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Decay Formula and Fission Barrier

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Abstract. For any theory of fission, which construes the last step as a one-dimensional tunnelling process through a potential barrier, we show that the observed pre-neutron emission kinetic energies are inconsistent with the Bohr–Wheeler–Strutinski barrier used in computing the spontaneous fission half-lives. As regards these data, we establish that the significant feature of any model which attempts to describe the last stage of fission as a single channel tunnelling phenomenon is the presence of a potential hole whose minimum is lower than the observed kinetic energy. Only such a hole can contain a resonant state decaying through the last stage of the barrier with appropriate kinetic energies. Besides, the observed half-lives indicate that this last stage of the barrier is likely to be thinner than the one usually used. We also formulate an exact expression relating the half-life of a decay process through a barrier to the phase shift of the scattering by the associated potential.

I. Purpose

The purpose of this note is to investigate conclusions that could be drawn on the nature of the fission barrier from our knowledge of the asymptotic relative kinetic energies of the daughter pair. We shall restrict our discussion to spontaneous and binary fission. In spontaneous fission a parent nucleus of mass number A and atomic number Z decays automatically to a daughter pair $(A_1 Z_1)$ and $(A_2 Z_2)$ after a characteristic time, and the energy balance equation may be symbolically written as

$$M(A, Z) = M(A_1 Z_1) + M(A_2 Z_2) + Q, \tag{1}$$

where Q is the amount of energy released and is a positive quantity. M stands for the mass of a nucleus. In practice, however, after the scission each member of the daughter pair is in an excited state. Denoting these excited states with asterisks we have

$$M(A, Z) = M(A_1 Z_1)^* + M(A_2 Z_2)^* + Q^*. \tag{2}$$

[1]) On leave of absence from the Physics Department, Indiana University, Bloomington, Indiana USA. Visiting Professor of Physics under the auspices of the Commission of Physics, Switzerland (C.I.C.P.), 1971–72. This work was carried out during that period.

Since excitation is a positive quantity,

$$Q^* < Q.$$

This Q^* is manifested in relative kinetic energy T_R. Thus

$$T_R < Q$$

because of the excitation of the daughter pair. Experimental information on pre-gamma and pre-neutron emission kinetic energy supports this claim, and T_R is 20 to 40 MeV less than Q. An example is shown in Figure 1. The dashed line in that figure

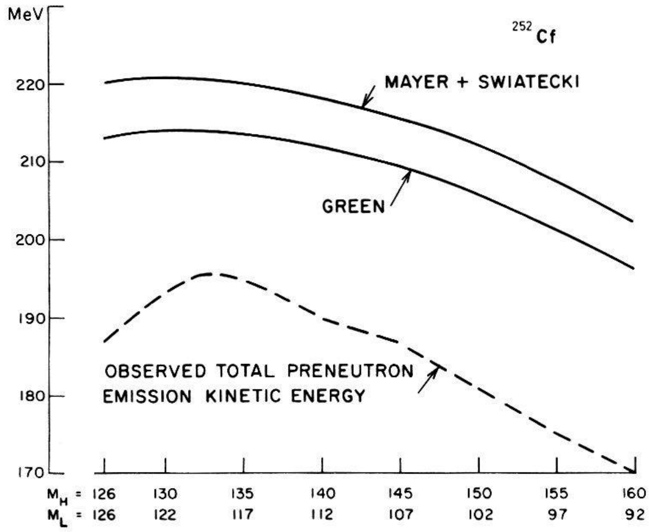

Figure 1
An example of the difference between the observed pre-neutron emission kinetic energies and the energy released (the Q-value) in the spontaneous fission. The case refers to the decay of ^{252}Cf and observed pre-neutron emission kinetic energies are taken from Ref. [20]. The computed Q-values labelled as Myers–Swiatecki and Green, are obtained from the mass formulas of Refs. [3] and [2], respectively. M_H and M_L refer, respectively, to the mass numbers of the heavy and the light fragments.

represents the measured kinetic energy spectrum for the spontaneous decay of ^{252}Cf [1] in various decay modes. The two solid lines refer to computed Q-values for the decay of ^{252}Cf to a series of daughter pairs given by equation (1) using a) the old mass formula of Green [2], and b) the recently revised mass formula of Myers and Swiatecki [3]. We see that the observed kinetic energy spectrum is at least 20 MeV lower than the Q-values. This is not peculiar to ^{252}Cf but a general feature of all known cases to date. Apart from this knowledge of the kinetic energy spectrum, we have two other pieces of information. Since the spontaneous fission is a barrier penetration problem, the maximum of the interaction potential *must be higher* than the asymptotic kinetic energy, and the nuclear state of the parent nucleus is metastable.

Our aim now is to examine the nature of inferences that can be drawn within the context of a one-dimensional barrier penetration theory. (If the potential is spherical symmetric, this holds also for a three-dimensional case.)

II. Theory

Both the Bohr–Wheeler theory [4, 5] (henceforth referred to as B–W), including its present-day modification suggested by Strutinski [6, 7, 8] (referred to as B–W–S) and the recently proposed theory of fission [9, 10, 11, 12], (henceforth referred to as QMF) treat the tunnelling stage of the fission problem with a Schrodinger equation of the type

$$\left[d^2/dx^2 - q^2(x) - \frac{l(l+1)}{x^2}\right] u_l(x) = 0 \tag{3}$$

with

$$q^2(x) = \frac{2\mu}{\hbar^2}\ (\text{Potential } V(x) - \text{Energy } E). \tag{4}$$

Here μ is the reduced mass of the problem, E the total energy and $V(x)$ the potential. E is equal to the kinetic energy, as measured in the range where $V(x)$ is negligible, i.e. E is asymptotically equal to the kinetic energy. Although the B–W model in principle computes a fission barrier in a space of multiple deformation parameters, the actual barrier penetration calculation in their model is often done using a one-dimensional barrier. In this simplified version of the B–W or the B–W–S model the variable x is the deformation parameter β which is the fractional increase of the nuclear radius $R(\beta)$ from its non-deformation (or equilibrium radius) radius R_0, i.e. $\beta = |R(\beta) - R_0|/R_0$.

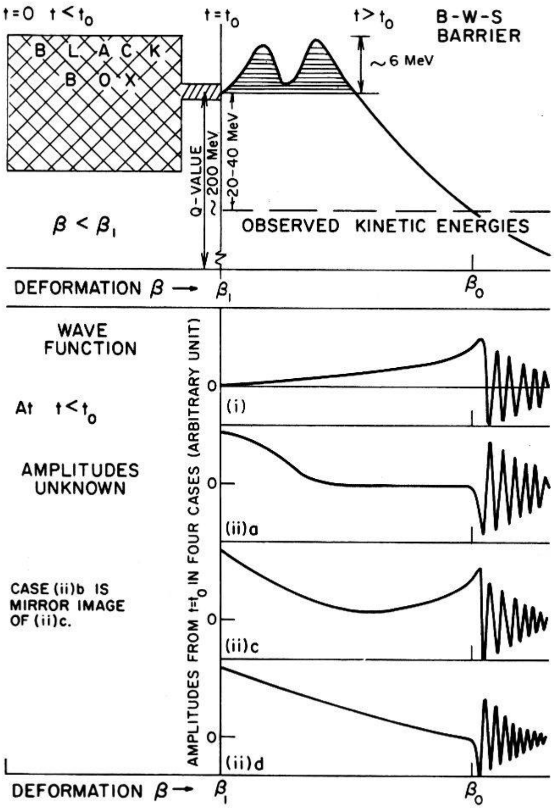

Figure 2
The upper half of the figure represents schematically the observed pre-neutron emission kinetic energies in relation to the potential energy relevant at the tunnelling stage in each of the three models. B–W, B–W–S and QMF refer, respectively to Bohr–Wheeler, Bohr–Wheeler–Strutinski and barrier of Ref. [9] through [12]. The lower half of the figure plots schematically the relevant wave functions corresponding to the potential barrier drawn above it.

In principle the QMF [9, 10, 11, 12] is not only a multidimensional but also a multichannel theory. However, actual computation of the barrier penetration usually has been done using a one-dimensional model. In this simple approximation, the variable x in (3) is the relative distance R between the centers of masses of the daughter pair.

Barriers of the B–W, the B–W–S, and the QMF theory along with the observed kinetic energy are plotted schematically in Figure 3. We note that in the B–W and the B–W–S models $q^2(x)$ is *always positive* for $x < x_0$, where x_0 is the only turning point there (irrespective of double or multiple humps or their details). The barrier of the QMF model has at least three turning points, x_1, x_2 and x_0, and $q^2(x) < 0$ in the region $x_1 < x < x_2$, and positive otherwise.

Let us first study the effect of $q^2(x)$ on the regular solution $u(x)$ of (3). For any value of x the regular solution $u(x)$ and its derivative $u'(x)$ are given by the Voltera equation,

$$u(x) = x + \int_0^x (x - t)\, u(t)\, q^2(t)\, dt \tag{5}$$

$$u'(x) = 1 + \int_0^x u(t)\, q^2(t)\, dt. \tag{6}$$

This is valid for $l = 0$, whereas, for $l \neq 0$, we have

$$u(x) = x^{l+1} + (2l+1)^{-1} \int_0^x [(x/t)^{l+1/2} - (t/x)^{l+1/2}]\, (xt)^{1/2}\, u(t)\, q^2(t)\, dt \tag{7}$$

$$u'(x) = (l+1)\, x^l + (2l+1)^{-1}\, x^{-1} \int_0^x [(l+1)\, (x/t)^{l+1/2} + l(t/x)^{l+1/2}](xt)^{1/2}\, u(t)\, q^2(t)\, dt. \tag{8}$$

Clearly, $u(x)$ and $u'(x)$ are known up to an arbitrary multiplicative constant. They are chosen here in such a way that $u(x) \sim x^{l+1}$ as x goes to zero. In the following we normalize $u(x)$ in such a way that it behaves at the origin like x^{l+1}. Our conclusion does not depend on this normalization. Now, both in the B–W and the B–W–S models, $q^2(x)$ is positive everywhere up to the external (here unique) turning point x_0. Therefore, all the iterated terms in (5) and (7) are positive, and when they are inserted in (6) and (8) respectively, the result is that $u'(x)$ is positive. Let $\psi(x) = x^{-1}\, u(x)$ be the wave function. Clearly, $d/dx(\psi^2(x))$ is positive inside the B–W or the B–W–S potential. Therefore, $|u(x)|$ and the probability $\psi^2(x)$ of finding the system at a given value of x are *steadily increasing from $x = 0$ to the turning point*. The complete absence of any extremum or of any node (apart from the one at the origin) of $\psi(x)$ before the turning point (i.e., in the region $x < x_0$) implies that no metastable ground or isomeric state is compatible with the observed kinetic energies within the context of a one-dimensional tunnelling through a potential barrier prescribed in the B–W or the B–W–S model.

It is sometimes customary to use an effective mass parameter $B(x)$ instead of $(2\mu/\hbar^2)$ in the barrier penetration problem within the context of the B–W and the B–W–S. This $B(x) > 0$. Therefore *the above conclusion remains valid even in cases which use this effective mass parameter B*. The above discussion also proves that a *necessary condition for a metastable state to exist is that $q^2(x)$ must be negative on a certain interval inside the barrier.*

Clearly, the barrier proposed in the QMF theory [9–12] fulfills this necessary condition. $q^2(x)$ in this case is negative in the region $x_1 < x < x_2$.

III. Expression for the Decay Constant and Information from Half-Lives

Let us now study the information on the structure of the barrier contained in the observed half-lives of the actinides. To derive a convenient expression suitable for our theoretical analysis of the half-life T associated with the barrier penetration problem in a fission channel, let us follow for example Goldberger and Watson [13]. To this end, we evaluate the time during which a wave packet remains localized in an interaction region S. The half-life T of the metastable state is the probability of finding the system in the interaction region S minus the probability of finding a non-interacting wave packet in the same region and is given by [13, 14],

$$T = 2\hbar d\lambda(E)/dE = \sqrt{2\mu/E}\, d\lambda(k)/dk \tag{9}$$

where λ is the phase shift associated with the outgoing wave packet with the wave number k and energy E.

The expression (9) is not convenient for analysis in many cases. For example, it is not possible to use it in the usual JWKB expression for the phase shift because such a phase shift [15, 16] has a sudden jump of π, when the energy goes through a resonant value corresponding to a metastable state. In addition, (9) must be applied with some care for a long-range Coulomb potential which is pertinent to the fission process. To deal with the Coulomb potential, we choose a large enough distance R in a region when the wave functions of both the interacting and non-interacting systems have attained their asymptotic values. We then cut off the potential at R. For simplicity we consider only the S-wave. For the case with an interaction the wave function with a wave number $k(k^2 = 2\mu E/\hbar^2)$ may be written as

$$u(k, R) = R\psi(k, R) = A(k) \sin(\varphi(k, R) + \lambda) \tag{10}$$

with

$$\varphi(k, x) = kx - \eta \log kx + \sigma + 0\left(\frac{\eta}{kR}\right) \tag{11}$$

(η and σ are, respectively, the Coulomb parameter and phase-shift).

Noting that $u(k,x)$ satisfies a wave equation of the form (3), we obtain after a partial integration

$$|u(k_2, x)\, u'(k_1, x) - u'(k_2, x)\, u(k_1, x)|_0^R + (k_1^2 - k_2^2) \int_0^R u(k_2, x)\, u(k_1, x)\, dx = 0, \tag{12}$$

where k_1 and k_2 correspond to two different energies for the same wave equation.

Hence we get

$$A(k_1)\, A(k_2) \left[\frac{k_1 + k_2}{2} \sin(\phi(k_2, R) - \phi(k_1, R) + \lambda_2 - \lambda_1) + (\tfrac{1}{2})(k_1 - k_2) \sin(\phi(k_2, R) + \phi(k_1, R) + \lambda_2 + \lambda_1)\right] = (k_2^2 - k_1^2) \int_0^R u(k_1, x)\, u(k_2, x)\, dx. \tag{13}$$

Taking the limit $k_1 \to k_2$, and dropping the subscripts we obtain

$$A^2(k)\{[(\tfrac{1}{2})\, d\phi(k,R)/dk + (\tfrac{1}{2})\, d\lambda/dk]\,[1-\eta/2kR] - (1/4k)\sin(2\phi(k,R)+2\lambda)\} = \int_0^R u^2(k,x)\, dx. \quad (14)$$

Similarly, for the case of a pure Coulomb field (with a cut-off radius R) we get

$$A^2(k)[(\tfrac{1}{2})\,(d\phi(k,R)/dk)\,(1-\eta/2kR) - (1/4k)\sin(2\phi(k,R)] = \int_0^R u_c^2(k,x)\, dx, \quad (15)$$

where u_c is the Coulomb wave function. We choose its normalization in such a way that $A(k)$ is the same as in (14). Subtracting (15) from (14) we obtain an equation determining $d\lambda/dk$.

$$A^2(k)[(\tfrac{1}{2})\,(d\lambda/dk)\,(1-\eta/2kR) - (1/4k)\,[\sin 2(\phi(k,R)+\lambda) - \sin 2\phi(k,R)] + (\eta/4kR)(d\phi/dk)] = \int_0^R [u^2(k,x) - u_c^2(k,x)]\, dx. \quad (16)$$

The half-life associated with the decay through a barrier in the fission channel can now be obtained from (9) and (16). These are *exact* expressions and can be evaluated given a potential and a computer.

It is easy to estimate that the second and third terms on the left side of (16) contribute only about 10^{-22} sec to T for the observed kinetic energies in the actinide region. In fact, all terms of the order of k^{-1} on the left-hand side of (16) can always be neglected so long as we are interested in the half-lives of resonant states. This is because they constitute the collision time taken to transverse the interaction region. As for the right-hand side, it is easy to see that, in the interval $x_0 < x \leqslant R$ (i.e., from the outer turning point to R), the integral $\int_{x_0}^R (u^2 - u_c^2)\, dx$ contributes only terms of the order $A^2 k^{-1}$. The reason is the oscillatory character of both u and u_c. (In the JWKB approximation, the envelopes of u and u_c are equal.) We can, for our purpose, neglect this contribution. Hence [17, 18]

$$\frac{d\lambda}{dk} \simeq (2/A^2) \int_0^{x_0} u^2(k,x)\, dx. \quad (17)$$

In the region $0 < x < x_1$ (i.e., from the origin to the first turning point) both u and u_c grow very slowly because $q(x)$ there is very large and positive in each case. For the purpose of a simple estimation, we can, therefore, use the approximate expression

$$\frac{d\lambda}{dk} \simeq (2/A^2) \int_{x_1}^{x_0} u^2(k,x)\, dx. \quad (18)$$

We now resort to the JWKB approximation to obtain further information on the half-lives. The integral in (18) consists of two parts.

a) The region of the formation of the metastable state, i.e., the region $x_1 < x < x_2$.

In this region marked as II, $q^2(x) < 0$ and for a metastable state (n = an integer)

$$\int_{x_1}^{x} q_{\mathrm{II}}(x)\,dx \simeq (n + \tfrac{1}{2})\,\pi. \tag{19}$$

Besides, the wave function u_{II} is given by [19]

$$u_{\mathrm{II}} = 2\sqrt{2/\pi q_{\mathrm{II}}\,A_2}\cos(\pi/6)\cos\left[\int_{x_1}^{x} |q_{\mathrm{II}}(x)|\,dx - \pi/4\right] \tag{20}$$

where

$$|q_{\mathrm{II}}| = \sqrt{(2\mu/\hbar^2)\,(E - V(x))}. \tag{21}$$

These formulas have been obtained by matching the wave function in the region II to the wave function in the range $(0, x_1)$ at x_1. As long as $q^2(x)$ is positive (and not too small) in that range, the JWKB method in the first approximation enables us to see that the only effect of this region is to impose the form (20) on the wave function in the region II.

b) The region $x_2 < x < x_0$ (i.e., region III of Fig. 3) which is the region of penetration through (essentially) the barrier. From the matching condition at $x = x_2$, the wave function in this region is

$$u_{\mathrm{III}} = A_2\cos(\pi/6)\sqrt{2/\pi q_{\mathrm{III}}}\exp\left(-\int_{x_2}^{x} q_{\mathrm{III}}\,dx\right). \tag{22}$$

Since (22) must match to the solution in the region $x > x_0$,

$$A = k^{-1/2} A_2 \exp\left[-\int_{x_2}^{x} q_{\mathrm{III}}\,dx\right]. \tag{23}$$

It should be noted that (23) is only approximate because, for a wave function decaying from region III to region IV, proper connection formulas must be obtained using the Bessel function $J_{1/3}$. However, an estimate has been done showing that a correction to (23) is not significant.

The only important contribution to (18) is the contribution from the region II to $d\lambda/dk$ because, the exponent of (22) being very large, the contribution to the integral of (18) from region III is very small. Thus

$$\frac{d\lambda}{dk} \simeq (A_2^2/A^2)\,k\int_{x_1}^{x_2} \cos^2(\omega_1 - \pi/4)\,dx/q_{\mathrm{II}} \tag{24}$$

with

$$\omega_1 = \int_{x_1}^{x} q_{\mathrm{II}}\,dx. \tag{25}$$

For our estimation we can extend the integration (25) to x_2 and assume $V(x)$ to be nearly a constant in the region II. In this approximation

$$\frac{d\lambda}{dk} \simeq (A_2/A)^2 \frac{x_2 - x_1}{q_{\mathrm{II}}} \,. \tag{26}$$

(In deriving (26), we have taken care of the difference in definitions of A_2 and A.) Hence the half-lives of the tunnelling processes are given by

$$T = \exp\left[2 \int_{x_2}^{x_0} q_{\mathrm{III}}\, dx\right] \cdot \sqrt{(2\mu/E)\,(x_2 - x_1)^2} \cdot \sqrt{E/(E - V)_{\mathrm{II}}}\,, \tag{27}$$

where V_{II} is understood to be independent of x in the region II. The factor $\sqrt{E/(E - V)_{\mathrm{II}}}$ can at the most be $\sqrt{2} \cdot 10$ for $E \simeq 200$ MeV and $(E - V)_{\mathrm{II}} \simeq 1$ MeV. Since we are interested only in an order of magnitude estimate, we shall not take this factor into account. Omitting this term we get

$$T \simeq \exp\left[2 \int_{x_2}^{x_0} q_{\mathrm{III}}\, dx\right] \cdot \sqrt{(2\mu/E)\,(x_2 - x_1)^2}. \tag{28}$$

Bohr and Wheeler [4, 5] proposed to use

$$T \simeq \exp\left[2 \int_{x_2}^{x_0} q_{\mathrm{III}}\, dx\right]. \tag{29}$$

We note that

a) For the fission half-lives the shape of the potential in the region II, i.e. the region where the metastable state is formed, is not critical. The critical point is that there should be such a region characterized by $E > V(r)$, so that one can form a metastable state which, upon entering the barrier in the region III, can decay. Without such a region one cannot have a resonant state decaying through the region III. This provides the qualitative insight as to why the QMF model has been successful.

b) The width $|(x_2 - x_1)|$ does not influence the half-lives to any reasonable extent, provided that it is broad enough to contain a metastable state (or rather broad enough for the wave function to have a negative slope at $x = x_2$).

c) The critical parameter governing the half-lives is the difference $(V - E)$ in the region III, i.e. effectively the maximum of the barrier height and the available kinetic energy of a daughter pair in a given mode. Since the half-lives are sensitive to this parameter, it is easy to understand why fission in various decay modes have different half-lives. This is because both the Coulomb energy and the available kinetic energy are different for various mass splittings of the parent nucleus.

d) Of course, the more appropriate expression is (24) or (27), which depends on the detailed shape of the potential in the region II. If we are satisfied with an accuracy of two orders of magnitude these expressions are to be used.

e) Clearly, the energy E in any decay formula including the B–W one, i.e. (29), must be equal to asymptotic kinetic energy.

f) It is impossible to determine barrier heights uniquely from the measured half-lives alone without other pieces of information.

IV. Synthesis and Discussion

A. Compatibility of the half-life and the kinetic energy

It is worth emphasizing that the assumption underlying the above analysis is that 'the parent' nucleus has already been 'prepared' to a particular stage from which it can be described by a simple tunnelling phenomenon. Therefore the total half-life $t_{1/2}$ associated with the decay in a particular daughter pair is given by

$$t^{1/2} = P^{-1}\, T, \tag{30}$$

Where P is the probability for a nucleus to come to a fissioning channel from which its motion is governed by a simple tunnelling process of the type described by equation (3). A model must be built to estimate P but in any model $\max(P)$ should be unity. In the QMF model P is designated as the preformation probability and estimated to be larger than 10^{-7}. A reasonable value seems to be 10^{-5}.

Clearly in (27) the factor $\sqrt{2\mu/E}(x_2 - x_1) = (1/c\sqrt{2\mu c^2/E}(x_2 - x_1)$ (c: velocity of light) is of the order of 10^{-22} sec for the fission of the actinides because $E \simeq 150$ to 200 MeV and $\mu \simeq 50$ times the nucleon mass. This cannot change to any appreciable amount (i.e. by more than one order of magnitude) for the fission of different heavy elements and is not sensitive either to the reduced mass of the decay channel or to its kinetic energy.

The other factor, $\exp(2 \int q\,dx)$, is sensitive to E and must be evaluated explicitly in every case. However, it is easy to see that the observed total half-lives of the spontaneous decay in the actinide region and the observed kinetic energies are compatible with each other. The value of $(V - E)$ in this region is between 20 and 40 MeV and $2\mu/\hbar^2 \simeq 3F^{-2}$ MeV^{-1}. Consequently, the average value of $\sqrt{(2\mu/\hbar)(V-E)}$ is 8 to $10F^{-1}$. If the mean value of $(x_0 - x_2) \simeq 4$ to $5F$ the exponent is 64 to 100 yielding $T \simeq (10^{27}$–$10^{43})$. 10^{-22} sec $\simeq 10^5$ to 10^{21} years.

A preformation probability between 1 and 10^{-5} yields spontaneous fission half-lives

$$t^{1/2} \sim 10^0 \text{ to } 10^{21} \text{ years.} \tag{31}$$

This is in good agreement with the observed spontaneous fission half-lives of even–even nuclei from uranium to californium. Thus, we conclude that a) the values of the spontaneous fission half-lives and of the observed kinetic energies are compatible in our model, and b) the observed half-lives demand a relatively thin barrier of an average width of about $5F$. The QMF model parameters are in accord with this.

B. Discussion on the initial condition

It is legitimate to argue that we have presented a somewhat simplified description of the tunnelling step in the B–W or the B–W–S mode. As a first refinement, one should include an effective mass parameter $B(\beta)$ which varies with deformation. *However, this parameter, which is in general positive, multiplies in a first approximation the difference $(V - E)$ in the equation and therefore cannot change the sign and hence does not nullify the above objections to the last step of the B–W and the B–W–S model.*

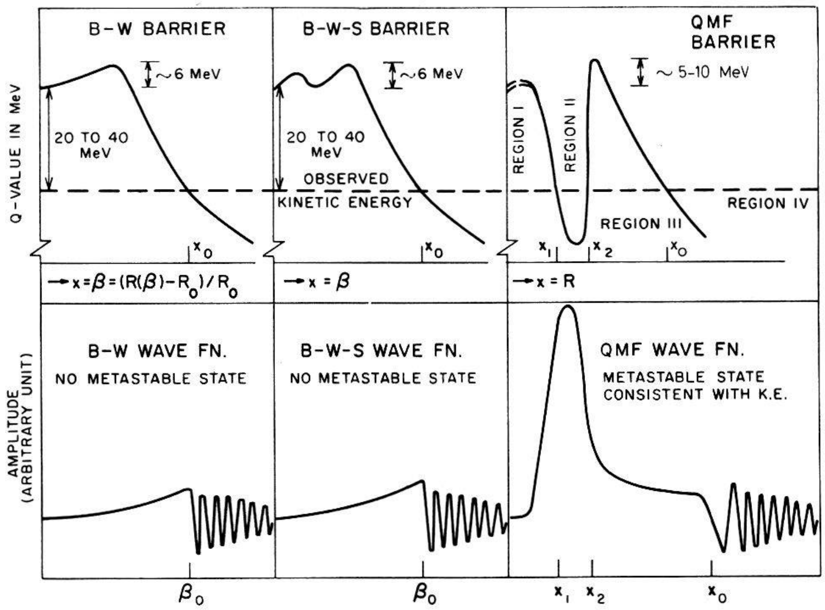

Figure 3
Schematical plots of the wave function for the B–W or the B–W–S barrier in a model where the boundary conditions at the start of the tunnelling phase are changed. The tunnelling starts at a time $t = t_0$ when the nucleus has a deformation $\beta = \beta_1$. Prior to t_0, the detail of the fission process is arbitrary but for $t \geqslant t_0$ the fission is described as a tunnelling through a potential surface of the B–W or the B–W–S type. As discussed in the text, five initial conditions on the wave functions are possible. The wave functions corresponding to these five cases are schematically shown in figures below the potential energy surface. The case (ii)b has not been plotted separately because it is essentially a mirror image of (ii)c. There is no metastable state at a deformation β.

Another way out in restoring the B–W or the B–W–S fission barrier is to question our boundary condition at a time $t = 0$ when the tunnelling starts. As shown in Figure 2, one may argue that a sophisticated version of the B–W and the B–W–S tunnelling process starts at a certain time t_0 when the system reaches β_1 which is not far from the saddle point. Therefore the potential between 0 (or equilibrium deformation $\beta_{eq.}$) and β_1, which is computed at a time previous to t_0, is irrelevant for the computation of the tunnelling process, and so is any wave function computed at $t < t_0$ and between $\beta_{eq.}$ and β_1. At the time $t = t_0$ and $\beta = \beta_1$, when it is possible to conceive a simple tunnelling model, one has to choose boundary conditions on $u(\beta_1)$ and $u'(\beta_1)$ which are imposed by 'the history of the system' at all previous times. In simple words we may say that at $t < t_0$ we have a complicated many-body system which is difficult to treat, i.e. we have a black box at $t < t_0$. But from $t = t_0$ we may treat the system as a simple tunnelling phenomenon epitomized by a Schrodinger equation and it may be possible to choose $u(\beta_1)$ and $u'(\beta_1)$ in such a way that the half-life in the fission channel is correct.

Unfortunately, this model of a 'black box' at $t < t_0$ still presents several difficulties with regard to the experimental values of the kinetic energy and the half-life as long as $q^2(\beta)$ remains positive in the interval $\beta_1 \leqslant \beta \leqslant \beta_0$, β_0 being the turning point. The reason is that there cannot be any maximum of $|u(x)|$ since $u''(x)/u(x)$ is positive everywhere. Suppose that $u(\beta_1)$ is positive, we have either i) $u'(\beta_0) \geqslant 0$ or ii) $u'(\beta_0) < 0$.

For the case i), the wave function will still grow if $(V - E) > 0$ everywhere up to the external turning point β_0 and there cannot be any metastable state in the region $\beta < \beta_0$, and the objection raised above still holds.

In the case of the alternative boundary condition ii), we may have effectively four situations: a) The wave function may decrease rapidly and be nearly zero before the penetration can take place – in that case there can hardly be any fission (the lifetime is extremely large). b) The wave function goes to zero, has then an inflexion point with the β-axis as a tangent and becomes negative. This case is analogous to the following case. c) The wave function and its second derivative remain positive for $\beta < \beta_0$, can decrease for a while and then turn around and start increasing. Since $u''(x) > 0$ everywhere inside the barrier if $(V - E) > 0$ everywhere, the maximum of the wave function will then be *outside the barrier* in the region $\beta > \beta_0$ and, hence, the probability of the presence of the system is larger outside the barrier than inside of it. This is obviously not compatible with the concept of a metastable state. d) The wave function continues to decay to β_0, and the particle comes out with the observed kinetic energy. However, in this case the tunnelling is through a very large barrier. From Figure 3 we see that the area of the barrier to be included in computing the penetration factor is at least 3 to 4 times larger than that usually taken and yields a half-life of 10^{80} to 10^{300} sec. This is far in excess of experimental measurements. (The use of the Bohr–Wheeler penetration formula (29) in conjunction with a factor to take care of the number of assaults also yields a large number in this case, e.g. for a parabolic barrier shape

$$t_{1/2} = (2\pi\omega)^{-1} \exp(2\pi(V - E)/\hbar\omega) \text{ sec} = 10^{-20} \exp(300 \text{ to } 100) \text{ sec} \simeq 10^{100} \text{ to } 10^{300} \text{ sec}$$

since the largest $\hbar\omega \simeq 0.3$ to 0.4 MeV and the average value of $(V - E) > 20$ MeV within the barrier. To get a correct $t_{1/2}$ in this case, we must use $\hbar\omega = 2$ to 4 MeV which is in sharp contrast to the β-vibration frequency currently admitted in heavy nuclei). The other alternative will be to extend 'the black box' well beyond the saddle point and close to 'the scission point' (whose abscissa is smaller than the outermost turning point). This procedure, however, renders meaningless the attempts to compute the potential surface starting at $\beta_{eq.}$ (or $\beta = 0$) and using a minimum principle because the relevant potential surface lies beyond the saddle point.

Clearly, there is no room for any metastable state of the Strutinski type in this kind of barrier penetration problem in all cases discussed above because the wave function cannot have any maximum inside the barrier.

C. Complex barrier

The B–W and the B–W–S model may be induced to accord with the observed half-lives and kinetic energies by using a complex barrier, which is a manifestation of the effect of other channels. This reflects the well-known fact that even if all channels but one are closed, a quantum mechanical problem is not the same as if there was only one channel (e.g. in some typical case one can even have bound states of positve energy). Naturally a complex barrier may also be ascribed to the thin QMF barrier. In the context on the QMF model, the off-diagonal coupling terms between the different channels are expected to yield an effective complex potential similar to the one used to describe the ion–ion collision. Although empirically there is some indication that the imaginary part of this potential is small in magnitude, conclusive evidence is lacking. Non-local potentials can also be used to describe the fission phenomenon.

V. Conclusion

If one wants to describe the later stage in the spontaneous fission as a single-channel tunnelling process through a static and real potential barrier, it seems difficult to reconcile the observed kinetic energies and half-lives with the B–W or the B–W–S barrier.

These experimental quantities clearly demand a 'thin barrier with a hole', i.e. a relatively thin barrier inside of which the asymptotic kinetic energy is larger than the potential energy at some point. Once such a barrier is assumed, it is easy to perform model calculations which can correctly predict the fission half-lives and kinetic energies of daughter fragments simultaneously. For example, the barrier used in the QMF model of fission meets this criterion.

On the other hand, our analysis of the fission barrier is invalid if one demands that in a refined version of the B–W theory there is no important region prior to the fission which admits a simple tunnelling description in one variable. In fact, it is a decay process of a '*coupled system as a whole through a multidimensional surface*'. In that case, however, the decay formula (29) of Bohr–Wheeler demands further investigation. Similarly, if the coupling terms between different channels in the QMF theory are large both in magnitude and in range, our analysis cannot make a definite statement. However, if one can uncouple such a set of equations through a similarity transformation, as is done in [11] and [12], our analysis holds for the effective diagonal potential barrier obtained after the system is uncoupled.

The present analysis further indicates that in its entirety the fission phenomenon, despite being a complex many-body problem, can be described in three distinct steps.

The first step deals with the preparation of the nucleus to prefission in a given channel – thus we need to estimate 'the probability of bringing the nucleus up to the final stage of tunnelling'. A nuclear model will always be needed to estimate this – in the QMF model this is called preformation probabilities – in the B–W model this is synonymous with 'the number of assaults'.

The second stage refers to the metastable state formed in a 'hole' in the barrier – there is no mechanism in the B–W or the B–W–S model to describe this stage. In the QMF model this part exists and originates from the attraction of the two daughter nuclei prior to the separation because the density in the overlapped region is less than the saturation density of the nuclear matter. Contribution of this stage to the half-life is $\sim 10^{-22}$ sec and this number is not very sensitive to the details of this attractive well.

The third stage consists of tunnelling through the barrier and is sensitive to the kinetic energy of the process.

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