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On the Existence of Light-Like Charges in Quantum Field Theory

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Abstract. We investigate the existence of light-like charge operators in the framework of general quantum field theory. The light-like charges present an important tool for a mathematically consistent formulation of the connection between local field theory and the generators of broken internal symmetry transformations. We find that light-like charges exist as unbounded operators for models with an appropriate behaviour of the long-range correlations in light-like directions provided the null plane restriction of the current commutator for the good components is defined.

1. Introduction

The purpose of the present paper is to investigate some properties of light-like charges [1, 2] in the context of quantum field theory. The main interest in these objects arises from the difficulties for space-like charges belonging to non-conserved currents. In one form these difficulties are expressed in Coleman's theorem [3]. In fact the original idea of current algebra and broken symmetries [4] soon met with serious trouble related to Coleman's statement: 'The invariance of the vacuum is the invariance of the world'. It had been hoped that the non-conserved charges have similar properties as the conserved ones: Annihilation of the vacuum, existence as unbounded operators when represented as a space integral over a non-conserved charge density etc. It turned out, however, that none of these properties are satisfied in the case of non-conserved currents [6].

A way out of these difficulties was drawn by the observation [5] that the saturation of the space-like charge algebra by a few low intermediate states is only possible in a non-trivial manner between states having infinite momentum, i.e. for matrix elements

$$L(\mathbf{p}', \mathbf{p}, \mathbf{a}) = \lim_{\kappa \to \infty} \langle \mathbf{p}' + \kappa\mathbf{a} | Q | \mathbf{p} + \kappa\mathbf{a} \rangle.$$

By a formal interchange of the singular boost of the states to a boost on the operator Q one gets a reinterpretation [1] of the infinite momentum limit as a matrix element of a transformed operator Q^l between states of finite momentum:

$$L(\mathbf{p}', \mathbf{p}, \mathbf{a}) = \langle \mathbf{p}' | Q^l | \mathbf{p} \rangle.$$

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When Q, as usual, is represented as an integral of a charge density over a space-like plane,

$$Q = \int_{\Sigma:\, x^0 = \text{const.}} d^3x j^0(x) \quad \text{('space-like charge')},$$

then

$$Q^l = \lim_{\kappa \to \infty} U(\Lambda_\kappa) Q U^+(\Lambda_\kappa)$$

has a representation

$$Q^l = \int_{\Sigma^l:\, x^0 + x^3 = \text{const.}} d\sigma_\mu j^\mu(x) \quad \text{('light-like charge')}$$

as an integral of $(j^0 + j^3)$ over a plane tangential to the light cone.

Formal arguments suggest that the light-like charges Q^l might have properties one would like any charge to possess irrespective of current conservation. So we may hope that a non-conserved charge algebra is meaningful when formulated in terms of charges Q^l.

In this paper we define and investigate light-like charges in the framework of general field theory. It will turn out, however, that dynamical requirements beyond Wightman's axioms are necessary in order to prove the existence of Q^l as an unbounded operator in Hilbert space, contrary to the case of *conserved* space-like charges where the Wightman axioms are sufficient to prove the existence of the charge operator [6, 7]. These requirements, which are compatible with the basic assumptions of field theory, restrict the class of Wightman field theories where l-charges can be defined.

We will not discuss problems related to the possible self-adjointness of Q^l. Hence, it remains an open question whether Q^l can be identified with the generator of unitary internal symmetry transformations. Furthermore our investigation is restricted to theories with no zero mass particles.

2. Light-Like Charges in Quantum Field Theory

2.1. Light-like coordinates

For our discussion it will be convenient to use the following conventions for the coordinates [2]: A light-like plane (l-plane) is characterized by an equation of the form Σ^l: $n_\mu x^\mu =$ constant, where the normal n_μ is light-like. As a particular choice we take

$$n_\mu = \frac{1}{\sqrt{2}}(1, 0, 0, 1).$$

It is then convenient to introduce new coordinates

$$x_\perp = n_\mu x^\mu = \frac{1}{\sqrt{2}}(x^0 + x^3); \quad x_\parallel = \frac{1}{\sqrt{2}}(x^0 - x^3); \quad \underline{x} = (x^1, x^2).$$

The surface Σ^l is determined by a constant value of $x_\perp$. In these coordinates the scalar product reads

$$yx = y^0 x^0 - \mathbf{y} \cdot \mathbf{x} = y_\perp x_\parallel + y_\parallel x_\perp - \underline{y} \cdot \underline{x}.$$

Similarly, we define the new momentum variables as

$$p_\perp = \frac{1}{\sqrt{2}}(p^0 - p^3); \quad p_\parallel = n_\mu p^\mu = \frac{1}{\sqrt{2}}(p^0 + p^3); \quad \underline{p} = (p^1, p^2)$$

such that the scalar product takes the form

$$px = p_\perp x_\perp + p_\parallel x_\parallel - \underline{p}\cdot\underline{x},$$

i.e. $P_\parallel$ generates translations parallel to Σ^l. Furthermore, we will write $\partial_\parallel = n_\mu \partial^\mu$ for the derivative and $j_\parallel = n_\mu j^\mu$ for the current and correspondingly for the other components. We will also use the abbreviations

$$\hat{x} = (x_\parallel, \underline{x}) \quad \text{and} \quad \hat{p} = (p_\parallel, \underline{p})$$

for the coordinates and momenta in the planes $x_\perp = \text{const.}$ and $p_\perp = \text{const.}$ respectively. With this notation the formal light-like charge (l-charge) may be expressed in the form

$$Q^l(\tau) = \int_{\Sigma^l:\, nx=\tau} d\sigma_\mu j^\mu(x) = \int d^4x\, \delta(x_\perp - \tau)\, j_\parallel(x).$$

2.2. *Basic assumptions*

We want to formulate our problem within the Wightman approach to quantum field theory. As a basic quantity the hermitean current $j^\mu(x)$ represents a tempered Wightman field, local and relatively local to the other basic fields $\Phi_i(x)$ of the theory. $j^\mu(x)$ transforms according to a unitary representation of the Poincaré group. The vacuum $|0\rangle$ is a unique cyclic vector with respect to the polynomial algebra $\mathscr{P}\{\Phi_i(f)\}$ of the smeared fields, such that the domain

$$D_{qL} = \mathscr{P}\{\Phi_i(f)\}|0\rangle,$$

the set of quasilocal states, is dense in the Hilbert space $\mathscr{H}$ of physical states. The smearing function f belongs to the space $\mathscr{S}(R^4)$ of strongly decreasing test functions. Thus

$$j^\mu(f) = \int dx\, f(x)\, j^\mu(x)$$

defines an unbounded operator in $\mathscr{H}$ with the stable domain D_{qL}. We are explicitly interested in non-conserved currents.

$$\partial_\mu j^\mu(x) \neq 0.$$

The current is always adjusted to have vanishing vacuum expectation value

$$\langle 0|\, j^\mu(x)\,|0\rangle = 0.$$

Furthermore, we assume the strong spectrum condition to hold, i.e. the spectrum of the translation operator P_μ is contained in $\{0\} \cup \overline{V}^+_{m_0}$ where

$$\overline{V}^+_{m_0} = \{p;\, p^2 \geqslant m_0^2;\, p^0 > 0\} \quad \text{and} \quad P_\mu|\psi\rangle = 0 \quad \text{if and only if } |\psi\rangle = |0\rangle.$$

For later use we now specify some notation: The general form of the states $|\psi\rangle \in D_{qL}$ reads

$$|\psi\rangle = \sum_{n=1}^{N} |\psi\rangle_n = \sum_{n=1}^{N} \int dx_1 \ldots dx_n f(x_1, \ldots, x_n)\, \Phi_1(x_1) \ldots \Phi_n(x_n)|0\rangle$$

with $f \in \mathscr{S}(R^{4n})$. By linearity it suffices to consider only states $|\psi\rangle_n$ instead of $|\psi\rangle$ and we omit the index n. B will denote a quasilocal operator.

$$B = \int dx_1 \ldots dx_n f(x_1, \ldots, x_n)\, \Phi_1(x_1) \ldots \Phi_n(x_n)$$

such that $|\psi\rangle = B|0\rangle$ (B and $|\psi\rangle$ are called local if f has compact support, i.e. $f \in \mathscr{D}(R^{4n})$). To exploit the spectral condition, we write $|\psi\rangle$ in momentum space.

$$|\psi\rangle = \int dp_1 \ldots dp_n \tilde{f}(p_1, \ldots, p_n)\, \tilde{\Phi}_1(p_1) \ldots \tilde{\Phi}_n(p_n)|0\rangle$$

where the support of the vector valued distribution

$$\tilde{\Phi}_1(p_1) \ldots \tilde{\Phi}_n(p_n)|0\rangle$$

in the variables $q_k = \sum_{i=k}^{n} p_i$ is contained in $q_k \in \{0\} \cup \overline{V}^+_{m_0}$; $k = 1, \ldots, n$. We exhibit a transformation of variables $p_k \to q_k$ and since the space $\mathscr{S}$ is nuclear, we restrict ourselves to states with $\tilde{f}(q_1, \ldots, q_n) = \tilde{\psi}_1(q_1) \ldots \tilde{\psi}_n(q_n) \in \mathscr{S}(R^4)^{\otimes n}$. These states have a representation which we will always use subsequently

$$|\psi\rangle = \int dq_1\, \tilde{\psi}_1(q_1)|q_1\rangle. \tag{2.0}$$

Here we introduced the generalized eigenstate of P_μ:

$$|q_1\rangle = \int dq_2 \ldots dq_n \tilde{\psi}_2(q_2) \ldots \tilde{\psi}_n(q_n)\, \tilde{\Phi}_1(q_1 - q_2)\, \tilde{\Phi}_2(q_2 - q_3) \ldots \tilde{\Phi}_{n-1}(q_{n-1} - q_n) \times \tilde{\Phi}_n(q_n)|0\rangle.$$

If $|\psi\rangle \neq |0\rangle$ we have $q_1 \in \overline{V}^+_{m_0}$.

2.3 *The problem of defining a light-like charge*

The main question to be discussed in the present paper is: What meaning can be given to the formal l-charge

$$\text{‘}Q^l(x_\perp) = \int_{\Sigma^l} d\sigma_\mu j^\mu(x)\text{’}$$

within the scheme of our basic postulates? Because of the translation property

$$Q^l(x_\perp) = e^{iP_\perp x_\perp} Q^l(0)\, e^{-iP_\perp x_\perp}$$

we need only consider $Q^l(0)$ in the following. We write, formally:

$$Q^l = Q^l(0) = \int dx\, \delta(x_\perp) j_\parallel(x) = \int dx\, g(x) j_\parallel(x)$$

with $g(x_\perp, x_\parallel, \underline{x}) = \delta(x_\perp) \otimes 1$. The assumption of $j^\mu(x)$ being a Wightman field only guarantees the existence of $Q^l(g)$ as an operator for $g \in \mathscr{S}(R^4)$. Therefore the distribution $\delta(x_\perp) \otimes 1$ has to be approximated by a sequence of test functions of $\mathscr{S}$ or $\mathscr{D}$. This is always possible because $\mathscr{D}(R^4)$ and $\mathscr{S}(R^4)$ and by nuclearity also $\mathscr{D}(R)^{\otimes 4}$ and $\mathscr{S}(R)^{\otimes 4}$ are dense in the space of distributions $\mathscr{S}'$. We will choose parametrized 'sequences'

$$f_{dR}(x) = f_d(x_\perp)\, g_R(x_\parallel)\, g_R(\underline{x}) \in \mathscr{S} \quad \text{or} \quad \mathscr{D}$$

with the limit

$$\lim_{\substack{d \to 0 \\ R \to \infty}} f_{dR}(x) = \delta(x_\perp)$$

where the sense of convergence will be specified later on. Within this context the problem of defining an l-charge operator is formulated as a question of convergence of the regularized charges

$$Q_{dR} = Q(f_{dR}) = \int dx\, f_{dR}(x)\, j_\parallel(x).$$

First of all there is no reason to hope that the convergence properties of the Q_{dR} are better than those of space-like charges in the case of *conserved* currents [6–8]. Therefore we restrict ourselves to the study of convergence in the sense of densely defined sesquilinear forms [7]. This means that on the dense set of quasilocal states $\langle\psi|Q_{dR}|\phi\rangle$ should converge to a sesquilinear form $Q(\psi, \phi)$ (i.e. the form is linear in $|\phi\rangle$ and anti-linear in $|\psi\rangle$) The sesquilinear form $Q(\psi, \phi)$ will define an operator Q provided it is continuous in one argument, when the other is fixed, as stated by the representation theorem of Riesz: A densely defined sesquilinear form $Q(\psi, \phi)$ has a representation as a scalar product

$$Q(\psi, \phi) = \langle\psi|\bar{\phi}\rangle \quad |\bar{\phi}\rangle \in \mathscr{H}$$

if and only if $Q(\psi, \phi)$ satisfies the boundedness condition

$$|Q(\psi, \phi)| < \|\Psi\| K_\phi$$

for some positive constant K_ϕ depending in general on $|\phi\rangle$. An operator Q is then uniquely defined by the mapping

$$|\bar{\phi}\rangle = Q|\phi\rangle$$

such that

$$Q(\psi, \phi) = \langle\psi|Q|\phi\rangle.$$

We point out that the convergence of $\langle\psi|Q_{dR}|\phi\rangle$ and the separate continuity of the limit form $Q(\psi, \phi)$ are the weakest requirements necessary in order to define a limit operator Q out of the Q_{dR}'s. Now Schwartz's inequality applied to the regularized charge matrix elements reads

$$|\langle\psi|Q_{dR}|\phi\rangle| \leqslant \|\psi\| \|Q_{dR}|\phi\rangle\|$$

or, taking the limit, we have

$$|Q(\psi,\phi)| \leqslant \|\psi\| \cdot \lim_{d,R} \|Q_{dR}|\phi\rangle\|$$

provided

$$Q(\psi,\phi) = \lim_{\substack{d\to 0\\ R\to\infty}} \langle\psi|Q_{dR}|\phi\rangle \tag{2.1}$$

exists. This shows that the boundedness of $Q(\psi,\phi)$ may be assured by the existence of the limit

$$K_\phi = \lim_{\substack{d\to 0\\ R\to\infty}} \|Q_{dR}|\phi\rangle\|. \tag{2.2}$$

We conclude that the existence of an l-charge operator corresponding to the formal light-like charge may be guaranteed by requiring the two limits (2.1) and (2.2) to exist for arbitrary quasilocal states.

2.4. *Admitted sequences of testing functions; r-convergence*

We have not yet considered a possible dependence of the limits on the sequences of testing functions f_{dR} [7]. In fact it turns out that the existence of the limits (2.1) and (2.2) depends on the choice of sequences and, in the case (2.2), on the order in which the two limits $d \to 0$ and $R \to \infty$ are taken. This will be shown in the subsequent section. It is not unexpected, since this sequence corresponds to a measuring procedure to be carried out to measure the charge. We expect a reasonable charge operator

i) to annihilate the vacuum weakly $Q|0\rangle = 0$ on D_{qL} (2.3a)
ii) to have zero vacuum fluctuation $\|Q|0\rangle\| = 0$ (2.3b)

provided this assumption does not contradict the former basic assumptions. (It turns out in the case of non-conserved space-like charges, that these assumptions are contradictory (Coleman's Theorem) [6, 7].) Of course ii) implies i); we will however investigate the two conditions separately as they impose different conditions on the theory.

Thus, we have to specify a class of suitably parametrized test functions $\{\hat{\mathscr{S}}_{dR}\} \subset \{\mathscr{S}_{dR}\}$ and the corresponding notion of convergence

$$f_{dR}(x) \xrightarrow{\ r\ } \delta(x_\perp); \quad f_{dR} \in \{\hat{\mathscr{S}}_{dR}\}.$$

First of all we restrict the class of functions f_{dR} to have the same symmetry and normalization properties as their limit

$$f_{dR}(x) = f_d(x_\perp)\, g_R(\hat{x}) \quad \text{symmetric, real}$$

(it would be natural even to take f_{dR} in addition to be positive and monotonic in $x_i \geqslant 0$).

$$\int dx_\perp f_d(x_\perp) = 1; \quad f_d(x_\perp) \xrightarrow{\ r\ } \delta(x_\perp) \quad (d \to 0)$$

$$g_R(0) = 1; \quad g_R(\hat{x}) \xrightarrow{\ r\ } 1 \quad (R \to \infty). \tag{2.4}$$

The functions $f_{dR} \in \mathscr{S}$ may have compact support either in configuration space (the parameters d and R may in this case be taken essentially as the corresponding bounds to the support) or in momentum space (with d^{-1} and R^{-1} as bounds to the support).

The convergence of a δ-sequence $f_n(z) \xrightarrow{r} \delta(z)$ must be defined so as to pick out from an arbitrary distribution $T \in \mathscr{S}'$ only contributions in the vicinity of $z = 0$ [11]. This is not guaranteed by simply demanding convergence in $\mathscr{S}'$. Rather we have to require that

i) $\alpha(z) f_n(z) \to 0$ in $\mathscr{S}$

for all $\mathscr{C}^\infty$ functions α which are polynomially bounded together with all the derivatives (i.e. $\alpha \in \mathscr{O}_M$) and $\operatorname{supp} \alpha \cap \{0\} = \varnothing$. This is equivalent to the condition

$$\int dz f_n(z)\, T(z) \to 0 \tag{2.5}$$

for all $T \in \mathscr{S}'$ with $\operatorname{supp} T \cap \{0\} = \varnothing$. Furthermore, we require, that in Fourier space the convergence of the 1-sequence $\tilde{f}_n(u) \xrightarrow{r} 1$ fulfills the condition

ii) $\tilde{\beta}(u) \tilde{f}_n(u) \to \tilde{\beta}(u)$ in $\mathscr{S}$

for all $\tilde{\beta} \in \mathscr{S}$. This is equivalent to the requirement

$$\tilde{f}_n(u)\, \tilde{T}(u) \to \tilde{T}(u) \quad \text{in } \mathscr{S}' \tag{2.6}$$

for all $\tilde{T} \in \mathscr{S}'$. The conditions i) and ii) are independent and compatible. They define a subspace of sequences $\{\hat{\mathscr{S}}_n\}$ of the space $\{\mathscr{S}_n\}$ of all $\mathscr{S}'$-convergent δ-sequences. An element $f_n \in \{\hat{\mathscr{S}}_n\}$ is called r-convergent to a δ-function respectively to 1 in Fourier space. (The notion of r-convergence is of course easily generalized to sequences f_n tending to any distribution with support at one point.) In general n will be a real positive parameter with $n \to \infty$. Examples of sequences belonging to $\{\hat{\mathscr{S}}_n\}$ are:

$${}_1f_n(z) = n^\gamma f_1(n^\gamma z) \quad \text{with} \quad f_1(z) \in \mathscr{S}; \quad \int f_n(z)\, dz = 1; \quad \gamma > 0$$

$${}_2f_n(z) \in \mathscr{D} \quad \text{with} \quad \operatorname{supp} f_n \subset [-n^{-\gamma}, n^{-\gamma}]; \quad \int f_n(z)\, dz = 1; \quad \gamma > 0.$$

Correspondingly we have in Fourier space $\tilde{f}(u) = \int dz\, e^{iuz} f(z)$

$${}_1\tilde{f}_n(u) = \tilde{f}_1(n^{-\gamma} u) \quad \text{with} \quad \tilde{f}_n(0) = 1.$$

${}_2\tilde{f}_n(u)$ has in general no simple representation; it is an entire analytic function with $\tilde{f}_n(0) = 1$. In particular:

$${}_1f_n(z) = \sqrt{\frac{n}{\pi a}} \exp\left(-\frac{nz^2}{a}\right) \quad \text{with} \quad a > 0$$

and

$${}_2f_n(z) = nC \exp\left(-\frac{1}{1-(nz)^2}\right) \quad \text{in} \quad |z| \leqslant n^{-1} \quad \text{with}$$

$$C^{-1} = n \int\limits_{|z| \leqslant n^{-1}} dz \exp\left(-\frac{1}{1-(nz)^2}\right)$$

are sequences of this kind. With these notions we are able to introduce now the class of admitted sequences f_{dR} to be used in the definition of l-charges:

$$f_{dR}(x) = f_d(x_\perp)\, g_R(\hat{x}) \text{ is admitted exactly if } \begin{cases} f_d \text{ is a } r\text{-convergent} \\ \quad \delta\text{-sequence for } d \to 0 \\ g_R \text{ is a } r\text{-convergent} \\ \quad 1\text{-sequence for } R \to \infty. \end{cases}$$

We will often write simply $f_{dR} \in \{\hat{\mathscr{S}}_{dR}\}$.

If not otherwise stated, we will only consider convergence with respect to $f_{dR} \in \{\hat{\mathscr{S}}_{dR}\}$ in the following, taking further f_{dR} to satisfy (2.4)

3. The Annihilation of the Vacuum

We first investigate the consistency of the requirements of vacuum annihilation (2.3) with the assumption of general field theory [12, 13].

i) Weak vacuum annihilation on D_{qL}:

$$\lim_{\substack{d \to 0 \\ R \to \infty}} \langle \psi | Q_{dR} | 0 \rangle = 0.$$

Using the notation introduced in (2.0) for $|\psi\rangle$ and translation invariance, we have

$$J_{dR} = \langle \psi | Q_{dR} | 0 \rangle = \int dq \tilde{\psi}^*(q)\, \tilde{f}_{dR}(q) \langle q | j_{\parallel}(0) | 0 \rangle; \quad f_{dR} \in \{\hat{\mathscr{S}}_{dR}\}.$$

The matrix element is a tempered distribution in q with support in $\overline{V}^+_{m_0}$. In momentum space $\tilde{f}_{dR}(u) \xrightarrow{r} (2\pi)^3 \delta(\hat{u})$ we observe that the d-limit can be taken trivially by (2.6). On the other hand in the R-limit $\tilde{f}_{dR}(q)$ tends to a distribution with support on the line $\hat{q} = 0$ and vanishes in $\mathscr{S}(R^4 - \{\hat{q} = 0\})$ by (2.5). Now because $\langle q | j_{\parallel}(0) | 0 \rangle$ has support in $\overline{V}^+_{m_0}$ the supports of the two distributions touch only at $q_\perp = \infty$ where $\tilde{\psi}^*$ vanishes faster than any power. So by the temperedness of $\langle q | j_{\parallel}(0) | 0 \rangle$ the 'point' $p_\perp = \infty$ cannot contribute. Therefore the two limits may be taken in any order and for arbitrary sequences $f_{dR} \in \{\hat{\mathscr{S}}_{dR}\}$ with $f_{dR} \xrightarrow{r} \delta(x_\perp)$ and the vacuum is weakly annihilated on D_{qL}: $r - \lim_{d,R} J_{dR} = 0$.

ii) Vanishing of the vacuum fluctuation:

$$\lim_{\substack{d \to 0 \\ R \to \infty}} \| Q_{dR} | 0 \rangle \|^2 = 0.$$

In this case we meet more subtle convergence properties than for weak vacuum annihilation. A general treatment is however possible by use of the Källen–Lehmann representation for the two-point function:

$$\langle 0 | j_\mu(y)\, j_\nu(x) | 0 \rangle = (2\pi)^{-3} \int d^4p\, \theta(p^0) \left\{ \left(\frac{p_\mu p_\nu}{p^2} - g_{\mu\nu} \right) \tilde{\rho}_1(p^2) + p_\mu p_\nu \tilde{\rho}_0(p^2) \right\} \cdot e^{-ip(y-x)}.$$

The regularized expression then reads

$$K_{dR} = \|Q_{dR}|0\rangle\|^2 = \int dy\, dx\, f_{dR}(y)\, f_{dR}(x) \langle 0| j_{\parallel}(y)\, j_{\parallel}(x)|0\rangle$$

$$= (2\pi)^{-3} \int dp\, \theta(p_{\parallel}) \left\{ \frac{p_{\parallel}^2}{p^2}\, \tilde{\rho}_1(p^2) + p_{\parallel}^2\, \tilde{\rho}_0(p^2) \right\} |\tilde{f}_{dR}(p)|^2.$$

The spectral functions $\tilde{\rho}_0$ and $\tilde{\rho}_1$ are positive tempered distributions with support in $p^2 \geqslant m_0^2$, so we may write

$$K_{dR} = \tfrac{1}{2}(2\pi)^{-3} \int_{m_0^2}^{\infty} ds\, \tilde{\rho}(s) \int_0^{\infty} dp_{\parallel} \int d^2\underline{p}\; p_{\parallel} \left| \tilde{f}_d \left(\frac{s + \underline{p}^2}{2p_{\parallel}} \right) \right|^2 |\tilde{g}_R(p_{\parallel})|^2 |\tilde{g}_R(\underline{p})|^2$$

with $\tilde{\rho}(s) = s^{-1}\tilde{\rho}_1(s) + \tilde{\rho}_0(s)$. It is evident that the d-limit cannot be executed for fixed finite R because we would get the factorized expression

$$K_{0R} = \tfrac{1}{2}(2\pi)^{-3} \int_{m_0^2}^{\infty} ds\, \tilde{\rho}(s) \times \int_0^{\infty} dp_{\parallel}\, p_{\parallel} |\tilde{g}_R(p_{\parallel})|^2 \times \int d^2\underline{p} |\tilde{g}_R(\underline{p})|^2 \tag{3.0}$$

where $\int ds \tilde{\rho}(s)$ need not converge. Even if

$$\int_{m_0^2}^{\infty} ds\, \tilde{\rho}(s) < \infty \tag{3.1}$$

the R-limit of K_{0R} diverges if not

$$\int_{m_0^2}^{\infty} ds\, \tilde{\rho}(s) = 0 \quad \text{i.e. } j_{\mu}(x) \equiv 0.$$

We may parametrize the divergence, e.g. by taking $g_R \in \{\mathscr{D}_R \cap \hat{\mathscr{S}}_R\}$ real symmetric with $g_R(x_i) = 1$ for $|x_i| \leqslant R$, such that

$$\int_0^{\infty} dp_{\parallel}\, p_{\parallel} |\tilde{g}_R(p_{\parallel})|^2 = O(1) \quad \text{and} \quad \int d^2\underline{p} |\tilde{g}_R(\underline{p})|^2 = O(R^2) \quad \text{as } R \to \infty.$$

Hence even for $f_{dR} \in \{\hat{\mathscr{S}}_{dR}\}$ we have $K_{0R} \gtrsim O(R^2)$ as $R \to \infty$ and the d-limit can therefore not be taken before the R-limit. However, we may easily see that for fixed $d \neq 0$ the R-limit annihilates the vacuum, if one considers smearing functions f_{dR} with compact support in momentum space (Fig. 1a). By the geometry of the supports of $\tilde{f}_{dR}$ and $\tilde{\rho}$ there is always a constant $R_0(d)$ such that

$$K_{dR} = 0 \quad \text{for} \quad R > R_0(d).$$

In order to enlarge the class of sequences $\tilde{f}_{dR}$ to non-compact supports it is useful to consider the behaviour of the R-limit ($d \neq 0$, fixed) for the $p_{\parallel}$ and $\underline{p}$ integrals individually.

The behaviour of $\int d^2\underline{p} \ldots$ is not affected by the function $\tilde{f}_d[(s + \underline{p}^2)/2p_{\parallel}]$ hence

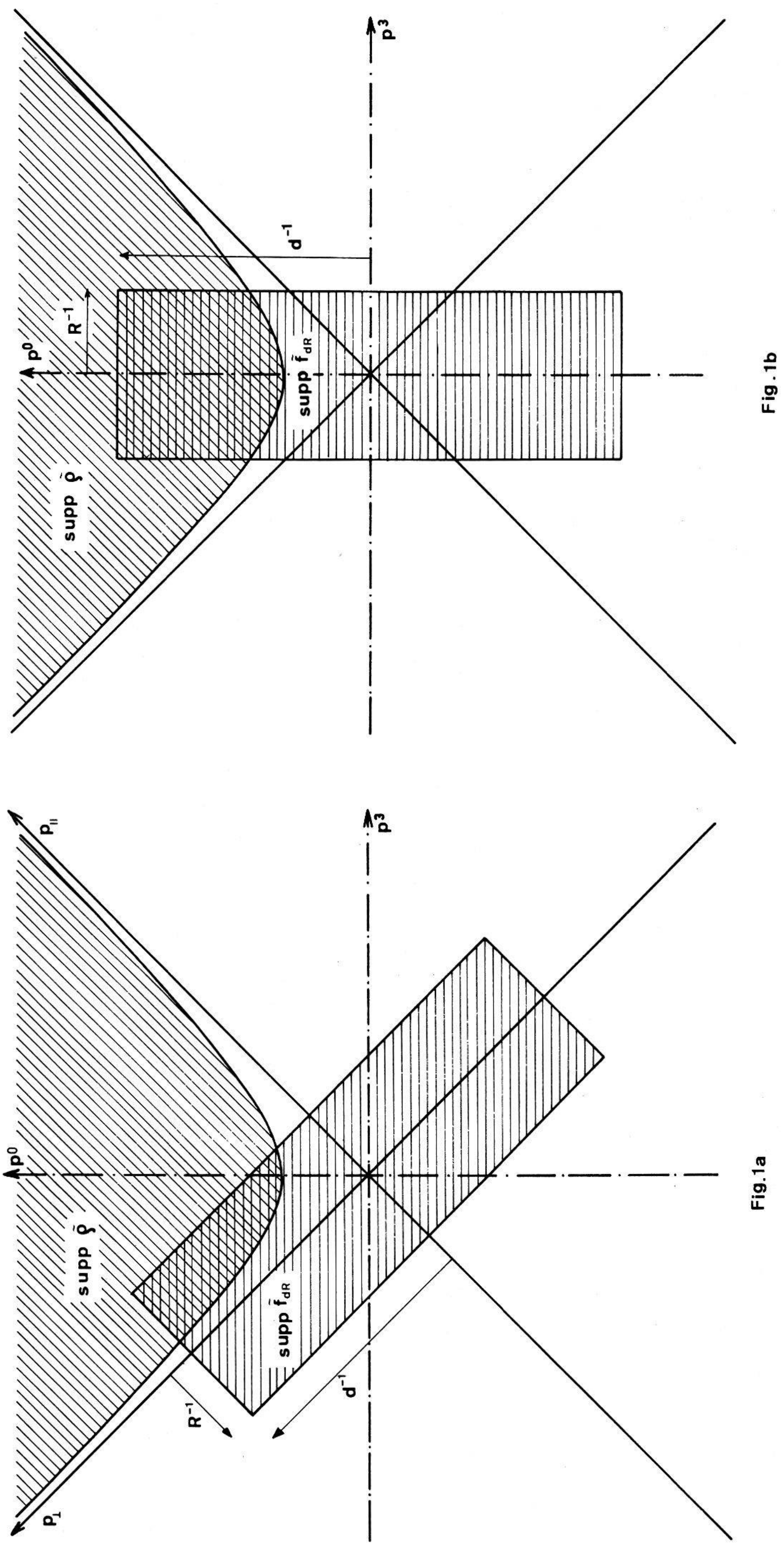

Figure 1
Momentum space supports of the Kallen–Lehmann spectral function $\tilde{\rho}$ and the test function $\tilde{f}_{dR}$. (a) Light-like case, (b) space-like case.

this integral diverges, e.g. as $O(R^2)$ in the parametrization used above. On the other hand $\tilde{f}_d$, having a zero of infinite order at $p_{\parallel} = 0$, may modify the behaviour of $\int_0^\infty dp_{\parallel} \ldots$ depending on the sequence $\tilde{g}_R(p_{\parallel})$. This sequence has to be chosen such as to compensate for the divergence of the $\underset{\sim}{p}$-integral. In fact for an r-convergent sequence $\tilde{g}_R \in \{\hat{\mathscr{S}}_R\}$ the $p_{\parallel}$ integral vanishes faster than any power of R as $R \to \infty$, $d \neq 0$ because, by construction, in the limit $\tilde{g}_R$ will pick out only contributions at $p_{\parallel} = 0$ where $\tilde{f}_d$ has a zero of infinite order. In this case, the vacuum is again annihilated.

It is essential that $\tilde{f}_{dR}$ is taken in the space $\{\hat{\mathscr{S}}_{dR}\}$. The requirement that $\tilde{f}_d$ modifies the behaviour of $\int dp_{\parallel} \ldots$ is not fulfilled for arbitrary testing function $\tilde{g}_R \in \{\hat{\mathscr{S}}_R\}$ as we will show now. (Note that in the space-like case $\tilde{f}_d(\sqrt{s+\mathbf{p}^2})$ does not modify the behaviour of the integral $\int d^3p |\tilde{g}_R(\mathbf{p})|^2 \ldots$ which diverges also for every $\tilde{g}_R \in \{\hat{\mathscr{S}}_R\}$ [7]. This is due to the fact that the point $\mathbf{p} = 0$ always gives a non-vanishing contribution for $d < m^{-1}$, as can easily be seen if $\tilde{f}_{dR}$ has compact support (Fig. 1b).)

Let $g_R(x_{\parallel})$ be a test function of the type used (among others), e.g. in the standard articles on space-like charges [6–8].

$$g_R(x_{\parallel}) \in \mathscr{D} \text{ real, symmetric}; \quad g_R(x_{\parallel}) = \begin{cases} 1 \text{ for } |x_{\parallel}| < R \\ 0 \text{ for } |x_{\parallel}| > R + c. \end{cases} \tag{3.2}$$

Then $\tilde{g}_R(p_{\parallel})$ may be represented in the form

$$\tilde{g}_R(p_{\parallel}) = \frac{2}{p_{\parallel}} \{g_c(p_{\parallel}) \sin p_{\parallel} R + g_s(p_{\parallel}) \cos p_{\parallel} R\}$$

where

$$g_c(p_{\parallel}) = \int_0^c dx_{\parallel} \cos p_{\parallel} x_{\parallel} g_0'(x_{\parallel}) \quad \text{and} \quad g_s(p_{\parallel}) = \int_0^c dx_{\parallel} \sin p_{\parallel} x_{\parallel} g_0'(x_{\parallel})$$

are elements of $\mathscr{S}$. This leads to a $p_{\parallel}$ integral of the form

$$4 \int_0^\infty dp_{\parallel} p_{\parallel}^{-1} \{\tfrac{1}{2}(g_s^2 + g_c^2) + \tfrac{1}{2}(g_s^2 - g_c^2) \cos 2p_{\parallel} R + g_s g_c \sin 2p_{\parallel} R\} |\tilde{f}_d|^2.$$

For $d \neq 0$ the first term is finite and independent of R whereas the non-leading terms vanish by the Riemann–Lebesgue Lemma. Thus with the testing functions (3.2) the divergence of the integral $\int d^2\underset{\sim}{p} \ldots$ is not compensated and we get $\|Q_{dR}|0\rangle\|^2 \gtrsim O(R^2)$ as $(R \to \infty,\ d \neq 0)$. This shows that a restriction of the class of sequences is necessary in order to get a finite limit. In fact the sequence (3.2) is not an element of $\{\hat{\mathscr{S}}_R\}$; it is ruled out by our notion of convergence.

It is instructive to consider an explicit example of a sequence that does converge in the class $g_R \in \{\mathscr{S}_R\}$. We chose

$$g_R(x_{\parallel}) \in \mathscr{D} \text{ real, symmetric}; \quad g_R(x_{\parallel}) = \begin{cases} 1 \text{ for } |x_{\parallel}| < R \\ 0 \text{ for } |x_{\parallel}| > R(1+c). \end{cases} \tag{3.3}$$

The Fourier transform $\tilde{g}_R(p_{\parallel})$ has the property

$$\tilde{g}_R(p_{\parallel}) = R\tilde{g}_1(Rp_{\parallel}).$$

After a change of variables $p_\parallel \to R^{-1} p_\parallel$ we have

$$\int_0^\infty dp_\parallel \, p_\parallel |\tilde{g}_1(p_\parallel)|^2 \left| \tilde{f}_d \left(\frac{s+p^2}{2p_\parallel} R \right) \right|^2 .$$

This integral vanishes faster than any power of R as $R \to \infty$ provided $d \neq 0$ and we have the desired result

$$r - \lim_{R \to \infty} \|Q_{dR}|0\rangle\| = 0 \quad \text{for } (d \neq 0 \text{ fixed}).$$

There is a simple physical meaning for the need of a restricted class of testing functions. As the charge measurement is performed within a finite box there are particle pairs produced at the wall, the density depending on the 'softness' of the wall. As the size R of the box increases, the wall has to be softened in such a way that the surface effects (which cause the divergence in the former case) vanish.

We summarize the results obtained in i) and ii) by the following:

Statement 1: Let Q_{dR} be a charge operator regularized by smearing functions $f_{dR} \in \{\hat{\mathscr{S}}_{dR}\}$. If the strong spectrum condition holds the R-limit for fixed $d \neq 0$ annihilates the vacuum irrespective of current conservation

$$\begin{aligned} &\text{i)} \quad r - \lim_{R \to \infty} \langle \psi | Q_{dR} | 0 \rangle = 0; \quad |\psi\rangle \in D_{qL} \\ &\text{ii)} \quad r - \lim_{R \to \infty} \|Q_{dR}|0\rangle\| = 0. \end{aligned} \tag{3.4}$$

From (3.0) on the other hand we have

$$\text{iii)} \quad r - \lim_{d \to 0} \|Q_{dR}|\psi\rangle\|^2 = \infty \tag{3.5}$$

except for the case when (3.1) holds.

iv) In the case (3.1) is satisfied however

$$\{r - \lim_{d \to 0} \|Q_{dR}|0\rangle\|^2\} \simeq O(R^2) \; (R \to \infty).$$

As a consequence an l-charge in general only exists if the R-limit on $\{\hat{\mathscr{S}}_R\}$ is executed first. The special case (3.1) where the d-limit exists for fixed R we call *superlocal* [10, 14, 16]. It corresponds to the absence of the leading light-cone singularity.

We have shown that it is possible to give a procedure defining an l-charge such that the vacuum annihilation requirements (2.3) in the case of a non-conserved current are compatible with the assumptions of general field theory. This is in contrast to the non-conserved space-like case.

4. Conditions for the Existence of an l-Charge Operator

We now consider the limits (2.1) and (2.2) for states different from the vacuum. There are essential differences also in this case between space-like and light-like charges, as may be seen from the following argument: Let B be a local operator and $|\psi\rangle =$

$B|0\rangle$ the corresponding local state. Then the action of the regularized charge on $|\psi\rangle$ may be written as

$$Q_{dR}|\psi\rangle = [Q_{dR}, B]|0\rangle + BQ_{dR}|0\rangle. \tag{4.1}$$

In the space-like case the first term achieves its R-limit for some finite value of R because of locality and therefore poses no problems in the R-limit. The only trouble then concerns the second term, i.e. vacuum annihilation. Ignoring the difficulties with the vacuum, we are left with the problem whether the d-limit exists for the first term. For the existence of this limit the behaviour of the commutator on the finite intersection of the plane $t = 0$ with the double cone, defined by the support of the smearing function of B, is relevant. Looking at (4.1) with a single field the behaviour of

$$[Q_{dR}, \Phi(x)]$$

in the vicinity of the tip of the light cone determines the behaviour in the d-limit. For l-charges however, the intersection of the l-plane $\tau = 0$ with the double cone is extending to infinity in light-like directions and the behaviour there becomes relevant. For a single field the plane $\tau = 0$ is tangent to the light cone, hence the behaviour of the commutator on the entire light cone must be known to evaluate the d-limit.

We will see in the sequel that the existence of the limits to be investigated here depends strongly on the light cone behaviour of the currents.

i) We first consider the limit (2.1) [13]

$$r - \lim_{d,R} \langle\psi|Q_{dR}|\phi\rangle = Q(\psi, \phi) \tag{2.1}$$

which should exist as a sesquilinear form on $D_{qL}(|\psi\rangle, |\phi\rangle \neq |0\rangle)$. Notice that in the space-like case a theorem of Borchers [9] immediately implies the existence of $Q(\psi, \phi)$. In our case, related to the former by a singular Lorentz transformation, the theorem is not sufficient for a corresponding conclusion, because of the light-like directions involved.

In fact the R-limit cannot exist for arbitrary field theories. By the theorem of Borchers [9]

$$F(x_\parallel) = \int d^2\underline{x}\langle\psi|j_\parallel(0, x_\parallel, \underline{x})|\phi\rangle \in \mathcal{O}_{M_{x_\parallel}}.$$

However, the R-limit only exists if $F(x_\parallel)$ is an integrable function, in the sense

$$\left|\lim_{N\to\infty} \int_{-N}^{+N} dx_\parallel\, F(x_\parallel)\right| < \infty, \tag{4.2}$$

i.e. the correlation of the current along light-like directions has to fall off appropriately. Using notation (2.0) condition (4.2) may be translated to momentum space: The restriction $\hat{q}' = \hat{q}$ of the truncated Wightman distribution

$$\tilde{W}^T(q', q) = \langle p'|j_\parallel(0)|q\rangle^T$$

with support $q', q \in \overline{V}^+_{m_0}$ is a distribution in the remaining variables. Physically speaking the light-like R-limit restricts the corresponding amplitude to zero momentum transfer

$t = (q' - q)^2 = 0$ between 'in' and 'out' particles. (For one-particle states and a conserved current it therefore measures the ordinary charge of the states.)

If we exploit locality and the vacuum annihilation property (3.4) for the matrix element

$$\langle\psi|Q_{dR}|\phi\rangle = \sum_a \langle\psi_a|[Q_{dR}, \Phi_a(\chi)]|\phi_a\rangle + \langle\psi_a|Q_{dR}|0\rangle,$$

we may use the Jost–Lehmann–Dyson representation for the commutators to translate (4.2) to a statement on the corresponding spectral functions $\tilde{\rho}$ defined by:

$$\langle q'|[j_{\parallel}(x), \Phi(z)]|q\rangle_{\mathrm{con}} = e^{iP(x+z)} \int du\, e^{-iu(x-z)} \times \int_0^\infty ds \int dk\, \epsilon(u-k)\, \delta((u-k)^2 - s)\, \tilde{\rho}_{\parallel}(q', q, k, s). \tag{4.3}$$

The spectral function $\tilde{\rho}$ has support in

$$\begin{aligned} &Q \pm k \in \overline{V}_0^+ \\ &\sqrt{s} \geqslant \sqrt{s_0} = \max\{0, m_0 - \sqrt{(Q \pm k)^2}\} \end{aligned} \tag{4.4}$$

where

$$Q = \frac{q' + q}{2} \quad \text{and} \quad P = \frac{q' - q}{2}.$$

It is immediately seen that (4.2) only restricts the behaviour of $\tilde{\rho}$ at $s = 0$. For $|q'\rangle$ and $|q\rangle$ 'one-particle' states it follows $s \geqslant s_0 = m_0/2$ and thus (4.2) is only a restriction to the continuous part of the spectrum. In the following we make the dynamical assumption (4.2). For the corresponding class of field theories

$$r - \lim_{d,R} \langle\psi Q_{dR}|\phi\rangle = Q(\psi, \phi) = (2\pi)^3 \int dq'\, dq\, \tilde{\psi}^*(q')\, \tilde{\phi}(q)\, \delta(\hat{q}' - \hat{q})\, \tilde{W}^T(q', q) \tag{4.5}$$

exists and defines a sesquilinear form on D_{qL}.

ii) The investigation of the continuity condition which must be satisfied in order that $Q(\psi, \phi)$ is the form of an operator is much more involved. The continuity of $Q(\psi, \phi)$ may be assured by requiring the existence of the limit (2.2)

$$K_\psi = r - \lim_{d,R} \|Q_{dR}|\psi\rangle\| < \infty \quad \text{on } D_{qL}.$$

Of course the limiting procedure should not depend on the state $|\psi\rangle$ hence by (3.3) the R-limit in general must be carried out first

$$K_\psi = r - \lim_d \{r - \lim_R \|Q_{dR}|\psi\rangle\|\}. \tag{2.2$'$}$$

For the vacuum we have $K_0 = 0$ as proven in Section 3. For states $|\psi\rangle$ different from the vacuum we are left to study the connected norm contribution

$$\{\|Q_{dR}|\psi\rangle\|^2 - \|\psi\|^2 \|Q_{dR}|0\rangle\|^2\} = \int dq'\, dq\, \tilde{\psi}^*(q')\, \tilde{\psi}(q) \times \int dy\, dx\, f_{dR}(y)\, f_{dR}(x) \langle q'|j_{\parallel}(y)\, j_{\parallel}(x)|q\rangle_{\mathrm{con}}. \tag{4.6}$$

A condition for the existence of the R-limit is immediately obtained. By the theorem of Borchers [9] the light-like correlation function

$$F(y_\parallel, x_\parallel) = \int dy_\perp dx_\perp \int d^2\underline{y}\, d^2\underline{x} f_d(y_\perp) f_d(x_\perp) \langle\psi| j_\parallel(y) j_\parallel(x) |\psi\rangle^{\text{con}} \tag{4.7}$$

has the properties

a) $F(y_\parallel, x_\parallel) \in \mathscr{S}'_{y_\parallel, x_\parallel}$

b) $F(y_\parallel, x_\parallel)$ is symmetric and positive

c) $\int dx_\parallel\, g(x_\parallel)\, F(x_\parallel, y_\parallel) \in \mathcal{O}_{M_{y_\parallel}}$ for $g \in \mathscr{S}$ (4.8)

by assumption (4.2)

d) $\lim_{N\to\infty} \int_{-N}^{+N} dx_\parallel\, F(x_\parallel, y_\parallel) \in \mathscr{S}'_{y_\parallel}$.

Thus a further condition must be imposed for the R-limit in (2.2) to exist:

$$\left| \lim_{N\to\infty} \int_{-N}^{+N} dy_\parallel\, dx_\parallel\, F(x_\parallel, y_\parallel) \right| < \infty. \tag{4.9a}$$

In momentum space we have a corresponding condition on the Wightman distribution $\tilde{G}(q', q, u)$ defined by

$$\langle q'| j_\parallel(y) j_\parallel(x) |q\rangle_{\text{con}} = e^{iP(y+x)} \int du\, e^{-iu(y-x)}\, \tilde{G}_{\parallel\parallel}(q', q, u). \tag{4.10}$$

The support of $\tilde{G}$ is restricted in u to $Q + u \in \bar{V}^+_{m_0}$, where $Q = (q' + q)/2$ and $P = (q' - q)/2$. (4.9) is then equivalent to the existence of the restriction

$$(2\pi)^6\, \delta^{(3)}(\hat{q}' - \hat{q})\, \tilde{G}_{\parallel\parallel}(q', q, u_\perp, 0) = \lim_{R\to\infty} \int d\hat{u}\, \tilde{g}_R(\hat{P} + \hat{u})\, \tilde{g}_R(\hat{P} - \hat{u})\, \tilde{G}_{\parallel\parallel}(q', q, u). \tag{4.11}$$

Provided we demand (4.7) to be absolutely integrable

$$\int dy_\parallel\, dx_\parallel\, |F(y_\parallel, x_\parallel)| < \infty, \tag{4.9b}$$

i.e. $F \in L^1(R^2)$ the restriction

$$\tilde{G}_{\parallel\parallel}(q', q, u)\big|_{\hat{q}'=\hat{q}, \hat{u}=0} \tag{4.12}$$

makes sense uniformly on $\{\hat{\mathscr{S}}_R^{\otimes 2}\}$ otherwise it has to be defined by (4.11). Our further investigations are restricted to field theories fulfilling (4.9a).

We are left with the consideration of the d-limit. As a distribution in q' and q we write

$$(2\pi)^6\, \delta(\hat{q}' - \hat{q}) \int du_\perp \tilde{f}'_d(u_\perp)\, \tilde{G}_{\parallel\parallel}(q', q; u_\perp, 0) \qquad u_\perp > -Q_\perp \tag{4.13}$$

where we have replaced $\tilde{f}_d(P_\perp + u_\perp)\tilde{f}_d(P_\perp - u_\perp)$ by $\tilde{f}'_d(u_\perp)$ simply. We see that the existence of the d-limit is a matter of the high $u_\perp$ behaviour of $\tilde{G}$. Because $f_d \in \{\mathscr{S}_d\}$ the d-limit converges if

$$\left| \int_a^\infty \tilde{G}(\psi^*, \psi; u_\perp, 0)\, du_\perp \right| < \infty \tag{4.14}$$

with an arbitrary constant $a > 0$, i.e. $\tilde{G}(u_\perp) \in L^1(R)$. Assuming a polynomial behaviour at high $u_\perp$ the condition reads

$$\tilde{G}(\psi^*, \psi; u_\perp, 0) \simeq u_\perp^{-(1+\alpha)}\,(u_\perp \to \infty) \quad \text{with } (\alpha > 0!).$$

If (4.14) holds the d-limit is uniform on $\{\hat{\mathscr{S}}_d^{\otimes 2}\}$. Again (4.14) cannot be derived from first principles. We have to take it as a condition on the models that allow for the existence of an l-charge. In the configuration space (4.14) is equivalent to the existence of the l-plane restriction:

$$\int d\xi_\perp\, \delta(\xi_\perp) \left\{ \int d\hat{\xi} \langle q' | j_\parallel \left(\frac{\xi}{2}\right) j_\parallel \left(-\frac{\xi}{2}\right) | q \rangle |_{\hat{q}'=\hat{q}} \right\} \in \mathscr{S}'.$$

We summarize the investigations of this section by the following:

Statement 2: Let $|\psi\rangle$ and $|\phi\rangle$ be quasilocal states different from the vacuum and Q_{dR} the regularized charge, then

i) $Q(\psi, \phi) = r - \lim\limits_{d,R} \langle \psi | Q_{dR} | \phi \rangle$

defines a sesquilinear form provided the current component $j_\parallel$ behaves in light-like directions $|x_\parallel| \to \infty$ such that

$$F(x_\parallel) = \int d^2\underline{x}\, \langle \psi | j_\parallel(0, \hat{x}) | \phi \rangle$$

is integrable in the sense

$$\left| \lim_{N\to\infty} \int_{-N}^{+N} dx_\parallel\, F(x_\parallel) \right| < \infty.$$

ii) $K_d = r - \lim\limits_R \|Q_{dR}|\psi\rangle\|$ exists if the distribution

$$F(y_\parallel, x_\parallel) = \int dy_\perp\, dx_\perp \int d^2\underline{y}\, d^2\underline{x}\, f_d(y_\perp) f_d(x_\perp) \langle \psi | j_\parallel(y)\, j_\parallel(x) | \psi \rangle_{\text{con}}$$

satisfies

$$\left| \lim_{N\to\infty} \int_{-N}^{+N} dy_\parallel\, dx_\parallel\, F(y_\parallel, x_\parallel) \right| < \infty.$$

iii) $K_\psi = r - \lim\limits_{d\to 0} K_d < \infty$ iff in addition

$$F(y_\perp) = \int d\hat{y}\, d\hat{x}\, \langle \psi | j_\parallel(y_\perp, \hat{y})\, j_\parallel(0, \hat{x}) | \psi \rangle_{\text{con}}$$

is continuous at $y_\perp = 0$.

iv) In general $r - \lim_{d\to 0}\|Q_{dR}|\psi\rangle\|$ for fixed R diverges even if all of the above conditions are satisfied. The special case where $r - \lim_{d\to 0}\|Q_{dR}|\psi\rangle\|$ converges is called superlocal [16].

If a model satisfies the above restrictions the statements 1 and 2 guarantee the existence of an l-charge operator annihilating the vacuum irrespective of current conservation. It should be remembered that the divergence of the limit (2.2) would not necessarily exclude the existence of Q^l since convergence of (2.2) is only a sufficient condition.

We verified the conditions (4.9) and (4.14) to be satisfied in perturbation theory for specific renormalizable models [15].

It is easily checked that the above conditions are trivially fulfilled for conserved currents and that in this case the light-like changes coincide with the ordinary ones. Thus our restrictions only concern the properties of the symmetry-breaking part of the current. The relation of the above conditions to properties of the divergence $\partial^\mu j_\mu(x)$ of the current will be discussed elsewhere. This will also give the bridge to the local approach to broken symmetries by means of Zimmermann's normal products and Ward–Takahashi identities. In the later framework [17–19] the 'softness' of the symmetry breaking has to be imposed, in order to have at high energies an asymptotically symmetric theory. To a certain extent the softness of symmetry breaking should be equivalent to our restrictions above.

Apart from the fact that the existence of 'non-conserved' l-charges annihilating the vacuum is compatible with the Wightman axioms it is important to have examples of models satisfying the existence conditions. As for interacting models in general only perturbative investigations are possible [15]. We will restrict ourselves here to give an example of a free field model with broken internal symmetry where exact non-trivial statements are possible.

5. Existence of Non-Conserved Charges in a Free Field Model

In this section we illustrate our general investigations for a free field model. The model is non-trivial as it shows all essential features discussed before (apart from a trivial existence of the R-limit for $d \neq 0$). We consider three free hermitean pseudoscalar meson fields $\phi_i(x)$, $i = 1, 2, 3$, satisfying the Klein–Gordon equation $(\Box + m_i^2)\,\phi_i(x) = 0$ with different masses m_i. The current is defined as usual by

$$v_{\mu i}(x) = \epsilon_{ikl} : \phi_k(x)\, \overleftrightarrow{\partial}_\mu \phi_l(x) :. \tag{5.1}$$

The divergence of this current does not vanish:

$$\partial_\mu v_i^\mu(x) = \epsilon_{ikl}(m_k^2 - m_l^2) : \phi_k(x)\,\phi_l(x) : \neq 0.$$

Formally the light-like charge takes the form

$$Q_i^l = \int_{x_\perp = 0} d\sigma_\mu v_i^\mu(x) = i\epsilon_{ikl}(2\pi)^{-3} \int_0^\infty \frac{dp_\parallel}{2p_\parallel} \int d^2 \underline{p}$$
$$\times \{a_l^+(p_\parallel, \underline{p})\, a_k(p_\parallel, \underline{p}) - a_k^+(p_\parallel, \underline{p})\, a_l(p_\parallel, \underline{p})\} \tag{5.2}$$

where we used the following conventions for the fields

$$\phi_i(x) = (2\pi)^{-3} \int_0^\infty \frac{dp_\parallel}{2p_\parallel} \int d^2\underline{p}\{a_i(p_\parallel,\underline{p})e^{-ipx} + a_i^+(p_\parallel,\underline{p})\,e^{ipx}\}$$

$$= (2\pi)^{-3} \int \frac{d^3p}{2p_0} \{a_i(\mathbf{p})\,e^{-ipx} + a_i^+(\mathbf{p})\,e^{ipx}\}. \tag{5.3}$$

The annihilation operators $a_i(p_\parallel,\underline{p})$ are related to the canonical ones by $a_i(p_\parallel,\underline{p}) = (p^0/p_\parallel)\,a_i(\mathbf{p})$ and satisfy the commutation relations

$$[a_i(p_\parallel,\underline{p}), a_k^+(q_\parallel,\underline{q})] = (2\pi)^3\,\delta(p_\parallel - q_\parallel)\,\delta^{(2)}(\underline{p}-\underline{q})\,2p_\parallel.$$

In calculating the momentum space expression of Q_i^l we have simply set

$$\int_0^\infty dp_\parallel'\,dp_\parallel\,\delta(p_\parallel' + p_\parallel)\ldots = 0.$$

A justification for this simple prescription is given by starting from the definition of Q_i^l as limit of a regularized charge. The right-hand side of (5.2) is an unbounded operator which maps n-particle states into n-particle states contrary to the corresponding space-like charge where terms $a_k a_l$ and $a_k^+ a_l^+$ are present in the case of different masses. The latter terms cause the non-existence of the corresponding space-like expression.

Applying our general investigations to this model we arrive at the following:

Statement 3: In the free field model defined above the l-charge of the non-conserved current (5.1) exists as a densely defined sesquilinear form which is continuous when defined in the specific sense $\lim_{d\to 0}\{\lim_{R\to\infty} Q_{dR}\}$ with $f_{dR} \in \{\hat{\mathscr{S}}_{dR}\}$ (see **3.1**) and hence exists as an operator. In the momentum representation the limit operator is given by

$$Q_i^l = i\epsilon_{ikl} \int_0^\infty \frac{dp_\parallel}{2p_\parallel} \int d^2\underline{p}\{a_l^+(p_\parallel,\underline{p})\,a_k(p_\parallel,\underline{p}) - a_k^+(p_\parallel,\underline{p})\,a_l(p_\parallel,\underline{p})\}.$$

Note that the formal configuration space expression

$$\tilde{Q}_i^l = \int dx\,\delta(x_\perp)v_{i\parallel}(x)$$

is not an operator. Operations with this object may lead to ambiguities. It would be even worse to try a smeared version like $v_{i\parallel}^l(g) = \int dx\,\delta(x_\perp)\,v_{i\parallel}(x)\,g(x)$ with $g \in \mathscr{S}$, this would mean that the d-limit was carried out before the R-limit. Indeed this leads to divergent expressions when matrix elements $\langle\psi|v_i^l(g)\,v_i^l(g)\,|\phi\rangle$ on D_{qL} are considered (i.e. the terms with aa and a^+a^+ do not drop out).

Remark: This shows that one has to be careful when studying l-plane restrictions of current densities. In fact $v_{\mu i}^l(g)$ can only exist as an operator for superlocal currents and even then, in general, only for the 'good' component $v_\parallel$. This will be discussed in detail in Refs. [15] and [16].

Clearly, for a free field the difference between the well-defined operator Q_i^l and the formal object $\tilde{Q}_i^l$ concerns only the pure creation and pure annihilation terms which are absent in Q_i^l but lead to ambiguities for $\tilde{Q}_i^l$. An alternative prescription for the construction of Q_i^l which avoids the use of regularizations is therefore the following: Write $\tilde{Q}_i^l$ in terms of creation and annihilation operators in momentum space and cut off the pure creation and pure annihilation terms:

$$Q_i^l = \text{cut}\,\{\tilde{Q}_i^l\}. \tag{5.4}$$

This cutting operation is of course much more convenient than defining Q_i^l by sequences of regularized charges. Whereas in our case this cutting operation is a local operation, an analogous procedure for space-like non-conserved charges is highly non-local. However,

$$Q_i(x^0) = \text{cut}\,\{\tilde{Q}_i(x^0)\}$$

defines a proper operator and the matrix elements of $Q_i(x^0)$ coincide with those of Q_i^l in the high-energy asymptote. The operators $Q_i(x^0)$ have recently been used in the discussion of internal symmetries of strongly interacting particles [20].

Finally we mention that the above statement has its complete analogue in the renormalized perturbation expansion of Q_i^l for the models we investigated [15]. Also the cutting operation extends to perturbation theory: Q_i^l (or $Q_i(x^0)$) are obtained by dropping all the pure creation and the pure annihilation terms in the corresponding formal expansion $\tilde{Q}_i$.

6. Conclusion

Our investigations have shown that for a certain class of Wightman field theories light-like charges exist as unbounded operators when defined in the sense

$$Q^l = r - \lim_{d\to 0}\{r - \lim_{R\to\infty} Q_{dR}\}.$$

On the other hand there are models where the l-charge cannot be defined. In fact the existence of Q^l requires an appropriate fall-off of long-range correlations in light-like directions and an appropriate light cone behaviour (strong locality) in addition to Wightman's axioms similar to the restrictions one sets on the physical spectrum of P_μ (strong spectral condition) in momentum space.

For superlocal models, which behave even smoother on the light cone than the strongly local ones, l-plane restrictions of the good components of current densities $Q^l(g)$ exist as *operators*. Q^l however is not an operator limit of $Q^l(g)$,

$$Q^l \neq r - \lim_{R\to\infty} Q^l(g_R) \quad \text{(statement 1 iii))}.$$

Light-like charges are not strange physical objects, in fact they coincide with the ordinary charges in the case of conserved currents. In any case Q^l annihilates the vacuum. We note that the restrictions to the models necessary to guarantee the existence of l-charges are in fact conditions on the strength of symmetry-breaking effects. For the strongly local models symmetry-breaking l-charge algebras have a precise mathematical meaning. They are therefore an important tool for the investigation of internal broken symmetries.

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