interval J_i and $\xi \in C_0^\infty([0,1])$ is independent of i. Then there is a constant σ, such that

$$\sup_x |\rho_i^{\#}(x)| \leqslant \sigma,$$

where $\rho_i^{\#}$ stands for ρ_i, ρ_i' or ρ_i''. Note that $x \in \hat{J}_i$, $y \in J_j$ and $i \neq j$ imply $|x-y| \geqslant 1$, by the assumption of the lemma.

Now we write

$$(P_i\varphi, P_j\psi) = (\mu_x^{-1/2}\rho_i \mu_x^{1/2} P_i\varphi, \quad \mu_x^{-1/2}\rho_j \mu_x^{1/2} P_j\psi) = A_1 + A_2 + A_3 + A_4,$$

$$A_1 = (\mu_x^{3/2}\rho_i \mu_x^{-3/2} P_i\varphi, \quad \mu_x^{3/2}\rho_i \mu_x^{-3/2} P_j\psi),$$

$$A_2 = (\mu_x^{3/2}\rho_i \mu_x^{-3/2} P_i\varphi, \quad \mu_x^{-1/2}[\rho_j, \mu_x^2]\,\mu_x^{-3/2} P_j\psi),$$

$$A_3 = (\mu_x^{-1/2}[\rho_i, \mu_x^2]\,\mu_x^{-3/2} P_i\varphi, \quad \mu_x^{3/2}\rho_j \mu_x^{-3/2} P_j\psi),$$

$$A_4 = (\mu_x^{-1/2}[\rho_i, \mu_x^2]\,\mu^{-3/2} P_i\varphi, \quad \mu_x^{-1/2}[\rho_j, \mu_x^2]\,\mu_x^{-3/2} P_j\psi).$$

Note that the operator ρ_i is multiplication by $\rho_i(x)$. Each of the four terms A_k will be estimated separately

$$|A_1| \leqslant \|\rho_i \mu_x^3 \rho_j\|\,\|\mu_x^{-3/2}\|^2\|\varphi\|\,\|\psi\|,$$

$$|A_2| \leqslant \|\rho_i \mu_x[\rho_j, \mu_x^2]\,\mu_x^{-3/2}\|\,\|\mu_x^{-3/2}\|\,\|\varphi\|\,\|\psi\|,$$

$$|A_3| \leqslant \|\rho_j \mu_x[\rho_i, \mu_x^2]\,\mu_x^{-3/2}\|\,\|\mu_x^{-3/2}\|\,\|\varphi\|\,\|\psi\|,$$

$$|A_4| \leqslant \|\mu_x^{-3/2}[\rho_i, \mu_x^2]\,\mu_x^{-1}[\rho_j, \mu_x^2]\,\mu_x^{-3/2}\|\,\|\varphi\|\,\|\psi\|. \tag{18}$$

In order to establish (17) we have to estimate the operator norms occurring on the right-hand side of the inequalities (18).

The operator μ_x^τ is multiplication by $\mu^\tau(p)$ in Fourier transform space and therefore convolution with $\mu_x^\tau(x) = \int e^{ipx}\mu^\tau(p)\,dp$ in x-space. For $\tau \neq$ even integer $\mu_x^\tau(x)$ is a smooth function for $x \neq 0$ which decreases exponentially at infinity. More precisely for all real τ, $\tau/2 \notin Z^+$, there is a constant γ_τ such that for $|x| \geqslant 1$, $\mu_x^\tau(x)$ is C^∞ and

$$|\mu_x^\tau(x)| < \gamma_\tau \exp\left(-\frac{m}{2}|x|\right). \tag{19}$$

This follows easily from an integral representation of $\mu_x^\tau(x)$ for $\tau < 0$ (see e.g. [6], p. 185) and the relation

$$\mu_x^{2n+\tau}(x) = \int \mu_x^{2n}(x-y)\, \mu_x^\tau(y)\, dy$$
$$= \left(-\frac{d^2}{dx^2} + m^2\right)^n \mu_x^\tau(x). \tag{20}$$

In (20) we have used the fact that μ_x^2 is the differential operator $[-(d^2/dx^2) + m^2]$. This also makes it possible to compute the commutator of μ_x^2 and ρ_i, namely

$$[\rho_i, \mu_x^2] = \rho_i'' + 2\rho_i' \partial_x = -\rho_i'' + 2\partial_x \rho_i', \tag{21}$$

where $\partial_x \rho_i'$ is the product of the operators $\partial_x = d/dx$ and ρ_i'.

Now we are prepared to estimate the operator norms occurring in (18). Obviously $\|\mu_x^{-3/2}\| = m^{-3/2}$. Denoting by $\|\cdot\|_{HS}$ the Hilbert Schmidt norm we get for $\rho_j^\# = \rho_j$ or ρ_j' or ρ_j'', and $\tau/2 \notin Z^+$

$$\|\rho_i \mu_x^\tau \rho_j^\#\|^2 \leqslant \|\rho_i \mu_x^\tau \rho_j^\#\|_{\mathrm{HS}}^2$$
$$= \int |\rho_i(y)\, \mu_x^\tau(y-x)\, \rho_j^\#(x)|^2\, dx\, dy$$
$$\leqslant \gamma_\tau^2 \sigma^2 \int_{\substack{y \in \hat{J}_i \\ x \in \hat{J}_j}} \exp(-m|x-y|)\, dx\, dy$$
$$\leqslant \text{const.} \exp(-m\Delta_{ij})$$
$$\leqslant \text{const.} \exp(-m|i-j|\Delta). \tag{22}$$

Furthermore,

$$\|\rho_i \mu_x [\rho_j, \mu_x^2]\, \mu_x^{-3/2}\| \leqslant \|\rho_i \mu_x \rho_j'' \mu_x^{-3/2}\| + 2\|\rho_i \mu_x \rho_j' \partial_x \mu_x^{-3/2}\|$$
$$\leqslant m^{-3/2}\|\rho_i \mu_x \rho_j''\| + 2\|\rho_i \mu_x \rho_j'\|\, \|\partial_x \mu_x^{-3/2}\|$$
$$\leqslant \text{const.} \exp\left(-\frac{m}{2}|i-j|\Delta\right), \tag{23}$$

where we have used (22). Note that $\partial_x \mu_x^{-3/2}$ is a bounded operator because it is multiplication by the bounded function $ip\mu^{-3/2}(p)$ in Fourier transform space. Finally

$$\|\mu_x^{-3/2}[\rho_i, \mu_x^2]\, \mu_x^{-1}[\rho_j, \mu_x^2]\, \mu_x^{-3/2}\| \leqslant m^{-3}\|\rho_i'' \mu_x^{-1} \rho_j''\|$$
$$+ 2m^{-3/2}(\|\rho_i'' \mu_x^{-1} \rho_j'\| + \|\rho_i' \mu_x^{-1} \rho_j''\|)\|\partial_x \mu^{-3/2}\|$$
$$+ 4\|\rho_i' \mu_x^{-1} \rho_j'\|\, \|\partial_x \mu^{-3/2}\|^2$$
$$\leqslant \text{const.} \exp\left(-\frac{m}{2}|i-j|\Delta\right). \tag{24}$$

Combining the estimates (22), (23) and (24) with (18) we prove (17) and hence (16).

Finally we have to estimate $\|D\|$. For $r \in Z$ let $B^{(r)}$ be the bounded operator on $l_2(Z)$ which is given by the matrix

$$b_{ij}^{(r)} = \begin{cases} d_{ij} & \text{if } i-j=r \\ 0 & \text{otherwise.} \end{cases}$$

Obviously

$$\|B^{(r)}\| \leqslant \max_{i\in Z} |d_{i\,i-r}| \leqslant c_3 \exp\left(-\frac{m}{2}|r|\Delta\right),$$

by (16) and

$$B^{(0)} = 1, \text{ since } \|P_i^2\| = \|P_i\| = 1.$$

On the other hand we have $D = \sum_r B^{(r)}$. Hence

$$\|D\| \leqslant \sum_r \|B^{(r)}\| \leqslant 1 + 2c_3 \sum_{r=1}^{\infty} \exp\left(-\frac{m}{2} r\Delta\right)$$

$$\leqslant 1 + c_2 \exp\left(-\frac{m}{2}\Delta\right).$$

This proves the lemma.

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