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The Maximal Kinematical Invariance Group of the Free Schrödinger Equation

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Abstract. The largest group of space-time transformations which leave invariant the free Schrödinger equation is found to be a 12-parameter Lie group containing the Galilei group, the dilations and a group of projective transformations. The group is discussed and it is shown that the Schrödinger functions carry a unitary irreducible projective representation.

1. Introduction

It is well known [1] that the largest group of space-time transformations which leave invariant the free Maxwell equations, or the massless Klein–Gordon equation, is not the Poincaré group but the 15-parameter conformal group isomorphic to $SO(4,2)$. The present paper is devoted to the analysis of a similar situation in non-relativistic quantum mechanics and attempts to find the maximal kinematical invariance group (MKI) of the free Schrödinger equation, i.e. the largest group of space-time transformations which leave this equation invariant. It turns out that the Schrödinger group, as the MKI is proposed to be called, is a 12-parameter group containing, in addition to the Galilei group, the group of dilations and a 1-parameter group of transformations, called expansions, which are, in some respects, very similar to the special conformal transformations of the conformal group.

The concept of invariance of a wave equation under a space-time transformation may be described as follows. Let

$$\Delta(t, \mathbf{x})\,\psi(t, \mathbf{x}) = 0 \tag{1.1}$$

be a wave equation where $\Delta(t, \mathbf{x})$ is any differential operator in the coordinates $(t, \mathbf{x})$, and let

$$(t, \mathbf{x}) \rightarrow g(t, \mathbf{x}) \tag{1.2}$$

be any invertible coordinate transformation g. Equation (1.1) is then said to be invariant under the transformation (1.2) if there exists, together with (1.2), a transformation T_g,

$$\psi(t, \mathbf{x}) \rightarrow (T_g \psi)(t, \mathbf{x}) = f_g[g^{-1}(t, \mathbf{x})]\,\psi[g^{-1}(t, \mathbf{x})], \tag{1.3}$$

of the wave function ψ with the property that $T_g \psi$ is again a solution of the wave equation (1.1). The totality of all such transformations g is called the MKI of Equation

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(1.1), the term 'kinematical' being meant to imply coordinate transformations as opposed to, say, gauge transformations or any internal symmetry transformation. The presence of the factor f_g in (1.3) may be necessitated by various reasons, e.g. the representation T of the MKI defined by (1.3) may be a projective representation only, as will indeed be the case for the Schrödinger group; in the general case of a vector function ψ the factor f_g would be a matrix. It should be noted though that the definition (1.3) is perhaps not the most general definition of kinematical transformations of a wave function as it obviously does not cover the case of time reversal.

To find the MKI of Equation (1.1) thus means to find all possible solutions (g, f_g) of the equation

$$\varDelta[g(t, \mathbf{x})]\{f_g(t, \mathbf{x})\, \psi(t, \mathbf{x})\} = 0 \tag{1.4}$$

for an arbitrary solution ψ of (1.1). Before this problem is solved in Sections 3 and 4 we briefly present, in Section 2, a method to find the Lie algebra of the MKI. In Section 3 the Schrödinger group [2] is determined directly by Equation (1.4) without reference to its Lie algebra and the formulas for product and inverse are given; the technical details of this section are dealt with in the appendix. In Section 4 the functions f_g are calculated and it is shown that the wave functions ψ carry a unitary irreducible projective representation of the Schrödinger group. Finally, Section 5 discusses the structure of the group and indicates a possibility to realize the Schrödinger group as a group of linear homogeneous coordinate transformations by introducing a scale parameter as a fifth coordinate.

2. The Lie Algebra of the Schrödinger Group

Rather than attacking the full problem of solving (1.4) we first want to show how the Lie algebra of the MKI can be obtained by a simple method. Let $G(t, \mathbf{x})$ be the generator of a transformation T_g defined in (1.3), i.e. let $G(t, \mathbf{x})$ be a linear differential operator of the form

$$-iG(t, \mathbf{x}) = a(t, \mathbf{x})\, \partial_0 + b_k(t, \mathbf{x})\, \partial_k + c(t, \mathbf{x}), \tag{2.1}$$

where $\partial_0 = \partial/\partial t$, $\partial_k = \partial/\partial x_k$. The condition of invariance of Equation (1.1) under the infinitesimal transformation $1 + i\,\epsilon\, G$ is then given by

$$\varDelta(t, \mathbf{x})[1 + i\,\epsilon\, G(t, \mathbf{x})]\, \psi(t, \mathbf{x}) = 0, \tag{2.2}$$

or, since $\varDelta(t, \mathbf{x})$ is the only annihilator of an arbitrary solution ψ of (1.1), by the ψ-independent equation

$$[\varDelta(t, \mathbf{x}), G(t, \mathbf{x})] = i\lambda(t, \mathbf{x})\, \varDelta(t, \mathbf{x}), \tag{2.3}$$

where λ is an arbitrary function. Inserting (2.1) in (2.3) with the Schrödinger operator ($\hbar = 1$)

$$\varDelta(t, \mathbf{x}) = i\, \partial_0 + \frac{1}{2m} \varDelta, \tag{2.4}$$

and comparing the coefficients of the differential operators of (2.3) we obtain a system of linear partial differential equations for the functions a, $\mathbf{b}$, c and λ. They are easily solved and the solution is

$$a(t, \mathbf{x}) = -\alpha t^2 + 2\delta t + b,$$

$$\mathbf{b}(t, \mathbf{x}) = (-\alpha t + \delta)\,\mathbf{x} + \mathbf{r} \times \mathbf{x} + \mathbf{v}t + \mathbf{a},$$

$$c(t, \mathbf{x}) = \tfrac{3}{2}(-\alpha t + \delta) + im(\tfrac{1}{2}\alpha \mathbf{x}^2 - \mathbf{v}\cdot\mathbf{x}) + k,$$

$$\lambda(t, \mathbf{x}) = \partial_0 a(t, \mathbf{x}), \tag{2.5}$$

where δ, α, b, $\mathbf{a}$, $\mathbf{v}$, $\mathbf{r}$ and k are real constants. Because a constant generator does not correspond to a space-time transformation the last constant, k, is uninteresting and we find the 12 generators

$$D = i(2t\,\partial_0 + \mathbf{x}\cdot\nabla + \tfrac{3}{2}),$$

$$A = -i(t^2\,\partial_0 + t\mathbf{x}\cdot\nabla + \tfrac{3}{2}t) - \frac{m}{2}\mathbf{x}^2,$$

$$H = i\,\partial_0,\quad \mathbf{P} = -i\nabla,\quad \mathbf{K} = it\nabla + m\mathbf{x},\quad \mathbf{J} = -i\mathbf{x} \times \nabla, \tag{2.6}$$

where the arbitrary generator is $i\delta D + i\alpha A + ibH - i\mathbf{a}\cdot\mathbf{P} + i\mathbf{v}\cdot\mathbf{K} - i\mathbf{r}\cdot\mathbf{J}$. The last ten of these generators are recognized as the generators of the Galilei group and the coordinate transformations corresponding to the first two generators can be obtained by an integration of the Lie differential equation for transformation groups [3] with the result

$$\delta\colon\ (t, \mathbf{x}) \to (d^2 t, d\mathbf{x}),\ \ d \equiv e^{\delta},$$

$$\alpha\colon\ (t, \mathbf{x}) \to \left(\frac{t}{1 + \alpha t}, \frac{\mathbf{x}}{1 + \alpha t}\right). \tag{2.7}$$

Thus D generates *dilations* and A generates a type of transformations which are very similar to the special conformal transformations of the conformal group although they only form a 1-parameter group instead of a 4-parameter group (a fact which shows that the Schrödinger group is not a contraction of the conformal group). In the following the α-transformations will be called *expansions*.

3. Determination of the Schrödinger Group

In the present section the problem of finding all possible pairs (g, f_g) satisfying the condition (1.4) is solved completely for the space-time transformations g while the final determination of the functions f_g is deferred to the next section. The operator $\Delta[g(t, \mathbf{x})]$ in Equation (1.4) may be converted into a differential operator in $(t, \mathbf{x})$ with coefficients related to the transformations g, and Equation (1.4) can be considered as a differential equation for the functions f_g; the initial arbitrariness of the transformations g is then restricted by integrability conditions and it is precisely this restriction which defines the Schrödinger group. This program involves an elementary but lengthy process of solving various differential equations and is carried out in the appendix. The result, as far as the transformations g are concerned, can be stated as follows:

The Schrödinger group G consists

i) of all elements g_0 of the form

$$g_0(t, \mathbf{x}) = [d^2(t + b), d(R\mathbf{x} + \mathbf{v}t + \mathbf{a})], \tag{3.1}$$

where $d \neq 0$, b, $\mathbf{a}$ and $\mathbf{v}$ are real parameters and $R \in \mathrm{SO}(3)$ is a rotation,

ii) of the discrete element

$$\Sigma(t, \mathbf{x}) = \left(-\frac{1}{t}, \frac{\mathbf{x}}{t}\right), \tag{3.2}$$

iii) of all combinations of g_0 and Σ.

The elements g_0 of (3.1) constitute the group of Galilei transformations enlarged by the dilations. The discrete element is the analogue of the inversion at the unit sphere which plays a similar role in the conformal group. We use it to define an expansion $[\alpha]$ by

$$[\alpha](t, \mathbf{x}) = [\Sigma^{-1}(-\alpha)\,\Sigma](t, \mathbf{x}) = \left(\frac{t}{1+\alpha t}, \frac{\mathbf{x}}{1+\alpha t}\right), \tag{3.3}$$

where $(-\alpha)$ is a time-translation with $b = -\alpha$. All elements $g \in G$ can now be characterized in a unified way by the symbol

$$g = (d, \alpha, b, \mathbf{a}, \mathbf{v}, R) = (d)[\alpha](b, \mathbf{a}, \mathbf{v}, R), \tag{3.4}$$

and the space-time transformation g is defined by

$$g(t,\ \mathbf{x}) = \left(d^2\frac{t+b}{1+\alpha(t+b)}, d\,\frac{R\mathbf{x}+\mathbf{v}t+\mathbf{a}}{1+\alpha(t+b)}\right). \tag{3.5}$$

In the notation (3.4) the element Σ appears as the limit

$$\Sigma = \lim_{\epsilon\to 0}\ (\epsilon,\ \epsilon,\ -1/\epsilon, 0, 0, 1), \tag{3.6}$$

where the limit is understood to be taken in the combination $\Sigma\ (t, \mathbf{x})$ only. Note that Σ^2 is the parity transformation; Σ generates the cyclic group of order 4.

Finally, let us mention the formulas for the product and the inverse in G as calculated from (3.5):

Product: $g_3 = g_2 g_1$

$$d_3 = \frac{d_2}{d_1}(d_1^2 + \alpha_1 b_2), \qquad \mathbf{a}_3 = R_2\,\mathbf{a}_1 + d_1 b_1\,\mathbf{v}_2 + \frac{1}{d_1}(1+\alpha_1 b_1)\,\mathbf{a}_2,$$

$$\alpha_3 = \frac{1}{d_1^2}(d_1^2 + \alpha_1 b_2)(\alpha_1 + \alpha_1\alpha_2 b_2 + d_1^2\alpha_2), \qquad \mathbf{v}_3 = R_2\,\mathbf{v}_1 + d_1\,\mathbf{v}_2 + \frac{1}{d_1}\alpha_1\,\mathbf{a}_2,$$

$$b_3 = b_1 + \frac{b_2}{d_1^2 + \alpha_1 b_2}, \qquad R_3 = R_2 R_1, \tag{3.7}$$

if $d_1^2 + \alpha_1 b_2 \neq 0$, and

$$g_3 = \Sigma\left(\frac{\alpha_1}{d_1 d_2}, 0, b_1 + \frac{1}{\alpha_1} - \frac{d_1^2\alpha_2}{\alpha_1^2}, \mathbf{a}_3, \mathbf{v}_3, R_3\right), \tag{3.8}$$

if $d_1^2 + \alpha_1 b_2 = 0$.

Inverse: $g' = g^{-1}$

$$d' = \frac{1}{d}(1+\alpha b), \qquad \mathbf{a}' = -dR^{-1}(\mathbf{a} - b\mathbf{v}),$$

$$\alpha' = -\frac{1}{d^2}(1+\alpha b)\,\alpha, \qquad \mathbf{v}' = -\frac{1}{d}R^{-1}[(1+\alpha b)\,\mathbf{v} - \alpha\,\mathbf{a}],$$

$$b' = -d^2\frac{b}{1+\alpha b}, \qquad R' = R^{-1} \tag{3.9}$$

if $1 + \alpha b \neq 0$, and

$$g^{-1} = \Sigma\left(-\frac{\alpha}{d}, 0, -\frac{d^2}{\alpha}, \mathbf{a}', \mathbf{v}', R'\right), \tag{3.10}$$

if $1 + \alpha b = 0$.

4. The Schrödinger Representation

The Hilbert space of Schrödinger wave functions with the inner product

$$(\psi_1, \psi_2) = \int d^3x\,\psi_1^*(t, \mathbf{x})\,\psi_2(t, \mathbf{x}) \tag{4.1}$$

carries a unitary irreducible projective representation of the Galilei group [4]. In the present section we want to show that it carries such a representation of the Schrödinger group G as well.

To each $g \in G$ there belongs an operator T_g acting on the wave functions ψ and defined by (1.3), where the functions f_g have yet to be specified. They have been calculated in the appendix for the expansion free elements g_0 and for the discrete element Σ, but they are only given up to a constant which may still be a function over G and which has to be determined by the requirement that the operators T_g form a projective representation of G, i.e. by the requirement

$$T_{g_2} T_{g_1} = \omega(g_2, g_1)\,T_{g_2 g_1}, \tag{4.2}$$

where ω is a factor system [5] with the property

$$\omega(g_3 g_2, g_1)\,\omega(g_3, g_2) = \omega(g_3, g_2 g_1)\,\omega(g_2, g_1), \tag{4.3}$$

derived from the associative law in G.

As a consequence of (4.2) the functions f_g have to satisfy the relation

$$f_{g_2}[g_1(t, \mathbf{x})] f_{g_1}(t, \mathbf{x}) = \omega(g_2, g_1) f_{g_2 g_1}(t, \mathbf{x}). \tag{4.4}$$

Using (4.4), the definitions (3.3), (3.4) and the expressions (A.17), (A.19) from the appendix we first obtain

$$f_g(t, \mathbf{x}) = C_g\,u_g(t)^{3/2} \exp\left[-\frac{im}{2u_g(t)}\,h_g(t, \mathbf{x})\right],$$

$$u_g(t) = 1 + \alpha(t + b),$$

$$h_g(t, \mathbf{x}) = \alpha\mathbf{x}^2 + 2R\mathbf{x}\cdot[\alpha\,\mathbf{a} - (1+\alpha b)\,\mathbf{v}] + \alpha\,\mathbf{a}^2 - (1+\alpha b)\,t\mathbf{v}^2 + 2\alpha t\,\mathbf{a}\cdot\mathbf{v}, \tag{4.5}$$

where C_g is a g-dependent constant. As can be seen from the easily verifiable relation

$$d_2^{-1} d_1^{-1} u_{g_2}(g_1 t)\, u_{g_1}(t) = d_3^{-1} u_{g_3}(t), \quad (g_3 = g_2 g_1) \tag{4.6}$$

it seems natural to choose

$$C_g = d^{-3/2}, \tag{4.7}$$

a choice which is confirmed by the term 3/2 in the generator D of (2.6). With this choice Equation (4.4) can be verified in a straightforward, though tedious way and the corresponding factor system turns out to be

$$\omega(g_2, g_1) = \exp[im\zeta(g_2, g_1)],$$

$$\zeta(g_2, g_1) = \tfrac{1}{2} d_1^2 b_1\, \mathbf{v}_2^2 + \tfrac{1}{2}\frac{1}{d_1^2}\alpha_1(1+\alpha_1 b_1)\,\mathbf{a}_2^2 + \frac{1}{d_1}\alpha_1 R_2\,\mathbf{a}_1\cdot\mathbf{a}_2$$

$$+\alpha_1 b_1\,\mathbf{a}_2\cdot\mathbf{v}_2 + d_1 R_2\,\mathbf{a}_1\cdot\mathbf{v}_2, \tag{4.8}$$

and satisfies (4.3). Note that the functions f_g agree with the generators (2.6), and that both f_g and the factor system (4.8) assume the correct Galilean values for $d = 1$, $\alpha = 0$.

With the operators T_g given by (1.3), (4.5) and (4.7) the map $g \to T_g$ defines a projective representation T of G which is irreducible because it is already irreducible when restricted to the Galilei group. That T is also unitary with respect to the inner product (4.1) follows from the quasi invariance [6] of the measure d^3x which compensates for the factors $d^{-3} u_g(t)^3$ from (4.5).

5. The Structure of the Schrödinger Group

The Lie algebra of the Schrödinger group can be calculated from the explicit form (2.6) of its generators

$$[J_i, J_k] = i\epsilon_{ikr} J_r, \qquad [K_i, K_k] = 0, \qquad [P_i, P_k] = 0,$$

$$[J_i, P_k] = i\epsilon_{ikr} P_r, \qquad [K_i, P_k] = (im\delta_{ik}), \qquad [P_i, H] = 0,$$

$$[J_i, K_k] = i\epsilon_{ikr} K_r, \qquad [K_i, H] = iP_i,$$

$$[J_i, H] = 0,$$

$$[D, \mathbf{J}] = 0, \qquad [D, \mathbf{K}] = i\mathbf{K}, \qquad [D, \mathbf{P}] = -i\mathbf{P},$$

$$[A, \mathbf{J}] = 0, \qquad [A, \mathbf{K}] = 0, \qquad [A, \mathbf{P}] = -i\mathbf{K}.$$

$$[D, H] = -2iH, \qquad [A, H] = iD, \qquad [D, A] = 2iA, \tag{5.1}$$

where the appearance of the bracketed term in the commutator $[K_i, P_k]$, which is absent in the Lie algebra of G, is due to the fact that the representation T of Section 4 is not a true but a projective representation of G, or, what comes to the same, a true representation of the extended [7] Schrödinger group G_e, i.e. the group G extended by the 1-parameter mass group generated by an operator M which commutes with G and which, in the representation T, takes the constant value m; this situation already occurs in the case of the Galilei group.

A closer inspection of the Lie algebra (5.1) shows that the subgroup generated by D, A and H is isomorphic to the group $SU(1,1) \simeq SL(2, R)$; in the customary form its generators are given by

$$\mathbf{L} = \{\tfrac{1}{2}(A + H), -\tfrac{1}{2}D, \tfrac{1}{2}(A - H)\}. \tag{5.2}$$

The group $SL(2, R)$ can be given a natural interpretation as a group of coordinate transformations by the introduction of a scale κ through

$$t = \frac{1}{\kappa}\eta, \qquad \mathbf{x} = \frac{1}{\kappa}\boldsymbol{\xi}, \tag{5.3}$$

where by definition $\boldsymbol{\xi}$ remains unchanged under a transformation (d, α, b). We then find a *linear* transformation of the 2-component vector (κ, η),

$$(d, \alpha, b)(\kappa, \eta) = \left(\frac{1}{d}(1 + \alpha b)\,\kappa + \frac{\alpha}{d}\eta, db\kappa + d\eta\right), \tag{5.4}$$

and the subgroup of the elements (d, α, b) appears explicitly as the group $SL(2, R)$ (note that in this realization the element Σ has lost its singular form). Moreover, the action of the full group G on the five coordinates $(\kappa, \eta, \boldsymbol{\xi})$ is *linear* and *homogeneous*,

$$(d, \alpha, b, \mathbf{a}, \mathbf{v}, R)(\kappa, \eta, \boldsymbol{\xi}) = \left(\frac{1}{d}(1 + \alpha b)\,\kappa + \frac{\alpha}{d}\eta, db\kappa + d\eta, R\boldsymbol{\xi} + \mathbf{v}\eta + \mathbf{a}\kappa\right) \tag{5.5}$$

and is completely separated into an action of $SL(2, R)$ on (κ, η) and an action of the group $\bar{G}$ of all elements $(\mathbf{a}, \mathbf{v}, R)$ on $\boldsymbol{\xi}$. This separation corresponds to the decomposition

$$G = \bar{G} \times SL(2, R), \tag{5.6}$$

where $\bar{G}$ is an invariant subgroup and $\times$ denotes the semidirect product.

There are various other decompositions of G besides (5.6) of which only one more should be mentioned, namely, the Levi–Malcev decomposition [8] into the radical and the semi-simple Levi factor,

$$G = T_6 \times (SL(2, R) \otimes SO(3)), \tag{5.7}$$

where $\otimes$ denotes the direct product and T_6 is the Abelian 6-parameter group of space-translations and boosts. Both decompositions (5.6) and (5.7) exhibit the essential structural difference between the Galilei group and the Schrödinger group: the time-translations of the Galilei group are replaced, in the Schrödinger group, by the 3-parameter simple group $SL(2, R)$.

APPENDIX

The Schrödinger operator $\Delta[g(t, \mathbf{x})]$ in Equation (1.4) can be written as

$$\begin{aligned} 2m\Delta[g(t, \mathbf{x})] &= \mathbf{c}^2\,\partial_0^2 + 2d_{ik}\,c_k\,\partial_0\,\partial_i + d_{ir}\,d_{kr}\,\partial_i\,\partial_k \\ &+ (c_i\,\dot{c}_i + d_{ik}\,\partial_i\,c_k + 2ima)\,\partial_0 \\ &+ (c_k\,\dot{d}_{ik} + d_{rk}\,\partial_r\,d_{ik} + 2imb_i)\,\partial_i, \end{aligned} \tag{A.1}$$

where $\dot{a} = \partial_0 a$ etc., and the real functions a, $\mathbf{b}$, $\mathbf{c}$ and d_{ik} are defined by

$$a(t, \mathbf{x}) = \frac{\partial t}{\partial t'}, \qquad c_i(t, \mathbf{x}) = \frac{\partial t}{\partial x_i'},$$

$$\mathbf{b}(t, \mathbf{x}) = \frac{\partial \mathbf{x}}{\partial t'}, \qquad d_{ik}(t, \mathbf{x}) = \frac{\partial x_i}{\partial x'_k},$$

$$(t', \mathbf{x}') \equiv g(t, \mathbf{x}). \tag{A.2}$$

Inserting (A.1) in (1.4) and replacing $\dot{\psi}$ by $(i/2m)\Delta\psi$ we obtain the four differential equations

$$\mathbf{c} = 0, \tag{A.3}$$

$$d_{ir} d_{kr} = a\delta_{ik}, \tag{A.4}$$

$$2a\,\partial_i f_g + (d_{rk}\,\partial_r d_{ik} + 2imb_i) f_g = 0, \tag{A.5}$$

$$a\Delta f_g + 2ima\dot{f}_g + (d_{rk}\,\partial_r d_{ik} + 2imb_i)\,\partial_i f_g = 0. \tag{A.6}$$

Due to the vanishing of $\mathbf{c}$ the definitions (A.2) can easily be inverted to differential equations for the transformed coordinates $(t', \mathbf{x}')$:

$$\frac{\partial t'}{\partial t} = a^{-1}, \qquad \frac{\partial t'}{\partial x_i} = 0$$

$$\frac{\partial x_i'}{\partial t} = -a^{-3/2} R_{ik} b_k, \qquad \frac{\partial x_i'}{\partial x_k} = a^{-1/2} R_{ik}, \tag{A.7}$$

where we have used (A.4) to write

$$d_{ik} = a^{1/2} R_{ik}^{-1} = a^{1/2} R_{ki} \tag{A.8}$$

with a rotation R. The integrability conditions for the Equations (A.7) are

$$\partial_i a = 0, \qquad \partial_r R_{ik} = \partial_k R_{ir}, \qquad \partial_k (R_{ir} b_r) = \tfrac{1}{2}\dot{a} R_{ik} - a\dot{R}_{ik}, \tag{A.9}$$

the second of which implies that the rotation R may at most depend on time. Equations (A.5) and (A.6) can now be written as

$$\partial_i f_g = -\frac{im}{a} b_i f_g,$$

$$\dot{f}_g = \left(\frac{im}{2a^2}\mathbf{b}^2 + \frac{1}{2a}\nabla\cdot\mathbf{b}\right) f_g, \tag{A.10}$$

with the integrability conditions

$$\partial_i b_k = \partial_k b_i, \tag{A.11}$$

$$\dot{a} b_i - a\dot{b}_i = b_k\,\partial_i b_k - \frac{ia}{2m}\,\partial_{ik}^2 b_k. \tag{A.12}$$

Equations (A.11) and the third Equation (A.9) yield

$$R_{ik} = \text{const.}, \qquad \partial_i b_k = \tfrac{1}{2}\dot{a}\delta_{ik}. \tag{A.13}$$

Hence we obtain the relation

$$\mathbf{b}(t, \mathbf{x}) = \tfrac{1}{2}\dot{a}\mathbf{x} + \mathbf{h}(t), \tag{A.14}$$

with the unknown function $\mathbf{h}(t)$, and, inserting (A.14) into (A.12), the conditions

$$\dot{a}^2 = 2a\ddot{a},$$
$$\dot{a}\mathbf{h} = 2a\dot{\mathbf{h}}. \tag{A.15}$$

For further investigation we have to distinguish two cases as to whether $\dot{a}$ vanishes or not. We finally obtain the following results:

Case 1: $(\dot{a} = 0)$

$$(t', \mathbf{x}') \equiv g_0(t, \mathbf{x}) = [d^2(t + b),\ d(R\mathbf{x} + \mathbf{v}t + \mathbf{a})], \tag{A.16}$$

$$f_{g_0}(t, \mathbf{x}) = C \exp[im(\tfrac{1}{2}\mathbf{v}^2 t + R\mathbf{x}\cdot\mathbf{v})], \tag{A.17}$$

where $d \neq 0$, b, $\mathbf{a}$, $\mathbf{v}$ are real parameters, R is a rotation and C is an arbitrary constant.

Case 2: $(\dot{a} \neq 0)$

$$(t', \mathbf{x}') \equiv \Sigma(t, \mathbf{x}) = -\left(\frac{1}{t}, \frac{\mathbf{x}}{t}\right), \tag{A.18}$$

$$f_\Sigma(t, \mathbf{x}) = C' t^{3/2} \exp\left(-\frac{im}{2t}\mathbf{x}^2\right), \tag{A.19}$$

where C' is an arbitrary constant and where, in (A.18), we have absorbed all constants of integration by transformations of the form (A.16).

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