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Analytic Representations of the Conformal Group in Four and Five Dimensions Connected with Differential Equations

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Abstract. The conformal group C_d is represented by analytic spinor representations in the complex symmetric domain corresponding to the group. In four dimensions for every spin there exists one irreducible representation solving invariantly a wave equation, thus describing a massless particle. For massive particles non-linear differential equations in five dimensions with the proper time as fifth coordinate are found which are covariant under the conformal group C_5 and invariant under the invariance group for massive particles $G \subset C_5$ containing the Poincaré group $P(4)$, dilatation, and proper time translation. The most simple system of equations is tentatively ascribed to the electron–positron system connected with electromagnetic and gravitational fields. The Lorentz group $L(4)$ can be extended by adding Lie elements from C_5 to the direct product of $L(4)$ and $SO(2,1)$ probably connected with isospin.

1. Introduction

The homogeneous Maxwell equations are invariant under the conformal group C_4, i.e. the group which leaves invariant isotropic differentials in Minkowski space [1, 2, 3]. It was shown that in fact most equations describing massless particles are conformally invariant [3, 4, 5]. Since the conformal group C_4 is the invariance group of a four-dimensional complex domain [6] it is expected that massless particles should be described by analytic tensor fields in this complex domain. In this domain, however, all Casimir operators operate on analytic fields as a constant and thus no invariant differential operator exists. It will be shown that the Casimir operator $\Delta = P_i P^i$ of the Poincaré group (a subgroup of C_4) determines a differential equation which is invariantly fulfilled by certain analytic fields. To every spin there exists one representation which is just the one described by Gross [3] allowing for a conformally invariant norm.

Wyler [7] suggested the use of the five-dimensional conformal group C_5 in order to describe massive particles. But beside a scalar field no analytic representations fulfil an invariant differential equation. It will be argued that the form of the differential equation need only be invariant under the group G consisting of the four-dimensional Lorentz group $L(4)$, the five translations, and the dilatation, but that the equations

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should be a 'representation' of the conformal group C_5. Such equations are then constructed, the most simple probably corresponding to the electron–positron system connected with electromagnetic and gravitational fields.

In the second section the conformal group C_d and its analytic representations in the complex domain are described. The representations for massless particles fulfilling the equation $\Delta f = 0$ invariantly are derived in the third section and it is shown in the following section that in five dimensions only a scalar field solves invariantly such an equation. Considering C_5 in more detail it is found that the Lorentz group $L(4)$ can be extended to the group $L(4) \times SO(2, 1) \subset C_5$, describing perhaps the isospin, since this new group is only an approximate symmetry. In Section 5 analytic functionals on analytic vector and spinor fields are considered. A non-linear differential functional equation is then found which is a representation of C_5. The lowest order in the parametric fields corresponds to equations containing fields of spin 0, $\frac{1}{2}$, and 1.

2. The Conformal Group and its Analytic Representations

The conformal group C_d of the d-dimensional Minkowski space M_d leaves invariant the isotropic differential forms $g_{ij}dx^i dx^j = 0$ with the metric $g_{ii} = (+--\ldots-)$ (summation convention is used). The group C_d contains as a subgroup the d-dimensional Poincaré group $P(d)$ which is the semi-direct product of the Lorentz group $L(d)$ with the Lie elements $L_{ij} = -L_{ji}$ and the translations (elements P_i). In surplus C_d contains the dilation D and the special conformal elements V_i. A representation as differential operators in Minkowski space is as follows

$$
\begin{aligned}
P_i &= \frac{\partial}{\partial x^i} \\
L_{ij} &= x_i P_j - x_j P_i \\
D &= x^i P_i \\
V_i &= 2x_i D - x_j x^j P_i
\end{aligned} \tag{1}
$$

The special conformal elements can be constructed from the translations with the help of the inversion I': $x_i \to -x_i/x^j x_j$ by the formula $V_i = I' P_i I'$ reflecting the symmetry between the translations and the special conformal elements. On the other hand the conformal group C_d is also the transformation group of the $(d-1)$-dimensional spheres [7]. Explicit construction of these transformations shows that it is the projective group $PO(d, 2)$ which is isomorphic to the orthogonal group $SO(d, 2)$ divided by its centre $Z = \{I, -I\}$ (I = identity). The groups C_d and $PO(d, 2)$ are isomorphic and the connection of the Lie elements of the two groups is [8]

$$
\begin{aligned}
J_{ij} &\equiv L_{ij} \\
J_{i\,d+1} &= -J_{d+1\,i} \equiv \tfrac{1}{2}(P_i - V_i) \\
J_{i\,d+2} &= -J_{d+2\,i} \equiv \tfrac{1}{2}(P_i + V_i) \qquad 1 \leqslant i, j \leqslant d \\
J_{d+2\,d+1} &= -J_{d+1\,d+2} \equiv D.
\end{aligned} \tag{2}
$$

The elements J_{ij} fulfil the commutation relations of the Lie algebra of $SO(d, 2)$ with the metric $g'_{ii} = (+--\ldots-+)$

$$
[J_{ij}, J_{kl}] = -g'_{ik} J_{jl} + g'_{il} J_{jk} - g'_{jk} J_{il} + g'_{jl} J_{ik}. \tag{3}
$$

The Casimir operators are constructed with the help of the totally antisymmetric tensor $\epsilon_{i_1 \ldots i_{d+2}}(\epsilon_{12\ldots d+2}=1)$.

$$C_2 = J_{i_1 i_2} J^{j_1 j_2} \epsilon^{i_1 \ldots i_{d+2}} \epsilon_{j_1 j_2 i_3 \ldots i_{d+2}}$$
$$C_4 = J_{i_1 i_2} J_{i_3 i_4} J^{j_1 j_2} J^{j_3 j_4} \epsilon^{i_1 \ldots i_{d+2}} \epsilon_{j_1 j_2 j_3 j_4 i_5 \ldots i_{d+2}} \tag{4}$$
$$\vdots$$

and so on. Indices are lowered and highered with the metric g'_{ij}. There are thus $[d+2/2]$ Casimir operators. In the case of four and five dimension this number is three. The most important operator C_2 is (beside the factor $2(d-2)!$)

$$C'_2 = J_{ij} J^{ij} = L_{kl} L^{kl} + P_i V^i + V_i P^i - 2D^2. \tag{5}$$

The largest compact subgroup of $PO(d, 2)$ is $SO_d x SO_2/Z$. It is natural to look at the symmetric space $S = PO(d, 2)/(SO_d \times SO_2/Z)$ because representations of $PO(d, 2)$ can be constructed as representations induced by the largest compact subgroup on the domain S [9]. This domain is described by Piatetsky-Chapiro [6]. It is symmetric and the largest group leaving invariant this domain is again $PO(d, 2)$. There exist compact and non-compact realizations of S. The compact realization, which is a symmetric space of type IV according to the classification of Cartan [6], is a domain in d-dimensional complex space

$$D_d = \{\mathbf{z} \,|\, |z_i z^i|^2 + 1 - 2\bar{z}_i z^i > 0, \quad |z_i z^i| < 1\} \tag{6}$$

with euclidian metric. The Silov boundary Q_d, i.e. that part of the boundary of D_d on which all analytic functions in D_d take the maximum of their module, is

$$Q_d = \{\mathbf{x} e^{i\varphi} \,|\, x_i x^i = 1, \mathbf{x}\,\text{real}\}. \tag{7}$$

The non-compact realization is a Siegel space of first kind [6] in d-dimensional complex space namely

$$T_d = M_d + iV_d, \quad V_d = \{\mathbf{y} \,|\, y_i y^i > 0, y_1 > 0\} \tag{8}$$

with the metric of Minkowski space g_{ij}. The Silov boundary of T_d is just the real Minkowski space M_d. The two realizations D_d with points $\mathbf{u}$ and T_d with points $\mathbf{z}$ are connected by an analytic transformation which takes the point $\mathbf{u} = 0 \epsilon D_d$ into the point $\mathbf{z} = Q \equiv (i, 0, \ldots, 0) \epsilon T_d$

$$u_1 = \Big(1 - z_1 + \sqrt{\sum_2^d z_i^2 - 2z_1}\Big)/T$$
$$u_k = z_k/T \quad k \geqslant 2 \tag{9}$$
$$T = \frac{1+i}{\sqrt{2}} - \frac{1-i}{\sqrt{2}} z_1 + \sqrt{2}\sqrt{\sum_2^d z_i^2 - 2z_1}$$

or by the inverse transformation

$$z_1 = \Big(-2\sqrt{2}\, i u_1 + 1 - \sum_1^d u_i^2 + i\Big(1 + \sum_1^d u_i^2\Big)\Big)/T'$$
$$z_k = 2\sqrt{2}\, i u_k / T'$$
$$T' = 2\sqrt{2}\, i u_1 + 1 - \sum_1^d u_i^2 - i\Big(1 + \sum_1^d u_i^2\Big) \tag{10}$$
$$T' T = 2\sqrt{2} i.$$

The representation of the elements of the conformal group be differential operators in Minkowski space (1) can be prolonged to a representation in the domain T_d in the following way. In expression (1) x_i is replaced by z_i and then the Lie elements are defined by

$$J'_{ij} \equiv J_{ij} + \bar{J}_{ij} \tag{11}$$

where $\bar{J}$ means the complex conjugate. These elements J' have the same commutation relations as $J(x)$ or $J(z)$. On analytic functions the second term of (11) has no effect. The Casimir operators can now be expressed as differential operators in T_d. The operator C_2 corresponds to a second-order differential operator Δ'

$$\begin{aligned}\Delta' = \tfrac{1}{2}C'_2 &= P_i \bar{V}^i + V_i \bar{P}^i + L_{ij}\bar{L}^{ij} - 2D\bar{D} \\ &= -(z_i - \bar{z}_i)(z^i - \bar{z}^i) P_j \bar{P}^j + 2(z_i - \bar{z}_i)(z_j - \bar{z}_j) P^i \bar{P}^j. \end{aligned} \tag{12}$$

Terms containing only derivatives according to z_i or to $\bar{z}_i$ disappear. This operator is just the Laplace operator described by Helgason [10], which corresponds to the conformally invariant line element in T_d

$$\begin{aligned} ds^2 &= \frac{1}{(y_i y^i)^2}(-y_i y^i dz_j dz_j + 2y_i y_j dz^i dz^j) \\ y_i &\equiv \operatorname{Im} z_i. \end{aligned} \tag{13}$$

Functions fulfilling $\Delta' f = 0$ are called harmonic. A special class of harmonic function is constituted by the analytic functions.

In the symmetric domain D_d and T_d there exist three kernel functions. For D_d they are given by Hua [11]. The Bergmann kernel k expresses the identity operation for analytic functions f by an integral on the domain, the Poincaré kernel p expresses every harmonic function g by an integral over the Silov boundary and the Szegö (or Cauchy) kernel h gives every analytic function f as an integral over the Silov boundary.

$$\begin{aligned} f(\mathbf{z}) &= \int_{D_d} k(\mathbf{z},\bar{\mathbf{w}}) f(\mathbf{w})\, dw, \quad k(\mathbf{z},\bar{\mathbf{w}}) = \frac{1}{V_D}(1 + |z_i z^i|^2 - 2\bar{z}_i z^i)^{-d} \\ g(\mathbf{z}) &= \int_{Q_d} p(\mathbf{z},\boldsymbol{\xi})\, g(\boldsymbol{\xi})\, d\xi, \quad p(\mathbf{z},\boldsymbol{\xi}) = \frac{1}{V_Q}\frac{(1 + |z_i z^i|^2 - 2\bar{z}_i z^i)^{d/2}}{|(z_i - \xi_i)(z^i - \xi^i)|^d} \\ f(\mathbf{z}) &= \int_{Q_d} h(\mathbf{z},\boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\xi, \quad h(\mathbf{z},\boldsymbol{\xi}) = \frac{1}{V_Q}[(x_i - e^{-i\varphi} z_i)(x^i - e^{-i\varphi} z^i)]^{-d/2} \\ V_D &= \frac{\pi^d}{2^{d-1} d!}, \quad V_Q = \frac{2\pi^{d/2+1}}{\Gamma(\frac{d}{2})} \end{aligned} \tag{14}$$

with the volumes V_D and V_Q of the domain and the Silov boundary, respectively. For the unbounded domain T_d the Szegö kernel is of the form $c((z_i - x_i)(z^i - x^i))^{-d/2}$ since it must be invariant under real translations (for $d = 4$ see [12]).

According to the construction of the domains the subgroup leaving invariant an interior point is the group $SO_d \times SO_2/Z$. Since the conformal group acts transitively on the domain it is sufficient to consider a special point, e.g. $O \in D_d$ and $Q \in T_d$. In D_d the group SO_d is just the linear orthogonal transformation of the whole domain. SO_2 is the multiplication by the phase factor $e^{i\varphi}$. In T_d the isotropy group of Q does not act linearly on the whole domain but only in the tangential space of the point Q. The group SO_d is established by the elements J_{ik} with $2 \leqslant i, k \leqslant d+1$ which in fact leave invariant the point Q. The only element of $SO(d, 2)$ which commutes with all elements of SO_d is then $J_{1d+2} = \frac{1}{2}(P_1 + V_1)$. It acts as SO_2. The transformation of the domain produced by the element J_{1d+2} is

$$\frac{dz_1}{d\varphi} = \frac{1}{2}\left(1 + \sum_1^d z_i^2\right)$$

$$\frac{dz_k}{d\varphi} = z_1 z_k \quad k = 2,\dots, d \tag{15}$$

which has the general solutions

$$z_1 = \frac{1}{2}\left(\tan\left(\frac{\varphi - \varphi_1}{2}\right) + \tan\left(\frac{\varphi - \varphi_2}{2}\right)\right)$$

$$z_k = c_k\left(\tan\left(\frac{\varphi - \varphi_1}{2}\right) - \tan\left(\frac{\varphi - \varphi_2}{2}\right)\right) \tag{16}$$

where φ_1, φ_2, c_k are complex numbers. Thus J_{1d+2} acts as SO_2 and in the tangential space at Q as the phase factor $e^{i\varphi}$. The group acting effectively in the tangential space at Q is then $SO_d \times SO_2/Z$ because the element $-I_d$ of SO_d has the same effect as the element $e^{i\pi} = -1$ of SO_2. The element of SO_2 leaving invariant the point $Q' = (\lambda + i\mu)\ 0,\dots, 0)$ $(\mu > 0)$ will be used, too. It is

$$F = \frac{1}{2\mu}((\lambda^2 + \mu^2)\, P_1 - 2\lambda D + V_1). \tag{17}$$

One can transform directly into the operator leaving invariant the point of the form $(\lambda + i\mu)\,(a_1, \dots)$ with real a_i fulfilling $a_i a^i = 1$. Defining $P = a_i P^i$ and $V = a_i V^i$ one has to change in expression (17) P_1 and V_1 into P and V, respectively.

Representations of the conformal group are constructed by determining the representations of the isotropy group. In the following only analytic representations are considered. That means that the isotropy group of an interior point has to be considered and not the isotropy of a boundary point, which is, e.g., for a point in M_d the Weyl group, i.e. the Poincaré group together with dilatation. For the representation induced from an interior point is also a representation on the boundary, but there could exist representations on the boundary which cannot be prolonged into the interior and thus would not be analytic. The irreducible representations of the compact group SO_d are well known. They consist of tensors with all traces vanishing and corresponding to a Young diagram [13]. In physics one considers ray representations which, however, are equivalent to vector representations for SO_d but they are not necessarily single-valued. According to Weyl the representations are at most double-valued. Single- and double-valued representations are described by spinor representations [14]. A spinor has the dimension $2^{[d/2]}$. General representations are then constructed by considering tensors with spinor indices and normal vector indices. The case of four and five dimensions will be dealt with in the following sections. The irreducible representations of SO_2 transform as $e^{in\varphi}$ and can be described by the one constant n which is integer for single-valued representations.

An irreducible representation of the isotropy group induces a representation on the domain. A spinor representation induces a field $f_\alpha(\mathbf{z})$ where α stands for all indices. In order to define the representation one has to find how the Lie elements act on the field. This is carried out in the unbounded domain where the action of the Lie elements on simple functions is given by (1) and (11). Translations and the Lorentz elements act in the usual way on spinor fields

$$L(f)_\alpha(\mathbf{z}) = f_\alpha(\mathbf{z}) - \epsilon L_{\beta\alpha} f_\beta(\mathbf{z}) - f_{\alpha,i}\,\delta z^i. \tag{18}$$

The indices α and β are composite indices and the expression $L_{\beta\alpha} f_\beta$ means the sum of the infinitesimal operators acting on the different indices in β. The minus sign and summation over the left index of L accounts for the fact that the tensor indices transform according to the transformation of the derivations $\partial/\partial z^i$, i.e. as the vectors in the tangential space. For the dilatation operator usually the dimension n' of the field is introduced

$$D(f)_\alpha = f_\alpha + \epsilon n' f_\alpha - f_{\alpha,i}\,\delta z^i. \tag{19}$$

The conformal elements can be divided into an antisymmetric part T acting as the Lorentz transformation and a dilatation

$$V_i(f)_\alpha = f_\alpha - \epsilon T_{\beta\alpha} f_\beta - f_{\alpha,i}\,\delta z^i + 2\epsilon n'' z_i f_\alpha. \tag{20}$$

The three constants n, n', and n'' are equal, as is shown by direct calculation. The element F for the group SO_2 leaving invariant the point Q' (17) transforms the spinor representation field at this point in the following way:

$$F(f)_\alpha = f_\alpha + \frac{\epsilon}{\mu}\left(\lambda(n'' - n') + in'\right) f_\alpha\big|_{Q'}. \tag{21}$$

The transformation according to the antisymmetric part T cancels out at this point. The element F is therefore the operator of SO_2 for the point Q' which is possible only for $n' = n''$. Clearly the dimension n' is then equal to the constant n of the representation of SO_2. Thus the representation induced by an irreducible representation of the isotropy group is constructed.

The question arises now if such an analytic representation is irreducible. Commonly it is assumed to be the case (e.g. [10, 12]). Tsu [15], however, has given an argument that this might not be true. The Casimir operators, which constitute the set of the invariant operators, do not act as differential operators on analytic representations. Thus there can be no splitting of the analytic representations by different eigenvalues of the Casimir operators in addition to the splitting by the Casimir operators of the isotropy group. A subgroup of the conformal group containing a Casimir operator which does act non-trivially on analytic representations is the Poincaré group. Its second-order Casimir operator is

$$\Delta \equiv P_i P^i. \tag{22}$$

The higher-order Casimir operators of $P(d)$ need not be considered since their eigenvalues are defined by the eigenvalue of Δ and by the irreducible representation of SO_d. The operator Δ is, of course, not invariant under the group C_d but one can construct a differential equation which might be invariant

$$\Delta f_\alpha = 0. \tag{23}$$

The problem is then to find those induced analytic representations for which equation (23) is invariant. These special representations then split into two parts. The large part not fulfilling (23) and the very small part solving equation (23). In the next section it is shown which representations in four dimensions fulfil (23) invariantly and can therefore be interpreted as massless particles. For the five-dimensional conformal group it will be shown then that only a special scalar field splits according to (23). This, however, does not prevent the group C_5 to be physically significant. It seems that the four-dimensional case is outstanding, since one can show that for all dimensions higher than four of all analytic tensor fields only the scalar field with dimension $n = 1 - d/2$ leaves invariant the equation (23) and is split thus into two irreducible parts. Spinor fields have not been considered in dimensions higher than five because of complexity. It is, however, improbable that there exists a double-valued representation of SO_d fulfilling (23) invariantly.

3. The Four-Dimensional Case and Massless Particles

Most equations describing massless particles are conformally invariant [3, 5, 16]. One can even begin by postulating that a theory of massless particles should be conformally invariant according to the following reasoning. A massless particle travels with the speed of light, i.e. on an isotropic trajectory in the four dimensional space-time. The group which transforms the set of all isotropic, not necessarily straight, trajectories into the same set is just the conformal group. On the other hand massless particles starting at time t_0 at the site $\mathbf{x}_0$ in any direction are found at a later time on a sphere in three-space. This suggests that the set of all three-dimensional spheres describes massless particles. The group transforming this set into the same set conserving tangential spheres tangential is again the conformal group [7]. In the following it will be shown that the analytic representations in T_4 are on the boundary M_4 the representations given by Gross [3] (beside the spin 0 field), for which Gross has shown that they are unitary. There are no other analytic representations fulfilling (23).

The spin representation has the following form in Minkowski metric after renumbering the coordinates [14]. Define the 4×4 matrices $\mathbf{S}_i$ and $\mathbf{S}_{ij}$ $(i, j = 0, 1, 2, 3)$

$$\mathbf{S}_0 \equiv \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \mathbf{S}_1 \equiv \begin{pmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}, \quad \mathbf{S}_2 \equiv \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad \mathbf{S}_3 \equiv \begin{pmatrix} 0 & i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}$$

$$\mathbf{S}_{ij} = -\mathbf{S}_{ji} \equiv \mathbf{S}_i \mathbf{S}_j, \quad i \neq j \tag{24}$$

where the σ_i are the usual Pauli matrices. The matrix $\mathbf{S}_0$ is hermitic whereas the remaining $\mathbf{S}_i$ are antihermitic. The spinor ψ_α is four-dimensional and a Lorentz transformation L_{ij} acts on the components as

$$\psi'_\alpha = \tfrac{1}{2} U_{\alpha\beta} \psi_\beta, \quad U_{\alpha\beta} = \tfrac{1}{2} S_{ij\alpha\beta} L^{ij}. \tag{25}$$

The matrices $\mathbf{S}_i$ and $\mathbf{S}_{ij}$ have the following properties

$$\begin{aligned} &\mathbf{S}_i \mathbf{S}_j + \mathbf{S}_j \mathbf{S}_i = 2 g_{ij} \\ &\mathbf{S}_{ij} S_{kl} = i \epsilon_{ijkl} \mathbf{K} - g_{ik} \mathbf{S}_{jl} + g_{il} \mathbf{S}_{jk} - g_{jl} \mathbf{S}_{ik} + g_{jk} \mathbf{S}_{il} - g_{ik} g_{jl} + g_{il} g_{jk} \\ &\mathbf{K} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{26}$$

It is useful to divide the matrices and the spinor into two-component parts

$$\boldsymbol{\psi} = \begin{pmatrix} \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \end{pmatrix}, \quad \mathbf{S}_i = \begin{pmatrix} 0 & \mathbf{S}_i^2 \\ \mathbf{S}_i^1 & 0 \end{pmatrix} \quad \mathbf{S}_{ij} = \begin{pmatrix} \mathbf{S}_{ij}^1 & 0 \\ 0 & \mathbf{S}_{ij}^2 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} \mathbf{U}^1 & 0 \\ 0 & \mathbf{U}^2 \end{pmatrix}. \tag{27}$$

The additional equations are then valid

$$S^l_{i s_{\alpha\beta}} S^{ljs}_{\gamma\delta} + S^l_{i s_{\gamma\delta}} S^{ljs}_{\alpha\beta} = 2\delta_i^j(-2\delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\alpha\beta}\delta_{\gamma\delta}), \quad l = 1,2. \tag{28}$$

Under pure rotations the irreducible representations are $D'_{p/2,q/2}$, i.e. tensors $f_{\alpha_1\ldots\alpha_p,\beta_1\ldots\beta_q}$ where the α_i transform as $\boldsymbol{\varphi}^1$ and the β_j as $\boldsymbol{\varphi}^2$ and the tensor is symmetrized separately in the α_i and in the β_j. If mirror operations are considered, too, the irreducible representations are $D_{p/2,q/2} = D'_{p/2,q/2} + D'_{q/2,p/2}$ for $p \neq q$ and for equal p and q there are two representations $D^+_{p/2,p/2}$ and $D^-_{p/2,p/2}$ which differ in sign under mirror operations [14].

The operator $\Delta = P_i P^i$ is invariant under the Poincaré group and the equation (23) is also invariant under dilatations. It is thus only necessary to consider invariance of (23) under a special conformal element $V = b^i V_i$. The transformation of the coordinates by V is

$$\delta z_i = \epsilon(z_i b^j - z^j b_i + z_k b^k \delta_i{}^j)\, z_j \equiv \epsilon t_i{}^j z_j. \tag{29}$$

The covariant transformation is thus effected by the tensor t_{ij} whereas contravariant vectors are transformed by the matrix

$$t'_{ij} = \frac{\partial \delta z_i}{\partial z^j} = 2t_{ij} \tag{30}$$

Writing the matrices $\mathbf{U}^l$ in terms of t_{ij} one finds the transformation law for the spinor representations

$$\begin{aligned} f'_{\alpha_1\ldots\beta_q} = f_{\alpha_1\ldots\beta_q} &+ \epsilon \sum_k U^1_{\alpha_k\gamma} f_{\alpha_1\ldots\overset{k}{\gamma}\ldots\alpha_p,\beta_1\ldots\beta_q} + \epsilon \sum_l U^2_{\beta_l\delta} f_{\alpha_1\alpha_p,\beta_1\ldots\overset{l}{\delta}\ldots\beta_q} \\ &- \epsilon f_{\alpha_1\ldots\beta_q,l}\, \delta z_l + 2\epsilon n b_i z^i f_{\alpha_1\ldots\beta_q}. \end{aligned} \tag{31}$$

One has now to show that the operator Δ applied to the function f' is zero under the condition that $\Delta f = 0$. Using this condition the following new equation is found which must be fulfilled in order that equation (23) is conformally invariant

$$\begin{aligned} \sum_k U^1_{\alpha_k\gamma,s} f_{\alpha_1\ldots\overset{k}{\gamma}\ldots\alpha_p,\beta_1\ldots\beta_q,}{}^s + \sum_l U^2_{\beta_l\delta,s} f_{\alpha_1\ldots\alpha_p,\beta_1\ldots\overset{l}{\delta}\ldots\beta_q,}{}^s \\ + (2+2n)\, b^s f_{\alpha_1\ldots\beta_q,}{}^s = 0 \end{aligned} \tag{32}$$

where $g,^s$ means the derivative according to z_s and in the last term it has been used $t_{i,k}^{\,k} = (2-d)\, b_i$ and $d = 4$. Equation (23) is thus only conformally invariant if equation (32) is fulfilled invariantly, too. Thus a new conformal transformation $V'(\mathbf{b}')$ is used. The transformed functions $f' = V'(f)$ are introduced in equation (32) instead of the old f and it is used that the tensors f fulfil equation (32). Then a new condition is found for the representation fields f to fulfil (23) involving no derivations.

$$\begin{aligned} \sum_{k \neq l}\sum U^1_{\alpha_k\gamma,s} U^{,1}_{\alpha_l\delta,}{}^s f_{\alpha_1\overset{k}{\gamma}\ldots\overset{l}{\delta}\ldots\alpha_p,\beta_1\ldots\beta_q} + \sum_k U^1_{\alpha_k\gamma,s} U^{,1}_{\gamma\delta,}{}^s f_{\alpha_1\ldots\overset{k}{\delta}\ldots\alpha_p,\beta_1\ldots\beta_q} \\ -2\sum_k (n b^{,s}\, U^1_{\alpha_k\gamma,s} + (n+1)\, b^s U^{,1}_{\alpha_k\gamma,s}) f_{\alpha_1\ldots\overset{k}{\gamma}\ldots\alpha_p,\beta_1\ldots\beta_q} \end{aligned} \tag{33}$$

$$+ \text{ analogous terms in } U^2 \text{ acting on } \beta_i$$

$$+ \sum_k \sum_l (U^1_{\alpha_k\gamma,s} U'^2_{\beta_l\delta,}{}^s + U'^1_{\alpha_k\gamma,}{}^s U^2_{\beta_l\delta,}{}^s) f_{\alpha_1 \ldots \overset{k}{\gamma} \ldots \alpha_p, \beta_1 \ldots \overset{l}{\delta} \ldots \beta_q}$$

$$+ 4n(n+1)\, b_i b'^i f_{\alpha_1 \ldots \beta_q} = 0.$$

This has to be fulfilled for any $\mathbf{b}$ and $\mathbf{b}'$. Equation $U^l_{\alpha\beta,s} = b^i S_{si_{\alpha\beta}}$ and equation (28) simplify (33) considerably. There remain mostly terms containing $b_i b'^i$. Only the mixed term U^1U^2 contains $b_i b'_j$ $(i \neq j)$. Choosing special values $\mathbf{b}$ and $\mathbf{b}'$, e.g. $b_i = \delta_{i0}$, $b'_j = \delta_{j1}$, a non-trivial linear equation is found. Since the representation f is irreducible under the isotropy group this would mean that f disappears. Thus non-zero solutions of (23) are possible only with either p or q zero [16]. The remaining equation, e.g. for $q = 0$, reads then

$$b_i b'^i (4n(n+1) - p(p+2)) f_{\alpha_1 \ldots \alpha_p} = 0. \tag{34}$$

This is fulfilled when the bracket disappears. Integer or half-integer solutions are first

$$n = -1 - \frac{p}{2} = -1 - s \tag{35}$$

where s is the spin. The irreducible representation, including mirror operations, is then $D_{s,0} = D'_{s,0} + D'_{0,s}$ the direct sum of the tensors $f^1_{\alpha_1 \ldots \alpha_p}$ and $f^2_{\beta_1 \ldots \beta_p}$.

Using the connection (35) between the spin and the dimension the first-order equation (32) has to be fulfilled for any $\mathbf{b}$. This gives

$$\sum_l S_{is}{}^{l'}_{\alpha_l\gamma} f^{l'}_{\alpha_1 \ldots \overset{l}{\gamma} \ldots \alpha_p,}{}^s + p f^{l'}_{\alpha_1 \ldots \alpha_p, i} = 0 \quad i = 0, \ldots, 3 \quad l' = 1, 2. \tag{36}$$

Equations (36) are easily shown to be equivalent to the equations

$$S^l_{s\alpha_1\gamma} f^l_{\gamma\alpha_2 \ldots \alpha_p, s} = 0 \tag{37}$$

because of the symmetry of the tensors according to the indices and because of the special value p as multiplication factor in the second term of (36). Equation (37) is the usual Dirac-form equation as given, e.g. by Gross [3]. The fields fulfilling (37) fulfil $\Delta f = 0$, too. Thus the tensors fulfilling (37) with dimension $n = -1 - s$ are the only possible analytic representations solving (23). The scalar field fulfils no first-order equation but invariantly equation (23). The equation (34) has a second solution $n = p/2$. However, equations (36) are then not equivalent to one equation. Direct calculation shows that all derivatives disappear, i.e. only a constant fulfils (23).

The wave equation has two different kinds of solution. First, the solution for free massless particles fulfilling equation (37) and, second, the solution coupled with a massive particle. If this massive particle is at rest at the origin the solutions of (37) are independent of time and a representation of the three-dimensional rotation group. Such a classical field is possible only for integer spin. The representations $D_{s,0}$ for integer spin have a natural representation as tensors of rank $2s$ corresponding to a Young diagram with two rows and s columns. The spin one, i.e. the photon field, is then a second-order antisymmetric tensor. Time-independent solutions $\mathbf{f}^1$ and $\mathbf{f}^2$ can be clearly combined since they fulfil the same differential equations. Space inversion corresponds to the operation $\mathbf{f}^1 \to \mathbf{f}^2$, $\mathbf{f}^2 \to \mathbf{f}^1$. Thus the linear combinations $f^{\pm}_{\alpha_1 \ldots \alpha_p} = f^1_{\alpha_1 \ldots \alpha_p} \pm f^2_{\alpha_1 \ldots \alpha_p}$ correspond to positive and negative parity. Both these fields are a representation d_3 of SO_3, i.e. they can be represented by symmetric tensors of rank s

with vanishing traces in three-space. The spin-one field, for example, is represented as spinor field $f^1_{\alpha\beta} \oplus f^2_{\gamma\delta}$ or as tensor $f_{ik} = -f_{ki}$, which splits under SO_3 into the electric field **E** and the magnetic field **H**. Static solutions fulfil the equations

$$\begin{aligned} &f_{2\alpha_2 \dots \alpha_p,1} - if_{2\alpha_2 \dots \alpha_p,2} + f_{1\alpha_2 \dots \alpha_p,3} = 0 \\ &f_{1\alpha_2 \dots \alpha_p,1} + if_{1\alpha_2 \dots \alpha_p,2} - f_{2\alpha_2 \dots \alpha_p,3} = 0 \end{aligned} \tag{38}$$

for all $\alpha_2 \dots, \alpha_p = 1, 2$ $(p = 2s)$. From (38) it follows that the tensors are solutions of the Laplace equation. Therefore they can be expanded into the forms $g_{lm} = Y_{lm} r^{-l-1}$ with the spherical harmonics Y_{lm}. Since the fields constitute a representation of SO_3 around the origin the series are divided according to different indices l. It is now shown that forms with $l < s$ cannot be combined to solve equation (38).

Equations (38) can be combined to

$$f_{11\alpha_3 \dots \alpha_p,1} + if_{11\alpha_3 \dots \alpha_p,2} + f_{22\alpha_3 \dots \alpha_p,1} - if_{22\alpha_3 \dots \alpha_p,2} = 0. \tag{39}$$

Expanding the tensor f into a series of functions g_{lm}

$$f_{11 \dots 1} = \sum_{m=-l}^{l} a_{1m} g_{lm}, \quad f_{2211 \dots 1} = \sum a_{2m} g_{lm} \cdots f_{22 \dots 2} = \sum a_{s+1m} g_{lm} \tag{40}$$

and using the fact that g_{lm} is of the form $g_{lm} = (x + iy)^n \, h_{lm}(r, z)$ equation (39) determines connections between the coefficients a_{rm}

$$a_{rm-1} = a_{r+1m+1}, \quad r = 1, \dots, s, \quad m = -s + 1, \dots, s - 1. \tag{41}$$

A special chain of equations is then

$$a_{1-l} = a_{2-l+2} = a_{3-l+4} = \dots = a_{s+1l}. \tag{42}$$

Obviously a non-zero solution is possible only for $l \geqslant s$. For other chains derived from (41) to be non-zero the index l must be even higher. In the case $l = s$ the elements entering in (42) are the only non-zero elements. The following solution of (38) is now found. The tensors f are not yet normalized. The normalization factor is $c = ((2r)!\,(2s - 2r)!)^{-1/2}$ where $2r = s - m$ is the number of 1 in $f_{\alpha_1 \dots 2s}$, i.e. $f'_{sm} \equiv f^{2r}_{1 \dots 1\,2 \dots 2} c_r$ transforms as a spherical harmonic Y_{lm}. Therefore it is expected that the tensor field $F_{sm} = f'_{sm} Y_{s-m} r^{-s-1}$ is a solution of (38). This tensor field transforms as the identity under rotations around the origin, i.e. it is the 'monopole' and all solutions of (38) can be found by differentiating the monopole field. It suffices now to show that, for example, F_{ss} and F_{ss-1} fulfil (38) since the remaining elements can be obtained by rotation. Using the forms of the spherical harmonics $Y_{ss} = c(x + iy)^s$ and $Y_{ss-1} = -c\sqrt{2s}z(x + iy)^{s-1}$ direct calculation shows that equation (38) is fulfilled. Thus every integer massless spin field can produce a monopole and higher multipole fields. The monopoles are, written in real space,

$$f \sim \frac{1}{r} \qquad \text{scalar}$$

$$f_i \sim \frac{x_i}{r^3} \qquad \text{vector}$$

$$f_{ij} \sim (x_i x_j - \tfrac{1}{3}\delta_{ij} r^2)/r^5 \qquad \text{rank-two tensor} \tag{43}$$

and so on. Possible interpretations of the scalar and vector fields are the gravitational and electric fields, respectively. Higher spin fields are not ruled out by the conformal group. However, even if they exist it would be hard to detect them, since it is impossible to create a homogeneous static field out of a spin-two or higher rank field. This can be seen, e.g. by looking at an equal distribution of such 'monopoles' on a plane. Then the resulting field above the plane cancels out.

Gross [3] has shown that the particles with non-zero spin produce a unitary representation of the conformal group on real Minkowski space. In the complex domain T_4 an invariant norm exists for scalars with dimension $n < -2$ [12]

$$\|f\|^2 = \int |f(\mathbf{z})|^2 (y_i y^i)^{-4-n} dz \tag{44}$$

which can be taken over to the domain D_4. In a limiting procedure Rühl shows that even for $n = -2$ the norm can be defined. According to Graev [17] the conditions for the dimension n remain the same for general spinor fields. Since the isotropy group is $SO_4 \times SO_2$ the expression entering the integral is positive definite. This is for integer spins expressed by tensors

$$\|f\|^2 = c_{sn} \int \sum_{i_1 \ldots i_s = 0}^{3} \bar{f}_{i_1 \ldots i_{2s}} f_{i_1 \ldots i_{2s}} (y_i y^i)^{-4-n} dz. \tag{45}$$

For spinor fields the expression looks just the same when interpreting the indices i_k as spinor indices. The analytic functions can be expressed by an integral on the boundary. In the case of spin one one can use the fact that the fields fulfil equation (23). Then the norm can be transformed into the double integral over three-space of Gross [3] using the formulae of Courant-Hilbert [18] for the solution of the wave equation. The two norms differ thus only by a constant for fields for which both norms are finite. The norm of Gross is equivalent to the usual norm defined with the help of the Fourier coefficients. The scalar field f has no invariant norm. However, its four gradient $f_{,i}$ is a vector field allowing for a norm since its dimension is the same as for the photon field. The half-spin field seems to have an invariant norm only as an integral on three-space [3].

The group $SO(4,2)$ is contained in the enveloping group $SU(2,2)$ with the same Lie algebra. On the other hand $SO(4,2)$ contains the group $PO(4,2) \simeq C_4$ as a subgroup. Thus one has the chain $PO(4,2) \subset SO(4,2) \subset SU(2,2)$ where the lower group is the factor group of the higher group divided by a two-component invariant group. This gives the following behaviour of the representations. In $SU(2,2)$ all representations are single-valued. In $SO(4,2)$ there are single- and double-valued representations, namely spinor representations. In $PO(4,2)$, however, there exist single-, double-, and four-valued representations in dependence of the behaviour of the representations of $SO(4,2)$ under the element $-I = -I_4 \times -I_2$. The inversions $-I_4$ and $-I_2$ are real rotations because of even dimension. The inversion $-I_2$ corresponds to a rotation of π, i.e. a multiplication with $e^{i\pi n}$. The inversion $-I_4$ is $\mathbf{S}_0 \mathbf{S}_1 \mathbf{S}_2 \mathbf{S}_3 = i\mathbf{K}$. The representation of $PO(4,2)$ of the spin fields solving equation (23) is then as follows. *Integer spin*: For odd spin the inversion $-I_4$ acts as -1 but the dimension n is even whereas for even spin $-I_4$ acts as the identity but n is odd. Thus in both cases a double-valued representation results unifying $(\mathbf{f}^1, \mathbf{f}^2)$ and $(-\mathbf{f}^1, -\mathbf{f}^2)$. *Half integer spin*: The representations of $SO(4,2)$ are already double-valued unifying $(\mathbf{f}^1, \mathbf{f}^2)$ and $(-\mathbf{f}^1, -\mathbf{f}^2)$. The operation $-I_4 \times -I_2$ transforms $\mathbf{f}^1$ into $\pm\mathbf{f}^1$ and $\mathbf{f}^2$ into $\mp\mathbf{f}^2$ where the upper and lower sign correspond to $(s - \frac{1}{2})$ even and odd, respectively. Thus a four-valued representation of the conformal group results unifying $(\mathbf{f}^1, \mathbf{f}^2)$, $(-\mathbf{f}^1, -\mathbf{f}^2)$, $(\mathbf{f}^1, -\mathbf{f}^2)$, and $(-\mathbf{f}^1, \mathbf{f}^2)$. The multivaluedness of half integer spin representations does not disturb physical interpretation. However, fields with classical interpretation should be single-valued. Considering the spinfields as representation of the conformal group C_4 this seems to be impossible. But in the following sections it will be shown how it is possible to interpret at least the scalar and spin-one field as single-valued representations using the conformal group C_5.

4. The Five-Dimensional Case and Massive Particles

According to the suggestion of Wyler [7] the group connected with massive particles is the five-dimensional conformal group C_5. There are different arguments for it. In relativity theory the proper time s of a particle plays an important role since it determines all time-dependent actions connected with the particle as, for example, atomic fission. The proper times of different particles travelling with different speed on not necessarily straight lines are comparable when the particles meet. Thus differences in proper time can be measured in general (e.g. the twin experiment, where one of the twins is flying fast for a while and remains, therefore, younger than the second twin at rest). It is therefore not unreasonable to consider representations in a five-dimensional space with the proper time $s = x_4/c$ as fifth dimension. The metric is that of five-dimensional Minkowski space since the proper time is determined by

$$dx_4^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2. \tag{46}$$

Particles travel on isotropic curves in this five-dimensional space. A single particle can be described, for example, by a plane wave

$$f = e^{i\mathbf{kx}}, \quad k^i k_i = 0, \quad \mathbf{k} = \frac{1}{\hbar}\left(\frac{E}{c}, p_1, p_2, p_3, mc\right). \tag{47}$$

Combining different $\mathbf{k}$ with $\mathbf{k}^2 = 0$, i.e. allowing also for an uncertainty in the mass, localized states can be constructed which travel in space-time–proper-time without spreading since there is no dispersion, just as for massless particles in four dimensions. The group leaving invariant the set of isotropic curves is the conformal group. Analogous as with the massless particles one can consider particles starting from a point $\mathbf{x}_0$ at time t_0 and proper time s_0 in any direction with any speed. They are found at a later proper time $s > s_0$ on a hyperboloid in four-space with 'radius' $c(s - s_0)$. The transformation group for hyperboloides conserving tangent hyperboloides tangent is again the conformal group [7]. Instead of the hyperboloides for given proper time one can look at the spheres in space–proper-time with given time. The transformation group is again C_5. Thus one could guess that the conformal group C_5 should be the invariance group for massive particles. However, this is not correct.

The movement of a particle can be described without knowing the proper time. The scattering of different particles at different times is well defined in space-time. Thus physically admittable transformations should not change the sequence of such events except an eventual turn in time. This condition means that straight lines parallel to the proper-time axis should remain parallel lines. The subgroup G in C_5 fulfilling this condition consists of the four-dimensional Poincaré group $P(4)$, the dilatation D and the translation P_4 along the proper-time axis. Thus it seems at first sight that little is won by introducing the five-dimensional space. But the conformal group is still important. It transforms the world into a different but equivalent one. The situation is similar to the problem of an electron around two nuclei (H_2^+ molecule) in quantum mechanics. The invariance group for the electron Hamiltonian is SO_2. However, the three-dimensional rotation group SO_3 transforms the Hamiltonian into a different but equivalent one. Thus the Schrödinger equation is a 'representation' of SO_3 which transforms as the identity under a SO_2. Wave functions are written in the space connected with SO_3 not SO_2. In a similar manner it seems meaningful to look for fields in five dimensions and for equations which are invariant under G but covariant under C_5.

The two Lie elements D and P_4, which the group G contains in addition to the Poincaré group, are connected with the dimension of the fields and with mass. The question arises which Lie groups exist in C_5 containing G and thus serving as an approximate symmetry. The group C_4 does not contain G because of the element P_4 in G. There is essentially only one possibility, namely to add an element $V = a^i V_i$ to the group G where $\mathbf{a}$ is any fixed real five-vector fulfilling $a^i a_i = 1$. The three elements V, D, and $P = a^i P_i$ can be added to leave the point $v\mathbf{a}$ invariant. Formula (17) gives this operator which is integrated to the Lie group SO_2 in the conformal group. The three elements constitute the Lie algebra of the group $SO(2, 1)$ as the commutation relations show.

$$[P, D] = P, \quad [V, D] = -V \quad [P, V] = 2D \tag{48}$$

which gives the normal basis for $SO(2, 1)$

$$I_x = iL_x = \tfrac{1}{2}(V - P), \quad I_y = iL_y = D, \quad I_z = L_z = \tfrac{1}{2}(P + V). \tag{49}$$

The elements L_i are the operators corresponding to SO_3. For fixed $\mathbf{a}$ to any complex v corresponds an operator F for SO_2. The numbers v can be thought of as points on a hyperboloid in three dimensions dual to the half-sphere. The operator F corresponding to this point is then the 'rotation' element of $SO(2, 1)$ leaving this point invariant. Instead of looking at these different operators V and P for different $\mathbf{a}$ interpolating operators can be constructed which act on all 'real' lines $v\mathbf{a}$ as the corresponding $SO(2, 1)$. They are

$$P' = \frac{z^i}{\sqrt{z^j z_j}} P_i = \frac{1}{\sqrt{z^j z_j}} D, \; V' = \frac{z^i}{\sqrt{z^j z_j}} V_i = \sqrt{z^j z_j}\, D, \; D. \tag{50}$$

These new operators constitute $SO(2, 1)$, too, but are no more Lie elements of C_5. For $SO(2, 1)$ can also be chosen the elements P_4, V_4 and D. These last two possibilities are in so far special as the thus constructed group $SO(2, 1)$ commutes with the Lorentz group $L(4)$. Solutions of conformally covariant equations are thus expected to be approximate representations of the direct product $L(4) \times SO(2, 1)$. This suggests the possible interpretation of $SO(2, 1)$ as the isospin group since representations of $SO_3 \subset SU_2$ can be transformed into representations of $SO(2, 1)$.

The analytic representations of the conformal group in the domain T_5 are spinor fields. A spinor for SO_5 has four components as for SO_4. There is one new antihermitian matrix $\mathbf{S}_4$ corresponding to the new coordinate x_4 and then $\mathbf{S}_{ij}$ is defined as before (24)

$$\mathbf{S}_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{S}_{ij} = -\mathbf{S}_{ji} = \mathbf{S}_i \mathbf{S}_j, \quad i \neq j$$

$$\mathbf{S}_i \mathbf{S}_j + \mathbf{S}_j \mathbf{S}_i = 2g_{ij}. \tag{51}$$

The matrices do not fulfil an equation of the form (28). The irreducible representations $E_{p/2,\ r}$ are tensors $f_{\alpha_1 \dots \alpha_p.\ i_1 \dots i_r}$ which are in the spin indices α_k symmetrized and in the vector indices i_k symmetrized with all traces vanishing [14]. Under the subgroup SO_4 the representation $E_{p/2,\ r}$ reduces out into

$$E_{p/2,r} = (D_{p/2,0} + D_{(p-1)/2,\,1/2} + \dots + D_{t/2,(p-t)/2}) \\ \times (D^+_{00} + D^+_{1/2,1/2} + \dots + D^+_{r/2,r/2}), \qquad t = [p/2] \tag{52}$$

which can be reduced out according to the formulae of Cartan [14].

Just as in four dimensions it is useful to determine those representations which fulfil equation (23) invariantly. The formula for the transformation of a field $f_{\alpha_1 \dots \alpha_p.\ i_1 \dots i_r}$ is similar to (31). The indices α_k run from one to four, the indices β_l have to be replaced by i_k running from zero to four, and $\mathbf{U}^2$ has to be replaced by $\mathbf{2T}$. The condition for invariantly fulfilling (23) then results in a formula analogous to (33), which, however, cannot be simplified in the same manner since an equation of the form of (28) does not hold. It suffices now to consider special cases. For $b_i = b'_i = \delta_{i3}$ the following equation is found for the indices $\alpha_i = 1$, $i_k = 4$

$$c f_{1\,1,\,4\,4} - 2pr(i f_{31\,1,\,34\,4} + i f_{31\,1,\,04\,4} + i f_{41\,1,\,14\,4} + f_{41\,1,\,24\,4}) = 0. \tag{53}$$

Since f is an irreducible representation of SO_5 it follows that $pr = 0$ and $c = 0$, i.e. either p or r disappears. For $p = 0$ a second equation is found considering the five cases $b_i = b'_i = \delta_{ij}$ ($j = 0, \dots, 4$) and adding the five equations. This new equation is proportional to $b_i b^i$ and its coefficient must disappear.

$$47r + 4r^2 - 15n - 10n^2 = 0 \\ c = 9r - 3n - 2n^2 = 0. \tag{54}$$

This can be solved only for $r = 0$, i.e. a scalar field. The dimension $n = -3/2$ gives then the field invariantly solving equation (23) whereas $n = 0$ gives only a constant because of the first-order equations. The case $r = 0$, $p \neq 0$ is treated by regarding $b_i = \delta_{ii'}$, $b'_i = \delta_{ij'}$, with $i' \neq j'$. There remains then an equation

$$\sum_{k \neq l} \sum S_{is_{\alpha_k}\gamma}\, S_{j\,\alpha_c\delta}^{\ s}\, f_{\alpha_1 \dots \overset{k}{\gamma} \dots \overset{l}{\delta} \dots \alpha_p} = 0. \tag{55}$$

In contrast to the four-dimensional case this gives in fact equations for the fields in default of an equation of the form of (28). Thus only scalar fields of dimension $n = -3/2$ fulfil invariantly $\Delta f = 0$.

As has been argued at the beginning of the section it is not necessary to find conformally invariant equations. The equations need only be invariant under G but it must be possible to prolong them continuously to equations after conformal transformations. Therefore the fields must be representations of C_5. In the next section differential equations are proposed which are indeed conformally covariant. They are then generalized to functional differential equations producing an infinite series of covariant differential equations for multipoint functions.

5. Covariant Differential Equations

Introducing the vector field $\mathbf{A}$ and the spinor field ψ' on five-dimensional space the following non-linear partial differential equations prove to be covariant.

$$
\begin{aligned}
&(S_{i_{\alpha\beta}} P^i + iS_{i_{\alpha\beta}} A^i)\,\psi'_\beta = 0 && \text{a)}\\
&f_{ik,}{}^{k} + \bar{\psi}'_\alpha(\bar{z})\, S_{0_{\alpha\beta}}\, S_{i_{\beta\gamma}}\, \psi'_\gamma = 0 && \text{b)} \qquad (56)\\
&f_{ik} \equiv A_{i,k} - A_{k,i}. && \text{c)}
\end{aligned}
$$

Under the five-dimensional Poincaré group equation a) transforms as a spinor, equation b) as a five-vector, and equation c) as an antisymmetric tensor. In equation b) it has to be used that $\bar{\mathbf{S}}_0\bar{\psi}$ transforms as $\mathbf{C}\psi$ ($\mathbf{C} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ in the representation (24) and (51) of the spin) [14] and thus $\bar{\psi}^*\mathbf{S}_0\psi$ as a scalar and $\bar{\psi}^*\mathbf{S}_0\mathbf{S}_i\psi$ as a vector. Therefore the set of equations (56) is invariant in the sense that with the fields $\mathbf{A}$ and ψ fulfilling (56) the transformed fields $\mathbf{A}'$ and ψ' fulfil the same set of equations. Under dilatation D the equation a) is invariant only if A_i has the same dimension as P_i, i.e. minus one. Then follows from c) the dimension minus two for f_{ik} and from b) that the dimension of ψ must be $-3/2$. The fields transformed by the infinitesimal special conformal element $V = b_i V^i$ do not fulfil (56) any more, but equations with A_i changed to $A_i + 2b_i$ in a) and $f_{ik}{}^{k}$, changed to $f_{ik,}{}^{k} + 2b^k f_{ik}$ in b). This transformation of the equations can be continued to a representation of the whole group C_5 by writing

$$
\begin{aligned}
&(S_{i_{\alpha\beta}} P^i + iS_{i_{\alpha\beta}} A^i)\;(g\cdot\psi'_\beta) = 0 && \text{a)}\\
&(g\cdot f_{ik}),{}^{k} + g\bar{\psi}'_\alpha S_{0_{\alpha\beta}} S_{i_{\beta\gamma}}\, \psi'_\gamma = 0 && \text{b)} \qquad (57)\\
&f_{ik} = A_{i,k} - A_{k,i}. && \text{c)}
\end{aligned}
$$

The function g is a simple scalar under the group G and has the dimension one under the special conformal elements V:

$$
g' = 2(b_i x^i)\, g - g_{,k}\, \delta z^k. \qquad (58)
$$

The representation g is, however, not an analytic representation, since the dimension is different under dilatation and a special conformal element. Thus equations (56) are invariant under G and are a 'representation' of the conformal group C_5.

The matrices $\mathbf{S}_i$ do not yet correspond to the usual matrices γ_i. The element $\mathbf{S}_4$ should be the identity matrix since it gives the element $I\cdot m$ for functions $\psi \sim e^{ims}$ ($m =$ mass). It is impossible to arrive at this matrix by unitary transformations of ψ alone. The following procedure leads to the usual form of the Dirac equation from a). With the matrix $\mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ equation a) is multiplied from the left and the new matrices $\mathbf{S}'_i = \mathbf{K}\mathbf{S}_i$ are defined. The vector in b) can be rewritten as $\bar{\psi}'^*\mathbf{S}_0\mathbf{S}_i\psi' = -\bar{\psi}'^*\mathbf{S}'_0\mathbf{S}'_i\psi'$ since $\mathbf{S}_0\mathbf{K} = -\mathbf{K}\mathbf{S}_0$ and $\mathbf{K}^2 = 1$. Then new spinors ψ are introduced by the unitary 4×4 matrix $\mathbf{U}$

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\boldsymbol{\psi} = \mathbf{U}\boldsymbol{\psi}'$$

$$\boldsymbol{\gamma}_i \equiv i\mathbf{U}\mathbf{S}_i'\mathbf{U}^* = i\mathbf{U}\mathbf{K}\mathbf{S}_i\mathbf{U}^*$$

$$\boldsymbol{\gamma}_i = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{59}$$

Equation (56) can now be rewritten with the help of the usual matrices $\boldsymbol{\gamma}_i$. This new form corresponds better to the form of the Dirac equation. The form is no more invariant under the five-dimensional Lorentz transformation but only under the four-dimensional one since multiplication with $\mathbf{K}$ corresponds only for $\mathbf{S}_0, \ldots, \mathbf{S}_3$ to a unitary transformation. It is possible to use gauge transformations for the field $\mathbf{A}$ and the spin field. Introducing the new field ψ' by $\psi = \psi e^{i\chi(z)}$ with the real function in real space χ the equations do not change the form if $\mathbf{A}$ is replaced by $A_i' = A_i + \chi_{,i}$. If one chooses such a gauge that $A_{k,}{}^{k}$ disappears the following equations result (the prime has been omitted)

$$\left.\begin{aligned} &(i\gamma_{i_{\alpha\beta}} P^i - \gamma_{i_{\alpha\beta}} A^i)\,\psi_\beta = 0 && \text{a)} \\ &A_{i,k}{}^{k} + \bar{\psi}_\alpha \gamma_{0_{\alpha\beta}} \gamma_{i_{\beta\gamma}} \psi_\gamma = 0. && \text{b)} \end{aligned}\right\} \tag{60}$$

Under the four-dimensional Poincaré group the five vector $\mathbf{A}$ is split into the scalar A_4 and the four vector $\mathbf{a} = (A_0, A_1, A_2, A_3)$. This four vector can directly be interpreted as the electromagnetic potential looking at equation (60a). For a solution of a) with the proper time dependence e^{ims} the Dirac equation results with the only difference that in addition a scalar field A_4 appears. It is natural to identify it with the gravitational field. The vector components of equation b) correspond to the inhomogeneous Maxwell equations. In particular the charge density is represented by $\rho = \bar{\psi}_\alpha \psi_\alpha$. In the usual representation of the γ_i the first and second components correspond to the electron, the third and fourth components to negative energy states which are interpreted as positrons after charge conjugation. Charge conjugation changes the relative sign of $\mathbf{a}$ and ρ thus justifying the above expression as charge density. The mass density corresponding to the inhomogeneous scalar equation for the gravitational field is $\rho_m = \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 - \bar{\psi}_3\psi_3 - \bar{\psi}_4\psi_4$ which is indeed a scalar under the group $P(4)$. The non-linearity of the equations does not allow the introduction of normalization constants for the fields. If one wants to normalize them in some way a constant factor has to be introduced, e.g. in equation b). It is expected that only for certain values (perhaps only one) of this factor non-zero solutions exist.

The vector field $\mathbf{A}$ and the spinor field $\boldsymbol{\psi}$ belong to analytic representations of the conformal group $PO(5, 2)$ with dimensions -1 and $-\frac{3}{2}$, respectively, as follows from the invariance condition for equations (56). They are connected with massless particles Setting $\mathbf{A}$ to zero in (60) and regarding massless particles, i.e. fields not depending on proper time, equation a) is just the Dirac equation for a massless spin-half particle whose dimension is $-\frac{3}{2}$ as it must. On the other hand the scalar field A_4 has the correct dimension -1 as the scalar massless field and the spin-one field corresponding to the antisymmetric tensor $f_{ik} = A_{i,k} - A_{k,i}$ has the dimension -2 of the spin-one massless field. Solutions of (60) seem to have therefore a strong connection to massless fields of spin 0, $\frac{1}{2}$, and 1. Since the subgroup G of the group $PO(5, 2)$ is considered the elements I and $-I_5 \times -I_2$ should be identified analogously as in the four-dimensional case. The inversion $-I_5$ turns the sign of a spinor and of a vector. The dimension of the vector $\mathbf{A}$ is -1 and thus I_2 turns the sign, too. Therefore the spin zero field A_4 and the spin-one field $\mathbf{a}$ are single-valued representations in contrast to the four-dimensional case. The difference lies in the additional inversion of the proper time. The spin half field remains four-valued but in a more trivial way. Spins multiplied by one of the four numbers $(1, -1, i, -i)$ belong to the identity.

It seems to be impossible to generalize directly equations (56) by introducing higher spin fields. E.g. if the field $\boldsymbol{\psi}$ contains more indices the infinitesimal element V introduces in (56a) new terms with factors b_i, where the indices i 'act' on all indices in $\boldsymbol{\psi}$. It is then impossible to prolong the equations as a 'representation' of C_5 by introducing the function g. New covariant equations involving multipoint functions which are coupled together are found by considering analytic functionals on spinor and vector fields.

Analytic functionals on analytic functions of one or more complex variables have been described by Pellegrino [19]. The general multidimensional case is much more complicated than the one-dimensional case since integrals over different curves have to be considered. This results in possible multivaluedness of the expressions. The situation is simplified in the case of symmetric spaces as T_d and D_d since a Szegö kernel exists which allows to look at volume integrals on the surface of the domain instead of independent integration over different curves. In spaces with a dimension higher than one as argument of a functional can serve not only a function but also analytic tensors or spinors. The case of functionals on analytic functions in five dimensions is considered first. All formulae are given as obvious generalizations of the formulae given by Pellegrino [19]. Exact proofs for these generalizations seem not to exist in the mathematical literature. For the domain of analyticity of the functions and tensors serving as arguments of the functionals the domain $T_5^- \equiv -T_5$ dual to T_5 is chosen ($\mathbf{z} \in T^- \Leftrightarrow \bar{z} \in T_5$).

A linear functional $F[f]$ can be expressed by an integral over the boundary, i.e. over real Minkowski space. In order to derive this form the value of F for the Szegö kernel (14) for T_5^- is considered. This function is called indicatrice and determines completely the linear functional

$$u(\mathbf{z}') \equiv F[h(\mathbf{z}, \mathbf{z}')], \quad \mathbf{z} \in T_5^-, \quad \mathbf{z}' \in T_5. \tag{61}$$

The function f can be written as an integral over Minkowski space with the help of the Szegö kernel. Because of the linearity of F integration and application of F can be reversed in order and thus with (61) the linear analytic functional has the form

$$F[f] = \int_{M_5} u(\mathbf{x}) f(\mathbf{x})\, dx. \tag{62}$$

The first variation $\delta G[f, \delta f]$ of a non-linear functional $G[f]$ with respect to the variation δf of the function is best defined as this part of the variation $(G[f + \delta f] - G[f])$ which is linear in δf [19]. Thus G is for any function f a linear functional in δf and can thus be expressed by an integral defining the functional derivative $G'[f, x]$

$$\delta G[f, \delta f] = \int_{M_5} G'[f, \mathbf{x}]\, \delta f(\mathbf{x})\, dx. \tag{63}$$

Comparing it with formula (61) the derivative can be determined in the following way

$$G'[f(\mathbf{z}),\mathbf{x}] = \frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0} G[f(\mathbf{z}) + \epsilon h(\mathbf{z},\mathbf{x})]. \tag{64}$$

The second derivative, e.g., is then given by

$$G''[f,\mathbf{x}_1,\mathbf{x}_2] = \frac{\partial^2}{\partial\epsilon_1\,\partial\epsilon_2}\bigg|_{\epsilon_1=\epsilon_2=0} G[f(\mathbf{z}) + \epsilon_1 h(\mathbf{z},\mathbf{x}_1) + \epsilon_2 h(\mathbf{z},\mathbf{x}_2)].$$

Using these derivatives a formula analogous to a Taylor expansion, called by Pellegrino Fantappiè expansion, is found

$$G[f] = G^0 + \sum_{n=1}^{\infty}\frac{1}{n!}\int G^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) f(\mathbf{x}_1)\ldots f(\mathbf{x}_n)\,dx_1\ldots dx_n \tag{65}$$

where in $G^{(n)}$ the argument $f_0 = 0$ has been omitted for simplicity. The functions $G^{(n)}(\mathbf{z}_1,\ldots,\mathbf{z}_n)$ are symmetric in the n arguments and are analytic for all $\mathbf{z}_i \in T_5$.

A functional $V[\mathbf{v}]$ on a vector field $\mathbf{v}$ can be looked at as a functional on five analytic functions (in $T_{\bar{5}}$). The partial functional derivative $V_i'[\mathbf{v},\mathbf{z}_1]$ is then defined by

$$V_i'[\mathbf{v},z_1] = \frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0} V[v^0,\ldots,v^i(\mathbf{z}) + \epsilon h(\mathbf{z},\mathbf{z}_1),\ldots,v^4] \tag{66}$$

and analogously for higher derivatives. The Fantappiè expansion then reads

$$V[\mathbf{v}] = V^0 + \sum_{n=1}^{\infty}\frac{1}{n!}\int V^{(n)}_{i_1\ldots i_n}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\,v^{i_1}(\mathbf{x}_1)\ldots v^{i_n}(\mathbf{x}_n)\,dx_1\ldots dx_n \tag{67}$$

where the functions V^n are symmetric for simultaneous interchange of $\mathbf{x}_k \leftrightarrow \mathbf{x}_p$ and $i_k \leftrightarrow i_p$ and transform under the Lorentz group as a tensor. Functionals $\phi[\boldsymbol{\varphi}]$ on spinor fields need more care. According to Cartan [14] the form $\boldsymbol{\psi}^*\mathbf{C}\boldsymbol{\psi}$ is a scalar ($\boldsymbol{\psi}$ = spinor) where $\mathbf{C}$ is of the form $\mathbf{C} = \begin{pmatrix}\sigma_2 & 0\\ 0 & -\sigma_2\end{pmatrix}$ in the representation (24) and (51). Therefore is is useful to choose for $\boldsymbol{\varphi}$ a field which does not transform as a spinor $\boldsymbol{\psi}$ but as the dual field $\mathbf{C}\boldsymbol{\psi}$. Then the indices of the partial derivatives transform as spinor indices since the resulting expression for the variation is a scalar. Defining the derivatives analogously as in (66) the Fantappiè expansion reads

$$\phi(\boldsymbol{\varphi}) = \phi^{(0)} + \sum_{n=1}^{\infty}\frac{1}{n!}\int \phi^{(n)}_{\alpha_1\ldots\alpha_n}(\mathbf{x}_1,\ldots,\mathbf{x}_1)\,\varphi_{\alpha_1}(\mathbf{x}_1)\ldots\varphi_{\alpha_n}(\mathbf{x}_n)\,dx_1\ldots dx_n. \tag{68}$$

In analogy to the equations (56) for fields there can now be written down equations for the functionals and their functional derivative.

$$(V[\mathbf{v}]\,S_{i_{\alpha\beta}}P^i + iS_{i_{\alpha\beta}}V'^{i}[\mathbf{v},\mathbf{z}])\,\phi_\beta'[\boldsymbol{\varphi},\mathbf{z}] = 0 \qquad \text{a)}$$

$$\phi[\boldsymbol{\varphi}]\,\bar{\phi}[\bar{\boldsymbol{\varphi}}]\,F_{ik,}{}^{k} + V[\mathbf{v}]\,\bar{\phi}_\alpha'[\bar{\boldsymbol{\varphi}}(\mathbf{z}_1),\bar{\mathbf{z}}]\,S_{0_{\alpha\beta}}S_{i_{\beta\gamma}}\phi_\gamma'[\boldsymbol{\varphi},\mathbf{z}] = 0 \qquad \text{b)} \tag{69}$$

The invariance under the group G and the covariance under the conformal group can be shown in the same way as for equations (56) assuming that functional and partial derivation can be interchanged. Invariance under dilatation requires the dimensions -1 and $-\frac{3}{2}$ for V' and ϕ', respectively. This means that the parametric fields $\mathbf{v}$ and $\boldsymbol{\phi}$ have the dimensions -4 and $-\frac{7}{2}$, respectively, since the functionals V and ϕ are of dimension zero.

Instead of equations (69) different equations with different powers of the functionals V and ϕ could be looked at. They seem to be less important. The most simple equations would be those without any factor ϕ or V. Introducing the Fantappiè series it is readily seen that then only equations (56) would result. Equations (69) are special since they are homogeneous in the functionals. It is then possible to introduce normalizing constants for the functionals to set, e.g., the constant terms V^0 and ϕ^0 to one representing thus a unique 'vacuum'. Equations (69) for the functionals are expanded into an infinite series of differential equations for the multipoint functions V^n and ϕ^n by introducing the Fantappiè series and setting zero the factors of the different powers of $\mathbf{v}$ and $\boldsymbol{\varphi}$ of the equations. The lowest order equations corresponding to the constant of the series of the functional equations are again equations (56) when setting

$$A=\frac{1}{V^0}V^1, \quad \psi'=\frac{1}{\phi^0}\phi^1, \quad \mathbf{f}=\frac{1}{V^0}F. \tag{70}$$

Thus the functional equations (69) produce a series of non-linear differential equations for multipoint functions. These equations are invariant under the group G and covariant to the conformal group C_5.

6. Conclusion

It has been shown that non-interacting massless particles can be described by analytic representations of the conformal group C_4 fulfilling invariantly the differential equation $\Delta f=0$. The direct generalization to massive particles as representations of the group C_5 is not possible. However, non-linear differential equations have been found which are invariant under the group G and covariant under C_5. They describe possibly interacting particles. The lowest order equations are differential equations for a spin half field coupled to spin zero and one fields. It is hoped that solving these equations and considering then higher orders of the functionals will lead to an interpretation of the formula for the fine structure constant given by Wyler [7]. It has been found that the group G can be extended essentially only in two ways. First by a group G' which leads to an approximate symmetry group $L(4)\times SO(2,1)$ and thus perhaps describing also isospin. The second extension is C_5 itself. A connection with the group SU_3 ,however, does not seem to be straightforward. States with a very high number of particles are probably best described by the functionals themselves, not expanded in a Fantappiè series. They are then described in terms of the fields $\boldsymbol{\phi}$ and $\mathbf{v}$ whose interpretation is not yet clear. Perhaps there exists a connection with the many calculations using the conformal group C_4 in the high energy (i.e. also high particle number) limit.

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