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Autor(en): **Lièvre, C. / Rothen, F.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **46 (1973)**

Heft 2

PDF erstellt am: **12.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-114480

Helvetica Physica Acta
Vol. 46, 1973.

Birkhäuser Verlag Basel

Stability of the Movement of a Planar Interphase Boundary Between Normal and Superconducting Domains

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Abstract. A study has been made of the stability and motion of an interphase boundary between normal and superconducting domains during the destruction of superconductivity by a greater than critical magnetic field, uniform, constant and parallel to the surface of a half-space. When the boundary shape or position is perturbed, we have shown that it is stable and in these two cases the motion is asymptotically given by the solution of Pippard and Lifshitz. A more general description of the boundary motion is also derived.

I. Introduction

The following problem has been studied by Pippard [1] and Liftshitz [2] (in the following we shall call it the P–L problem). Let us consider the half-space $x > 0$ occupied by a type I superconductor. Let us suppose that at time $t = 0$ we suddenly apply a greater than critical external magnetic field, $H_e = H_c(1 + p)$ uniform and parallel to

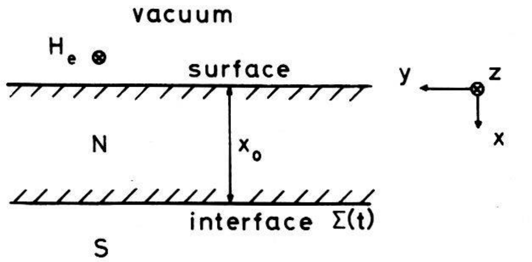

Figure 1
The Pippard–Lifshitz problem: the destruction of superconductivity in the half-space $x > 0$ by an external uniform magnetic field H_e.N = normal domain, S = superconducting domain, x_0 = thickness of the normal domain, satisfying $x_0/t^{1/2}$ = const.

the surface $x = 0$. The superconductivity is destroyed progressively. An interface $\sum(t)$, supposed to be planar, is formed between two phases; the domain N becomes normal, and the domain S remains a superconductor (see Fig. 1). Pippard and Lifshitz have shown that the depth x_0 of the boundary satisfies the equation

$$x_0^2 = \frac{2bp}{D} t, \tag{1}$$

with

$$D = \frac{4\pi\sigma}{c^2},$$

where σ is the electrical conductivity of the normal metal,
c the speed of light *in vacuo*,
b a constant, function of p, whose behaviour is indicated in [1] and satisfies the relation

$$b \int_0^1 \exp[\tfrac{1}{2} b p (1 - y^2)]\, dy = 1.$$

The following assumptions are necessary to obtain (1).

1) The penetration depth $\delta(\simeq 10^{-6}\,\mathrm{cm})$ is very much smaller than x_0.
2) The transition is isothermic, which implies either $T \ll T_c$, or, a sufficiently good contact with a thermal reservoir so that heat diffuses much more rapidly than the field in the metal. Thus

$$\frac{c^2}{4\pi\sigma} \ll \frac{K}{C},$$

where C is specific heat per unit volume and K the thermal conductivity. The Wiedemann–Frantz law

$$K = \frac{\pi^2 k_B^2}{3e^2} T\sigma$$

allows us to write this inequality in the form

$$\sigma^2 \gg \frac{3e^2 c^2 C}{4\pi^3 k_B^2 T}$$

which is easily realized for a pure metal.
3) The displacement current can be neglected. This is always the case, since the displacement velocities of the boundary involved here are of the order of $\mathrm{cm\,s^{-1}}$.
4) The Hall effect is not considered.

Under these conditions, the magnetic field $\vec{h} = \vec{H}/H_c$ satisfies the equation

$$\Delta \vec{h} = D \frac{\partial \vec{h}}{\partial t}. \tag{2}$$

Following Pippard, we shall designate by f the greater than critical fraction of field h in N and put $h = 1 + f$ with

$$f = \mathrm{p} \quad \text{for } x = 0$$

$$f = 0 \quad \text{for } x = x_0$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0} = -D \frac{dx_0}{dt}. \tag{3}$$

The following question presents itself: is the P–L solution stable? The answer is obtained by studying the boundary conditions at the interface N–S. We shall restrict our study to the case $bp < 1$ (which corresponds, see figure given in [1], to $p \lesssim 1.5$). Furthermore the existence of a surface tension between the two phases is not considered. Therefore, we are interested in the stability only from the point of view of electrodynamics.

The method of solving the problem studied here is more interesting than the result. In fact, if situations exist in which the boundary is actually unstable (see below) it is not the case here. But the problem considered has the merit of presenting the simplest geometry and as far as this is concerned the method is exemplary. Because of the special boundary conditions characterizing the problem, this study of stability differs considerably from the analogous studies made in hydrodynamics [3]. In any case, this study leads to a positive result which apparently can be generalized to systems presenting a different geometry: this is because the interphase boundary is deformed more easily in the direction normal to the magnetic field than parallel to it. Now, in

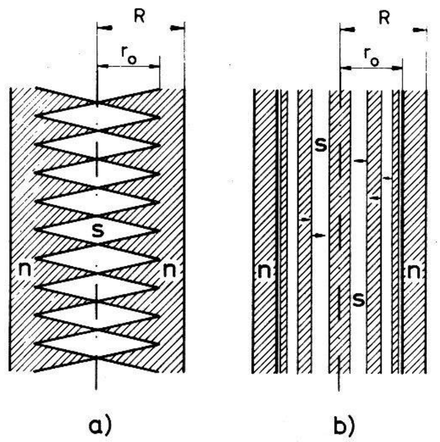

Figure 2
The intermediate kernel in a cylinder carrying an over-critical current: *a*) London's picture with static domains, b) Gorter's picture with domains moving towards the centre. R = radius of the cylinder, r_0 = radius of the intermediate kernel, n = normal domains, s = superconducting domains.

the case of a superconducting cylinder, in which a greater than critical current is suddenly set up, we know that the interphase boundary which penetrates into the cylinder at first keeps its cylindrical shape, until the appearance of the instability and then deforms to give finally an intermediate kernel [4]. The remark made above tends then to prove that London's structure [5] will be favoured at the expense of Gorter's [6] (see Fig. 2).[1])

Summary

In Section II we establish, in vectorial form, the boundary conditions at the interface between two phases. We consider in Section III the particular case of a planar boundary during the destruction of superconductivity in a half-space. In Section IV we study the stability of this boundary relatively to an infinitesimal periodical deformation. In Section V, finally, we give a more general solution to this problem, and show that the P–L solution is the asymptotic one.

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[1]) W. Bestgen had suggested the use of this particular method in order to determine the spatial period of the alternating domains in the intermediate kernel [7].

II. Establishment of Boundary Conditions at the Interface in Vectorial Form

A point in the normal domain will be designated by the vector $\vec{x}$, while a point at the boundary will be located by $\vec{r} = \vec{r}_0 + \vec{r}_1$ where $\vec{r}_0(t)$ corresponds to the unperturbed boundary of the P–L problem and $\vec{r}_1$ is the perturbation, which is assumed to be small (Fig. 3). The relation between $\vec{r}$ and $\vec{r}_0$ is unequivocal as shown by equation (6).

For any function defined at $\vec{r}_0$ we must determine its value at the perturbed boundary, at $\vec{r}$. Similarly, every vectorial field $\vec{c}$ will be the sum of two fields $\vec{c} = \vec{c}_0 + \vec{c}_1$ where $\vec{c}_1$ is the perturbed component, with $|\vec{c}_1| \ll |\vec{c}_0|$

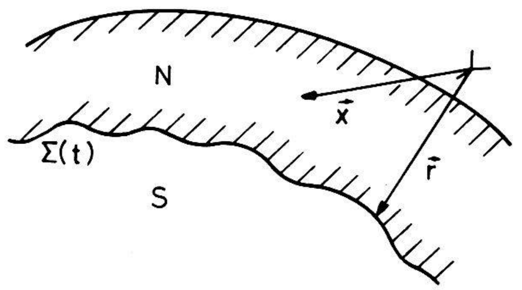

Figure 3
The perturbed interface $\Sigma(t)$ between the normal domain N and the superconducting domain S during the destruction of superconductivity in an arbitrary geometry of the specimen. $\vec{x}$ = arbitrary point in the normal domain, $\vec{r}$ = arbitrary point of the interface $\Sigma(t)$.

In the following we develop a linear approximation, retaining only the first degree terms in the perturbations. This has the consequence that all the perturbations have to be calculated on the unperturbed boundary. Otherwise, it seems natural to require that on $\vec{r}_0$ the primitive equations of the P–L problem are also satisfied.

Let us consider the normal to the boundary. At the instant t, $\vec{n}_0$ is evidently defined only on the interphase boundary $\sum(t)$: $\vec{n}_0 = \vec{n}_0(\vec{r}_0(t))$. If we vary t, the family $\sum(t)$ constitutes a three-dimensional domain $\mathscr{D}$. It will be convenient to consider $\vec{n}_0(\vec{r}_0)$ as a vectorial field on $\mathscr{D}$ without specification of the variable t. Under these conditions we have

$$\vec{n}_0(\vec{r}_0(t)) = \vec{n}_0(\vec{x})\big|_{\vec{x}=\vec{r}_0(t)}.$$

At the perturbed boundary, the normal is written

$$\vec{n}(\vec{r}) \cong \vec{n}_0(\vec{r}_0) + \vec{n}_1(\vec{r}_0) \quad \text{with } |\vec{n}_1| \ll |\vec{n}_0| = 1.$$

Since $\vec{n}$ must be unitary, and $\vec{n}_0$ is unitary by definition, we have $\vec{n}_1 \perp \vec{n}_0$.

The velocity $\vec{v}$ of the boundary is defined on the normal: $\vec{v} = v\vec{n}$ from which we have

$$v = \vec{n}\vec{v} = \vec{n}\frac{d\vec{r}}{dt} = v_0 + v_1 \tag{4}$$

with

$$v_0 = \vec{n}_0 \frac{d\vec{r}_0}{dt}. \tag{5}$$

Boundary conditions

The perturbation $\vec{r}_1$ can be written generally as

$$\vec{r}_1 = A(\vec{r}_0, t)\, \vec{n}_0(\vec{r}_0) \tag{6}$$

with $A(\vec{r}_0, t)$ small. Using (4) and (5) it is easy to verify that

$$v_1 = \frac{\partial A(\vec{r}_0, t)}{\partial t} + \vec{v}_0 \vec{n}_1 + (\vec{v}_0 \vec{\nabla}) A(\vec{r}_0, t)|_{\vec{x}=\vec{r}_0} \tag{7}$$

where the operator $\vec{\nabla}$ symbolizes the derivative with respect to $\vec{r}$

1) The continuity of the normal component of the magnetic field requires

$$(\vec{n} \cdot \vec{h})_{(\vec{r})} = 0 \tag{8}$$

which leads (see Appendix A) to the condition

$$(\vec{n}_0 \vec{h}_1 + \vec{n}_1 \vec{h}_0)_{(\vec{r}_0)} = 0. \tag{9}$$

2) The local equilibrium of the boundary introduces the condition

$$|\vec{h}(\vec{r})|^2 = 1. \tag{10}$$

A calculation given in detail in Appendix B leads to

$$\left[\vec{h}_0 \cdot \vec{h}_1 + A \vec{n}_0 \cdot \overrightarrow{\text{grad}} \left(\frac{h_0^2}{2}\right)\right]_{(\vec{r}_0)} = 0. \tag{11}$$

3) The continuity of the tangential component of the electrical field $\vec{E}$ in a reference system moving with the boundary at the velocity $\vec{v}$ requires

$$\left(\vec{E} + \frac{\vec{v}}{c} \wedge \vec{H}\right)_{(\vec{r})} = 0.$$

We obtain immediately the two relations

$$(\vec{E} \cdot \vec{H})_{(\vec{r})} = 0 \tag{12}$$

$$v = \frac{c}{H_c^2} [\vec{n} \cdot (\vec{E} \wedge \vec{H})]_{(\vec{r})} \tag{13}$$

which give respectively (see Appendix C and D)

$$(\vec{h}_0 \cdot \text{rot} \rightarrow \vec{h}_1 + \vec{h}_1 \cdot \overrightarrow{\text{rot}}\, \vec{h}_0)_{(\vec{r}_0)} = 0 \tag{14}$$

$$v_1 = \frac{1}{D} \vec{n}_0 \cdot [\overrightarrow{\text{rot}}\, \vec{h}_0 \wedge \vec{h}_1 + \overrightarrow{\text{rot}}\, \vec{h}_1 \wedge \vec{h}_0 + A\, \overrightarrow{\text{grad}} \{\vec{n}_0 \cdot (\overrightarrow{\text{rot}}\, \vec{h}_0 \wedge \vec{h}_0)\}]_{(\vec{r}_0)} \tag{15}$$

III. Application to the Half-Space

Let us regard the superconducting half-space of Figure 1 and suppose the boundary slightly distorted. Let us consider a particular perturbation, periodic in y and z, which does not restrict the generality because every perturbation can be unequivocally decomposed into a sum of such perturbations. Then we shall take[2])

$$A(\vec{r}_0, t) = a(x_0)\, e^{i\vec{k}\vec{r}_0} = a(x_0)\, e^{i(k_y y + k_z z)}$$

replacing the time t by the variable $x_0(t)$ which is more convenient. The magnetic field $\vec{h} = \vec{H}/H_c = \vec{h}_0 + \vec{h}_1$ will have the same periodicity

$$\vec{h}_0 = (0, 0, 1 + f(x, x_0))$$

$$\vec{h}_1 = (\alpha(x, x_0), \beta(x, x_0), \gamma(x, x_0)) \cdot e^{i(k_y y + k_z z)}$$

with

$$\alpha(0, x_0) = \beta(0, x_0) = \gamma(0, x_0) = 0 \quad \forall x_0. \tag{16}$$

A point at the interface will be located by

$$x = x_0(t) + a(x_0)\, e^{i(k_y y + k_z z)}.$$

Then the normal $\vec{n} = \vec{n}_0 + \vec{n}_1$ will be given by

$$\vec{n}_0 = (1, 0, 0)$$

$$\vec{n}_1 = (0, -ik_y, -ik_z)\, e^{i(k_y y + k_z z)}.$$

Using successively the relations (9) and (11) our boundary conditions become:

$$\alpha(x_0, x_0)\, e^{i(k_y y + k_z z)} = ik_z a(x_0)\, e^{i(k_y y + k_z z)} \tag{17}$$

$$\gamma(x_0, x_0)\, e^{i(k_y y + k_z z)} + a(x_0)\, e^{i(k_y y + k_z z)} \left.\frac{\partial f}{\partial x}\right|_{x = x_0} = 0. \tag{18}$$

From (5) and (13) it is easy to obtain two expressions for v_0. Using the relation (1) and the definition of $\vec{h}_0$ we have respectively

$$v_0 = \frac{dx_0}{dt} = \frac{1}{D} \frac{bp}{x_0} \tag{19}$$

$$v_0 = \frac{c}{H_c^2} \vec{n}_0 \cdot (\vec{E}_0 \wedge \vec{H}_0) = -\frac{1}{D} \left.\frac{\partial f}{\partial x}\right|_{x = x_0}$$

By comparison, we find again the relation (3) of the P–L problem.

Using (7) and the particular form of $A(\vec{r}_0, t)$ the relation (15) leads to

$$D \frac{da(x_0)}{dx_0} \frac{dx_0}{dt} e^{i(k_y y + k_z z)} = e^{i(k_y y + k_z z)} \left\{ -\gamma(x_0, x_0) \left.\frac{\partial f}{\partial x}\right|_{x = x_0} + ik_z\, \alpha(x_0, x_0) - \left.\frac{\partial \gamma}{\partial x}\right|_{x = x_0} \right.$$
$$\left. - a(x_0) \left.\frac{\partial^2 f}{\partial x^2}\right|_{x = x_0} - a(x_0) \left(\left.\frac{\partial f}{\partial x}\right|_{x = x_0} \right)^2 \right\}.$$

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[2]) It is clear that in the following only the real part of the equations must be taken into consideration.

The first and the last term on the right-hand side vanish by (18). Otherwise we have (cf. [1])

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x_0} = \frac{b^2 p^2}{x_0^2}$$

Whence, still using (17) and (19), we have

$$\frac{bp}{x_0}\frac{da(x_0)}{dx_0} + \left(\frac{b^2 p^2}{x_0^2} + k_z^2\right) a(x_0) + \left.\frac{\partial \gamma}{\partial x}\right|_{x=x_0} = 0. \tag{20}$$

IV. Study of the Stability

Here we study the behaviour of $a(x_0)$ as a function of the depth x_0 as we seek the solution of equation (20) with the knowledge that $\gamma(x,x_0)$ satisfies the relation (2)

$$\frac{\partial^2 \gamma}{\partial x^2} - (k_y^2 + k_z^2)\,\gamma = D\frac{\partial \gamma}{\partial t}$$

and the two boundary conditions (16) and (18).

We shall proceed in two steps. In the first step we will consider the particular case when $k_y = k_z = 0$. In this case the boundary is planar, but displaced with respect to $x_0(t)$. The phenomenon will be stable relative to an infinitesimal perturbation if this displacement approaches 0 as t goes to infinity. In the second step it will be necessary to show that the stability is improved if k_y and k_z differ from 0.

Our problem, then, is the following: calling $\tilde{a}(x_0)$ and $\tilde{\gamma}(x,x_0)$ the new unknown functions we must seek the function $\tilde{a}(x_0)$ which satisfies

$$\frac{bp}{x_0}\frac{d\tilde{a}(x_0)}{dx_0} + \frac{b^2p^2}{x_0^2}\,\tilde{a}(x_0) + \left.\frac{\partial \tilde{\gamma}}{\partial x}\right|_{x=x_0} = 0 \tag{21}$$

when $\tilde{\gamma}(x,x_0)$ is a solution of the heat equation

$$\frac{\partial^2 \tilde{\gamma}}{\partial x^2} = D\frac{\partial \tilde{\gamma}}{\partial t} \tag{22}$$

and satisfies the two boundary conditions

$$\tilde{\gamma}(0,x_0) = 0 \quad \forall\, x_0 \tag{23}$$

$$\tilde{\gamma}(x_0,x_0) = bp\,\frac{\tilde{a}(x_0)}{x_0}. \tag{24}$$

The relations (21) and (24) depend on time, which complicates the search for the solution of the equation (22). Let us put

$$\tilde{\gamma}(x,x_0) = \frac{\tilde{a}(x_0)}{x_0}\,g(u) \quad \text{with } u = \frac{x}{x_0} \tag{25}$$

The equations (21) and (22) are then written

$$\frac{\tilde{a}(x_0)}{x_0^2}\left[bp\,\frac{x_0}{\tilde{a}(x_0)}\,\frac{d\tilde{a}(x_0)}{dx_0}+b^2p^2+g'(1)\right]=0$$

$$\frac{\tilde{a}(x_0)}{x_0^3}\left[g''(u)+bpug'(u)+bpg(u)-bp\,\frac{x_0}{\tilde{a}(x_0)}\,\frac{d\tilde{a}(x_0)}{dx_0}\,g(u)\right]=0.$$

We see that it will certainly be possible to find a solution for $g(u)$ if

$$-\frac{x_0}{\tilde{a}(x_0)}\,\frac{d\tilde{a}(x_0)}{dx_0}=M, \tag{26}$$

where M is constant. The introduction of the function $g(u)$ is therefore compatible with our system of equations (21) and (22). We see without difficulty that $g(u)$ must be the solution of

$$g''(u)+\lambda ug'(u)+\mu g(u)=0 \tag{27}$$

where, for simplification, we put

$$\lambda=bp \quad \mu=bp(1+M). \tag{28}$$

Otherwise, $g(u)$ must satisfy the three conditions

$$g(0)=0 \tag{29}$$

$$g(1)=bp \tag{30}$$

$$g'(1)=bp(M-bp). \tag{31}$$

Let us seek a solution in a power series for $g(u)$

$$g(u)=\sum_{r=0}^{\infty}c_r u^r.$$

From the differential equation (27) we easily obtain the recurrence formula

$$c_{r+2}=-\frac{\lambda r+\mu}{(r+2)\,(r+1)}\,c_r.$$

The condition (29) leads to $c_0=0$ therefore $c_{2k}=0\ \forall k$. We have then

$$g(u)=\sum_{l=0}^{\infty}c_{2l+1}\,u^{2l+1}$$

with, for $l\geqslant 1$,

$$c_{2l+1}=(-1)^l\,\frac{(\lambda+\mu)\,(3\lambda+\mu)\ldots[(2l-1)\,\lambda+\mu]}{(2l+1)!}\,c_1.$$

The convergence of $g(u)$ depends on the behaviour of the general term and the ratio of two consecutive terms. By definition $u\leqslant 1$, it is then sufficient to consider $u=1$.

As p is positive by assumption, and b by definition, $\lambda = bp$ is positive. Let us suppose at first $\mu > 0$. By evident overestimation, we have:

$$|c_{2l+1}| \leqslant \frac{(\lambda+\mu)\,[3(\lambda+\mu)]\ldots[(2l-1)\,(\lambda+\mu)]}{(2l+1)!}\,|c_1|$$

from which

$$|c_{2l+1}| \leqslant \frac{1}{2l+1}\,\frac{\left(\dfrac{\lambda+\mu}{2}\right)^l}{l!}\,|c_1|. \tag{32}$$

Thus the general term tends to zero. The conclusion is the same if $\mu < 0$, for $(\lambda/2)^l$ appears in the numerator of (32) and we have in all cases

$$|c_{2l+1}| \xrightarrow[l\to\infty]{} 0.$$

The ratio of two consecutive terms is

$$\left|\frac{c_{2l+1}}{c_{2l-1}}\right| \leqslant \frac{(2l-1)\,(\lambda+\mu)}{2l(2l+1)} \quad \text{if } \mu \geqslant 0 \tag{33}$$

$$\left|\frac{c_{2l+1}}{c_{2l-1}}\right| \leqslant \frac{(2l-1)\,\lambda}{2l(2l+1)} \quad \text{if } \mu \leqslant 0. \tag{34}$$

Thus for any value of λ or $\lambda + \mu$ we are sure that beyond a certain point the terms decrease monotonically, and this occurs after a finite number of terms. This assures that the series converges, as otherwise the series determining $g(u)$ is alternating.

Differentiating $g(u)$ term by term, we see with (32) that the general term of the new series tends to zero too. With the aid of (33) and (34) we see that the ratio of two consecutive terms is, in this case, overestimated by

$$\begin{aligned} &\frac{\lambda+\mu}{2l} \quad \text{if } \mu \geqslant 0 \\ &\frac{\lambda}{2l} \quad \text{if } \mu \leqslant 0. \end{aligned} \tag{35}$$

Thus this new series converges too. As the convergence is uniform, the series becomes identical with $g'(u)$. We can then put

$$g'(u) = \sum_{l=0}^{\infty} (2l+1)\, c_{2l+1}\, u^{2l}.$$

Stability of the phenomenon

By the relation (26) we see that the evolution of the amplitude $\tilde{a}(x_0)$ of the perturbation with the depth x_0 depends on the sign of M. To prove that the phenomenon is stable in the considered conditions, we shall demonstrate by reduction to the absurd that M cannot be negative. Let us subdivide the negative real axis in three parts:

1) $M < -1$
With the aid of (28) and (31), we see that $\mu < 0$ and $g'(1) < 0$. Otherwise $g(1)$ is positive (30). Thus the equation (27) can be verified in $u = 1$ only if $g''(1) > 0$, which

gives $g(u)$ the following aspect (Fig. 4). Thus

$\exists\, u_0 \in]0, 1[$

such as

$g(u_0) > 0 \quad g'(u_0) = 0 \quad g''(u_0) < 0$

but the equation (27) cannot then be verified in u_0 with two negative terms. Thus M cannot be smaller than -1.

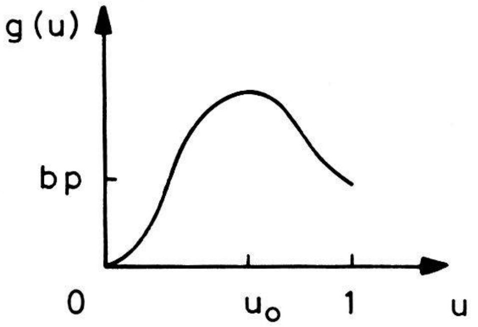

Figure 4
Behaviour of the function $g(u)$ in the case where $g'(1)$ is assumed to be negative.

2) $M = -1$
In this case $\mu = 0$. Taking (30) into account, the equation (27) leads to

$$g'(u) = G \exp\{-\tfrac{1}{2} bpu^2\}$$

$$g(u) = G \int_0^u \exp\{-\tfrac{1}{2} bpt^2\}\, dt.$$

As $g(1) > 0$, we must have $G > 0$ then $g'(1) > 0$ which is contrary to the assumption.

3) $-1 < M < 0$
We still have $g'(1) < 0$ but $0 < \mu < bp$, then $\lambda + \mu < 2bp$. Thus thanks to (35) we know that the ratio of two consecutive terms of $g'(1)$ is

$$\left|\frac{(2l+1)\, c_{2l+1}}{(2l-1)\, c_{2l-1}}\right| \leqslant \frac{\lambda + \mu}{2l} < \frac{bp}{l}.$$

As indicated in the introduction, we shall suppose $bp < 1$, in order that the ratio above is less than 1 even if $l = 1$. In this case, we are certain that the terms of $g'(1)$ decrease monotonically, and do so from the first term. We have *a fortiori* the same result for $g(1)$. In these conditions, the sum of the series is positive and not greater than the first term. We have thus

$0 < g(1) < c_1 \quad$ from which $c_1 > 0$

but

$g'(1) = c_1 \cdot Q \quad$ with $0 < Q < 1$

thus $g'(1) > 0$ which is contrary to the assumption.

Then in all cases we obtain a contradiction, therefore M is positive. We have thus

$$\tilde{a}(x_0) \propto x_0^{-M} \quad M > 0$$

which proves that the phenomenon is stable, when the boundary is planar and displaced.

General case, k_y and k_z not null

It is easy to show, all things equal, that if $k_z \neq 0$ the stability of the boundary is greater than for $k_z = 0$. For this purpose, let us put

$$a(x_0) = \hat{a}(x_0)\, e^{-(k_z^2/D)t} = \hat{a}(x_0)\, e^{-(k_z^2/2bp)x_0^2} \tag{36}$$

$$\gamma(x, x_0) = \hat{\gamma}(x, x_0)\, e^{-(k_z^2/D)t}$$

$\hat{a}(x_0)$ and $\hat{\gamma}(x, x_0)$ are then solutions of

$$\frac{bp}{x_0}\frac{d\hat{a}(x_0)}{dx_0} + \frac{b^2 p^2}{x_0^2}\hat{a}(x_0) + \left.\frac{\partial \hat{\gamma}}{\partial x}\right|_{x=x_0} = 0 \tag{37}$$

$$\frac{\partial^2 \hat{\gamma}}{\partial x^2} - k_y^2 \hat{\gamma} = D\frac{\partial \hat{\gamma}}{\partial t} \tag{38}$$

with the boundary conditions

$$\hat{\gamma}(0, x_0) = 0 \quad \forall x_0 \tag{39}$$

$$\hat{\gamma}(x_0, x_0) = bp\frac{\hat{a}(x_0)}{x_0}. \tag{40}$$

(37) to (40) constitute the general equations for the perturbations in the case $k_z = 0$. The presence of the decreasing exponential in (36) proves our affirmation.

In particular, we can find the solution $\bar{a}(x_0)$ for the case in which $k_y = 0$. We introduce in this case, too, an auxiliary function like that defined by (25) and the condition of compatibility (26) is replaced by

$$bp\frac{x_0}{\bar{a}(x_0)}\frac{d\bar{a}(x_0)}{dx_0} + k_z^2 x_0^2 = -\overline{M}, \quad \overline{M} = \text{constant}$$

which leads immediately to

$$\bar{a}(x_0) \propto x_0^{-\overline{M}/bp}\, e^{-(k_z^2/2bp)x_0^2}.$$

The sign of $\overline{M}$ is, in this case, without consequence, because the exponential is preponderating.

Rather than studying in detail the behaviour of a periodic perturbation in the direction of the axis Oy, we shall confine ourselves to consideration of the limiting case where $x_0(t)$ is large enough. More precisely, we suppose $x_0 \gg bp/k_y$. In these conditions we verify that

$$D\frac{\partial \hat{\gamma}}{\partial t} \ll \frac{\partial^2 \hat{\gamma}}{\partial x^2}.$$

Therefore, the equation (38) has for an approximate solution, using the boundary conditions (39) and (40),

$$\hat{\gamma}(x, x_0) \simeq bp\frac{\hat{a}(x_0)}{x_0}\frac{\operatorname{sh}|k_y|\,x}{\operatorname{sh}|k_y|\,x_0}.$$

Thus (37) has the following asymptotic form

$$\frac{d\hat{a}(x_0)}{dx_0} + |k_y|\, \hat{a}(x_0) = 0$$

of which the solution is clearly decreasing. In fact

$$\hat{a}_{k_y}(x_0) \cong \hat{a}_0(x_0)\, e^{-k_y x_0} = \hat{a}_0(x_0) \exp\left[-k_y \left(\frac{2bp}{D}\, t\right)^{1/2}\right]. \tag{41}$$

Comparing (36) with (41) we establish that the mean times τ_y and τ_z for the disappearance of a perturbation of the respective wave vectors, k_y and k_z, are given by the ratio

$$\frac{\tau_y}{\tau_z} \cong \frac{1}{2bp}\left(\frac{k_z}{k_y}\right)^2.$$

More general solution of the P–L problem

As we saw above, it is possible to obtain an exact solution of equations (21) to (24) which describe the evolution of the peturbations $\tilde{\gamma}$ of the field and $\tilde{a}$ of the boundary assumed to be planar ($k_y = k_z = 0$). In addition the ansatz

$$\tilde{\gamma}(x, x_0) = \frac{\tilde{a}(x_0)}{x_0}\, g\left(\frac{x}{x_0}\right)$$

is very analogous to the corresponding one

$$h_0(x, x_0) = h_0\left(\frac{x}{x_0}\right)$$

used by Pippard and Lifshitz to obtain the unperturbed solution.

This fact suggests the possibility of generalizing the process considered so far by regarding the movement of a planar boundary which occupies at time t a position such that the deviation $\Delta(t) = r_0(t) - [(2bp/D)\,t]^{1/2}$ between its real position $r_0(t)$ and its 'asymptotic' position $[(2bp/D)\,t]^{1/2}$ predicted by Pippard and Lifshitz is not small. More precisely, we introduce a relative deviation

$$\epsilon(t) = \frac{r_0(t) - \left(\dfrac{2bp}{D}\, t\right)^{1/2}}{\left(\dfrac{2bp}{D}\, t\right)^{1/2}} \tag{42}$$

and we assume that it is arbitrary. This leads us to regard the P–L problem under a more general form: we look for a family of solutions $h(x,t)$ of the equation

$$\frac{\partial^2 h}{\partial x^2} = D\frac{\partial h}{\partial t}$$

satisfying the boundary conditions

$$h(0,t) = 1 + p \quad \forall\, t$$

$$h(r_0(t), t) = 1$$

$$\left.\frac{\partial h}{\partial x}\right|_{x=r_0(t)} = -D\,\frac{dr_0}{dt}.$$

(25) and (42) suggest then the change of variables

$$w = \frac{x}{\kappa\sqrt{t}} \qquad \epsilon = \frac{r_0(t)}{\kappa\sqrt{t}} - 1 = \epsilon(t),$$

κ being still undetermined. We obtain, without much effort, the differential equation

$$\frac{\partial^2 h}{\partial w^2} + \frac{D\kappa^2}{2}\,w\,\frac{\partial h}{\partial w} = D\kappa^2\,\vartheta(\epsilon)\,\frac{\partial h}{\partial \epsilon}, \tag{43}$$

where we have put

$$\vartheta(\epsilon) = t(\epsilon)\,\frac{d\epsilon}{dt};$$

$\vartheta(\epsilon)$ being evidently unknown.

The boundary conditions become

$$h(0,\epsilon) = 1 + p \quad \forall\, \epsilon \tag{44}$$

$$h(1+\epsilon, \epsilon) = 1 \tag{45}$$

$$\left.\frac{\partial h}{\partial w}\right|_{w=1+\epsilon} = -D\kappa^2\left[\vartheta(\epsilon) + \tfrac{1}{2}(1+\epsilon)\right]. \tag{46}$$

If we introduce (46) into (43), it is possible to eliminate the unknown function $\vartheta(\epsilon)$; (43) is then subject only to the conditions (44) and (45).

The P–L solution h_0 is evidently a particular solution of this problem; more precisely, it is the solution of (43) independent of ϵ. It corresponds to the value $\epsilon = 0$ under the condition that κ satisfies the relation

$$p = D\kappa\, e^{D\kappa^2/4} \int_0^{\kappa/2} e^{-Dv^2}\, dv,$$

a choice that we shall make hereafter.

General remarks on the P–L problem

Equations (43) to (45) constitute the general equations of evolution of the field in the P–L problem. Once $h\,(w,\epsilon)$ is determined, the relation (46) allows us to obtain the movement of the boundary since it constitutes a differential equation of the first order for $\epsilon(t)$, thus for $r_0(t)$.

In the absence of an existence theorem for the equations (43) to (46), we can, however, have an idea of the richness of the class of solutions for this equation system. To do it, let us formally develop $h(w,\epsilon)$ and $\vartheta(\epsilon)$ relative to ϵ around the origin

$$h(w,\epsilon) = h_0(w) + h_1(w)\,\epsilon + h_2(w)\frac{\epsilon^2}{2!} + \dots$$

$$\vartheta(\epsilon) = \vartheta_1\,\epsilon + \vartheta_2\frac{\epsilon^2}{2!} + \vartheta_3\frac{\epsilon^3}{3!} + \dots,$$

$\vartheta(\epsilon)$ having, evidently, no constant term. If we introduce this formal development into the equations (43) to (46) we obtain separately for every power n of ϵ $(n = 0, 1, 2, \dots)$ a differential equation for $h_n(w)$ and the boundary conditions determining entirely $h_n(w)$ and ϑ_n as a function of the set of $h_i(w)$ and $\vartheta_j(i, j = 0, 1, \dots, n-1)$. As one would expect, $h_0(w)$ coincides with the P–L solution and $h_1(w)$ can be identified with the function $g(w)$ introduced in equation (25) of the present paper: $h_1(w)\epsilon$ constitutes in fact the linear approximation of the more general problem considered here.

Once we have determined the ϑ_k, and assuming that the formal series converge, we can use the definition of $\vartheta(\epsilon)$ for calculating $\epsilon(t)$:

$$t\frac{d\epsilon}{dt} = \sum_{k=1}^{\infty} \vartheta_k \frac{\epsilon^k}{k!}.$$

We establish that if we fix at time t_0 the relative deviation of the boundary in relation to its asymptotic position $[(2bp/D\ t_0]^{1/2}$, $h(w, \epsilon)$ and $r_0(t)$ are determined for any time $t \geqslant t_0$.

For example, we consider here the case where $p \ll 1$. We see immediately that $\kappa^2 \cong 2p/D$ may be neglected and to the first order of p the equation (43) becomes trivial,

$$\frac{\partial^2 h}{\partial w^2} = 0,$$

from which we draw without pain, using boundary conditions

$$\left.\frac{\partial h}{\partial w}\right|_{w=1+\epsilon} = -\frac{p}{1+\epsilon} = -2pt\frac{d\epsilon}{dt} - p(1+\epsilon).$$

Setting $\xi = 1 + \epsilon$, we have

$$\frac{\xi\,d\xi}{(\xi-1)(\xi+1)} = \left(\frac{\frac{1}{2}}{\xi-1} + \frac{\frac{1}{2}}{\xi+1}\right)d\xi = -\frac{dt}{2t}$$

from which, finally, we have

$$\frac{x_0^2(t)}{\kappa^2 t} - 1 = \frac{L}{t} \quad \text{where } L = \text{constant}.$$

Thus when $t \to \infty$ we have $x_0^2(t) \to \kappa^2 t \cong 2pt/D$ which corresponds to Pippard's solution, then if $p \to 0$ we have $b \cong 1$.

V. Conclusion

We have studied the stability of a planar interphase boundary during the destruction of type I superconductivity by a magnetic field applied tangentially to the surface of a superconducting half-space. If the boundary undergoes an arbitrary infinitesimal deformation or displacement relatively to the planar shape and the position occupied in the P–L solution, this deformation or displacement decreases in the course of time. The decrease is infinitely more rapid in the case where the boundary undergoes a periodic deformation in the direction parallel to the unperturbed field than in the case where that direction is perpendicular to it. In the case where the boundary is assumed to be planar but displaced to a finite distance from the position predicted by the P–L solution, it has been possible to calculate its movement. The P–L solution describes in this case the asymptotic position of this boundary which is approached relatively slowly.

VI. Appendix

A. The equation (8) can be written

$$[\vec{n}_0 + \vec{n}_1]_{(\vec{r}_0)} \cdot [\vec{h}_0 + A(\vec{n}_0\, \vec{\nabla})\, \vec{h}_0 + \vec{h}_1]_{(\vec{r}_0)} = 0.$$

Taking account of the fact that

$$(\vec{n}_0 \cdot \vec{h}_0)_{(\vec{r}_0)} = 0 \tag{47}$$

let us show that the term $\vec{n}_0(\vec{n}_0\, \vec{\nabla})\vec{h}_0$ is null, which will prove (9), using the following vectorial identity

$$2(\vec{\epsilon}\vec{\nabla})\, \vec{\eta} = \overrightarrow{\mathrm{rot}}(\vec{\eta} \wedge \vec{\epsilon}) + \overrightarrow{\mathrm{grad}}(\vec{\epsilon} \cdot \vec{\eta}) - \vec{\eta}\, \mathrm{div}\, \vec{\epsilon} + \vec{\epsilon}\, \mathrm{div}\, \vec{\eta} + \overrightarrow{\mathrm{rot}}\, \vec{\eta} \wedge \vec{\epsilon} + \overrightarrow{\mathrm{rot}}\, \vec{\epsilon} \wedge \vec{\eta}. \tag{48}$$

Let us recall that $\vec{n}_0$ is considered here as the value taken in $\vec{x} = \vec{r}_0(t)$ by the field $\vec{n}_0(\vec{x})$

$$2\vec{n}_0(\vec{n}_0\, \vec{\nabla})\, \vec{h}_0 = \vec{n}_0 \cdot \overrightarrow{\mathrm{rot}}(\vec{h}_0 \wedge \vec{n}_0) + \vec{n}_0 \cdot \overrightarrow{\mathrm{rot}}\, \vec{n}_0 \wedge \vec{h}_0.$$

The vectorial identity

$$\vec{w}\, \overrightarrow{\mathrm{rot}}\, \vec{u} = \mathrm{div}(\vec{u} \wedge \vec{w}) + \vec{u}\, \overrightarrow{\mathrm{rot}}\, \vec{w} \tag{49}$$

leads to

$$2\vec{n}_0(\vec{n}_0\, \vec{\nabla})\, \vec{h}_0 = \mathrm{div}[(\vec{h}_0 \wedge \vec{n}_0) \wedge \vec{n}_0] + 2\overrightarrow{\mathrm{rot}}\, \vec{n}_0(\vec{h}_0 \wedge \vec{n}_0) = 0 \quad \text{q.e.d.}$$

B. The developed equation (10) is written

$$|\vec{h}_0 + A(\vec{n}_0\, \vec{\nabla})\, \vec{h}_0 + \vec{h}_1|^2_{(\vec{r}_0)} = 1.$$

Since $|\vec{h}_0(\vec{r}_0)|^2 = 1$ we have

$$2\vec{h}_0 \cdot [\vec{h}_1 + A(\vec{n}_0\, \vec{\nabla})\, \vec{h}_0]_{(\vec{r}_0)} = 0. \tag{50}$$

Let us apply to the second term inside the brackets the identity (48) taking (47) into account

$$2\vec{h}_0(\vec{n}_0\, \vec{\nabla})\, \vec{h}_0 = \vec{h}_0 \cdot \overrightarrow{\mathrm{rot}}(\vec{h}_0 \wedge \vec{n}_0) - \vec{h}_0^2\, \mathrm{div}\, \vec{n}_0 + \vec{h}_0 \cdot \overrightarrow{\mathrm{rot}}\, \vec{h}_0 \wedge \vec{n}_0.$$

Using (49) we obtain easily

$$2\vec{h}_0(\vec{n}_0\,\vec{\nabla})\,\vec{h}_0 = \operatorname{div}[(\vec{h}_0 \wedge \vec{n}_0) \wedge \vec{h}_0] - \vec{h}_0^2 \operatorname{div}\vec{n}_0 = \vec{n}_0\,\overrightarrow{\operatorname{grad}}(\vec{h}_0^2).$$

Introducing this into (50) we obtain (11) immediately.

C. Equation (12) is equivalent to

$$[\vec{E}_0 + A(\vec{n}_0\,\vec{\nabla})\,\vec{E}_0 + \vec{E}_1]_{(\vec{r}_0)}\cdot[\vec{H}_0 + A(\vec{n}_0\,\vec{\nabla})\,\vec{H}_0 + \vec{H}_1]_{(\vec{r}_0)} = 0.$$

Taking into account that $(4\pi\sigma/c)\vec{E}_0 = \overrightarrow{\operatorname{rot}}\,\vec{H}_0$ is normal to $\vec{H}_0$ we have

$$A(\vec{r}_0,t)\cdot\{\overrightarrow{\operatorname{rot}}\,\vec{h}_0\cdot(\vec{n}_0\,\vec{\nabla})\,\vec{h}_0 + (\vec{n}_0\,\vec{\nabla})\,\overrightarrow{\operatorname{rot}}\,\vec{h}_0\cdot\vec{h}_0\}_{(\vec{r}_0)} + [\vec{h}_0\cdot\overrightarrow{\operatorname{rot}}\,\vec{h}_1]_{(\vec{r}_0)} + [\vec{h}_1\,\overrightarrow{\operatorname{rot}}\,\vec{h}_0]_{(\vec{r}_0)} = 0.$$

Let us use two times the relation (48) to show that the expression included within the braces { } is null, which proves the relation (14). We have to take into account that $\vec{n}_0$, $\vec{h}_0$, $\overrightarrow{\operatorname{rot}}\,\vec{h}_0$ are mutually perpendicular

$$2\,\overrightarrow{\operatorname{rot}}\,\vec{h}_0\cdot(\vec{n}_0\,\vec{\nabla})\,\vec{h}_0 = \overrightarrow{\operatorname{rot}}\,\vec{h}_0\cdot[\overrightarrow{\operatorname{rot}}(\vec{h}_0 \wedge \vec{n}_0) + \overrightarrow{\operatorname{rot}}\,\vec{n}_0 \wedge \vec{h}_0]$$

$$2h_0\cdot(\vec{n}_0\,\vec{\nabla})\,\overrightarrow{\operatorname{rot}}\,\vec{h}_0 = \vec{h}_0\cdot[\overrightarrow{\operatorname{rot}}(\overrightarrow{\operatorname{rot}}\,\vec{h}_0 \wedge \vec{n}_0) + \overrightarrow{\operatorname{rot}}\,\overrightarrow{\operatorname{rot}}\,\vec{h}_0 \wedge \vec{n}_0 + \overrightarrow{\operatorname{rot}}\,\vec{n}_0 \wedge \overrightarrow{\operatorname{rot}}\,\vec{h}_0]$$

which leads easily to

$$2\{\,\} = \vec{h}_0\cdot\overrightarrow{\operatorname{rot}}\,(\overrightarrow{\operatorname{rot}}\,\vec{h}_0 \wedge \vec{n}_0) + \overrightarrow{\operatorname{rot}}\,\vec{h}_0\cdot\overrightarrow{\operatorname{rot}}(\vec{h}_0 \wedge \vec{n}_0) - (\vec{h}_0 \wedge \vec{n}_0)\,\overrightarrow{\operatorname{rot}}\,\overrightarrow{\operatorname{rot}}\,\vec{h}_0.$$

We use (49) two times to transform the first term and to group the two others

$$2\{\,\} = \operatorname{div}[(\overrightarrow{\operatorname{rot}}\,\vec{h}_0 \wedge \vec{n}_0) \wedge \vec{h}_0] + (\overrightarrow{\operatorname{rot}}\,\vec{h}_0 \wedge \vec{n}_0)\,\overrightarrow{\operatorname{rot}}\,\vec{h}_0 + \operatorname{div}[(\vec{h}_0 \wedge \vec{n}_0) \wedge \overrightarrow{\operatorname{rot}}\,h_0]$$

but all terms are null, and $\{\,\} = 0$. q.e.d.

D. Equation (13) can be written

$$v_0 + v_1 = \frac{1}{D}(\vec{n}_0 + \vec{n}_1)_{(\vec{r}_0)}\cdot[\overrightarrow{\operatorname{rot}}\,\vec{h}_0 + A(\vec{n}_0\,\vec{\nabla})\,\overrightarrow{\operatorname{rot}}\,\vec{h}_0 + \overrightarrow{\operatorname{rot}}\,\vec{h}_1]_{(\vec{r}_0)}$$
$$\wedge[\vec{h}_0 + A(\vec{n}_0\,\vec{\nabla})\,\vec{h}_0 + \vec{h}_1]_{(\vec{r}_0)}.$$

Since $v_0 = (1/D)\vec{n}_0\cdot\overrightarrow{\operatorname{rot}}\,\vec{h}_0 \wedge \vec{h}_0$ and $\vec{n}_1 \perp \overrightarrow{\operatorname{rot}}\,\vec{h}_0 \wedge \vec{h}_0$ it remains

$$v_1 = \frac{1}{D}\vec{n}_0\cdot[\overrightarrow{\operatorname{rot}}\,\vec{h}_0 \wedge \vec{h}_1 + \overrightarrow{\operatorname{rot}}\,\vec{h}_1 \wedge \vec{h}_0]_{(\vec{r}_0)}$$
$$+\frac{A}{D}\vec{n}_0\cdot[\overrightarrow{\operatorname{rot}}\,\vec{h}_0 \wedge (\vec{n}_0\,\vec{\nabla})\,\vec{h}_0 + (\vec{n}_0\,\vec{\nabla})\,\overrightarrow{\operatorname{rot}}\,\vec{h}_0 \wedge \vec{h}_0]_{(\vec{r}_0)}.$$

Let us use the vectorial identity (48) to transform the second term of the right-hand side, designated by S

$$S = \frac{1}{2D}A(\vec{r}_0,t)\,\{(\vec{n}_0 \wedge \overrightarrow{\operatorname{rot}}\,\vec{h}_0)\cdot\overrightarrow{\operatorname{rot}}(\vec{h}_0 \wedge \vec{n}_0) - (\vec{n}_0 \wedge \overrightarrow{\operatorname{rot}}\,\vec{h}_0)^2$$
$$- 2(\overrightarrow{\operatorname{rot}}\,\vec{h}_0\cdot\vec{h}_0 \wedge \vec{n}_0)\operatorname{div}\vec{n}_0 - (\vec{h}_0 \wedge \vec{n}_0)\,\overrightarrow{\operatorname{rot}}(\vec{n}_0 \wedge \overrightarrow{\operatorname{rot}}\,\vec{h}_0)$$
$$- (\vec{n}_0 \wedge \vec{h}_0)\,(\overrightarrow{\operatorname{rot}}\,\overrightarrow{\operatorname{rot}}\,\vec{h}_0 \wedge \vec{n}_0)\}_{(\vec{r}_0)}.$$

Using the relation (49) and transforming the products of four vectors we obtain

$$S = \frac{1}{2D} A(\vec{r}_0, t) \{\mathrm{div}[(\overrightarrow{\mathrm{rot}}\, \vec{h}_0 \cdot \vec{h}_0 \wedge \vec{n}_0)\, \vec{n}_0]$$

$$- 2\mathrm{div}\, \vec{n}_0\, (\overrightarrow{\mathrm{rot}}\, \vec{h}_0 \cdot \vec{h}_0 \wedge \vec{n}_0) + \mathrm{div}(\overrightarrow{\mathrm{rot}}\, \vec{h}_0 \wedge \vec{h}_0)\}_{(\vec{r}_0)}.$$

Since $\overrightarrow{\mathrm{rot}}\, \vec{h}_0 \wedge \vec{h}_0 \parallel \vec{n}_0$ we can put

$$\overrightarrow{\mathrm{rot}}\, \vec{h}_0 \wedge \vec{h}_0 = \vec{n}_0 (\overrightarrow{\mathrm{rot}}\, \vec{h}_0 \cdot \vec{h}_0 \wedge \vec{n}_0)$$

from which

$$S = \frac{A(\vec{r}_0, t)}{D}\, \vec{n}_0 \cdot \overrightarrow{\mathrm{grad}} (\overrightarrow{\mathrm{rot}}\, \vec{h}_0 \cdot \vec{h}_0 \wedge \vec{n}_0)_{(\vec{r}_0)}$$

and finally we have equation (15).

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