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# Comment on a Paper of Amrein, Martin, and Misra

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(12. II. 73)

*Abstract.* We point out a gap in the proof of a proposition in a paper of Amrein, Martin and Misra and give a proof of their result.

The paper by Amrein, Martin and Misra [1] is an interesting and important contribution to the literature of time-dependent scattering theory. In this note we point out a gap in the proof of their Proposition 1 and furnish a proof of this result.

The scattering theory of AMM is based upon three conditions which they call (A1), (A2), and (A3). Using their notation we define the unitary groups  $V_t = \exp(-iHt)$  and  $U_t = \exp(-iH_0t)$  ( $-\infty < t < \infty$ ) which respectively describe the total evolution and free evolution of the scattering system, and the von Neumann algebra  $\mathcal{A}_0$  consisting of all bounded linear operators on the Hilbert space  $\mathcal{H} = L^2(R^3)$  that commute with all spectral projections of the positive self-adjoint operator  $H_0$ . We will be concerned with the following two conditions of AMM:

(A1) There exists a projection operator  $P$  on  $\mathcal{H}$  such that

$$a) [P, V_t] = 0, t \in (-\infty, \infty),$$

b) for every operator  $A \in \mathcal{A}_0$  there exist two operators  $A_{\pm}$  such that

$$s\text{-}\lim_{t \rightarrow \pm\infty} V_t^* A V_t P = A_{\pm} = \mu_{\pm}(A), \quad (1)$$

$$c) [P, A_{\pm}] = 0.$$

(A3) For every vector  $f \in \mathcal{H}$  there exists a vector  $g \in P\mathcal{H}$  such that for all  $A \in \mathcal{A}_0$

$$\lim_{t \rightarrow -\infty} (g, V_t^* A V_t g) = (f, Af).$$

In their Proposition 1 AMM want to prove that the images  $\mu_{\pm}(\mathcal{A}_0)$  of the mappings defined by (1) are von Neumann algebras by assuming only their first condition (A1). After establishing that  $\mu_{\pm}$  are \*-homomorphisms they quote a theorem of Feldman and Fell [2] thereby arriving at the conclusion that  $\mu_{\pm}$  are ultraweakly continuous on  $\mathcal{A}_0$ . However, the theorem of Feldman and Fell does not apply in this case, because, as one easily proves,  $\mathcal{A}_0$  is not properly infinite. This result was also discovered by AMM subsequent to the publication of their paper [3].

It is possible to prove that  $\mu_-(\mathcal{A}_0)$  is a von Neumann algebra by making use of (A3) as well as (A1). W. O. Amrein has informed the author that such a proof has been discussed in detail by Mourre [4]. However, in order to use this method to prove that  $\mu_+(\mathcal{A}_0)$  is a von Neumann algebra it is necessary to require the validity of (A3) also in the limit  $t \rightarrow +\infty$ . Call this new condition (A3)<sub>+</sub>. Alternatively, (A3)<sub>+</sub> can be derived by assuming the validity of (A1), (A2) and (A3) and that the scattering system is time reversal invariant [1]. With the additional condition (A3)<sub>+</sub> the scattering operator  $S$  is necessarily unitary when it exists, as is noted in [1]. Thus, this method of proof seems to be too restrictive (at least to the author).

In view of the above remarks it would appear to be useful to have a proof of the proposition in question assuming only the validity of (A1). We will give such a proof.

*Proposition.* Assume the validity of (A1). Then  $\mu_{\pm}$  are ultraweakly continuous on  $\mathcal{A}_0$ .

Once this much has been proved, the results stated by AMM as their Proposition 1 can be obtained by, for example, following the procedure in the latter part of their proof. In the proof given below we will use freely the result of AMM that  $\mu_{\pm}$  are \*-homomorphisms on  $\mathcal{A}_0$  without proving it again. We note that Lavine [5] proved that  $\mu_{\pm}$  are \*-isomorphisms on a certain operator algebra under more restrictive assumptions. In the present situation  $\mu_{\pm}$  are not necessarily isomorphisms as in the cases discussed by Mourre [4] and Lavine [5].

*Proof.* We write  $\mathcal{A}_0$  as the direct sum of finite and properly infinite von Neumann algebras  $(\mathcal{A}_0)_G$  and  $(\mathcal{A}_0)_{I-G}$  ([6], Proposition 8, p. 98), and consider the finite summand  $(\mathcal{A}_0)_G$  by assuming that  $\mathcal{A}_0$  is finite.

Let  $A$  denote an arbitrary non-zero element of  $\mathcal{A}'_0$ . Then  $A$  is a bounded normal operator and consequently has a polar decomposition

$$A = U|A|, \quad |A| \in \mathcal{A}'_0, \quad (2)$$

where  $U$  is unitary and  $|A|$  is bounded, positive, and self-adjoint. It follows that  $A \in \text{Ker } \mu_{\pm}$  if and only if  $|A| \in \text{Ker } \mu_{\pm}$ .

Since  $|A|$  is bounded and self-adjoint, and the spectrum of  $H_0$  is the non-negative real axis [1], we can write,

$$(\Psi, |A|\Psi) = \int_0^{\infty} f(\lambda) d(\Psi, E_0(\lambda)\Psi), \quad \Psi \in \mathcal{H}, \quad f \in \mathcal{B}, \quad (3)$$

where  $E_0$  denotes the resolution of the identity of  $H_0$  and  $\mathcal{B}$  the class of all real-valued bounded Borel measurable functions on  $[0, \infty)$ . It follows that

$$F(\delta) = E_0(f^{-1}(\delta)) \quad (4)$$

for every Borel set  $\delta$  of the spectrum of  $|A|$ , where  $F$  denotes the resolution of the identity of  $|A|$  ([7], Corollary X.2.10).

Now suppose  $A \in \text{Ker } \mu_{\pm}$ . From (1) and (2) one finds

$$\lim_{t \rightarrow \pm\infty} \|V_t^* |A| V_t g\| = 0, \quad \text{all } g \in P\mathcal{H}. \quad (5)$$

Since  $V_t^* F V_t$  is the resolution of the identity of  $V_t^* |A| V_t$ , we find, using (4), (5) and ([7], Corollary X.7.3), that

$$\lim_{t \rightarrow \pm\infty} \|V_t^* E_0(f^{-1}(\delta)) V_t g\| = 0 \quad (6)$$

for all  $g \in P\mathcal{H}$  and all Borel sets  $\delta$  of the spectrum of  $|A|$ .

Take  $f$  to be the characteristic function of a Borel set  $\Delta$  of  $[0, \infty)$ ,  $f = \chi_\Delta$ . Then  $f \in \mathcal{B}$  and we find from (3),  $|A| = E_0(\Delta)$ , which is non-zero by assumption. The set  $\delta = \{0, 1\}$  is a Borel set of the spectrum of this operator so that

$$E_0(f^{-1}(\delta)) = E_0([0, \infty)) = I = \text{identity operator}$$

and we obtain a contradiction with (6). Consequently,

$$E_0(\Delta) \notin \text{Ker } \mu_\pm \quad (7)$$

for all Borel sets  $\Delta$  of  $[0, \infty)$  such that  $E_0(\Delta) \neq 0$ .

Let  $f_n$  denote a simple function,

$$f_n = \sum_{i=1}^n \alpha_i \chi_{\Delta_i}, \quad n < \infty, \quad (8)$$

where the  $\alpha_i$  are non-zero real numbers and  $\{\Delta_i\}$  is a sequence of disjoint Borel sets of  $[0, \infty)$ . Then  $f_n \in \mathcal{B}$  and the corresponding operator is obtained from (3),

$$B_n = \sum_{i=1}^n \alpha_i E_0(\Delta_i).$$

Such operators will be called simple. Since  $\mu_\pm$  are linear we find

$$\mu_\pm(B_n) = \sum_{i=1}^n \alpha_i \mu_\pm(E_0(\Delta_i)). \quad (9)$$

From the disjointness of the sequence  $\{\Delta_i\}$  and the standard properties of a spectral measure one finds that  $\{E_0(\Delta_i)\}$  is a sequence of pairwise orthogonal projections. The homomorphisms  $\mu_\pm$  preserve this property so that  $\{\mu_\pm(E_0(\Delta_i))\}$  are also sequences of this type. If  $B_n \neq 0$  then there is at least one Borel set  $\Delta_k$  occurring in the sequence  $\{\Delta_i\}$  of (8) such that  $E_0(\Delta_k) \neq 0$  so that (7) obtains. Take  $\Psi_\pm \neq 0$  in the range of  $\mu_\pm(E_0(\Delta_k)) \neq 0$  so that (9) yields

$$\mu_\pm(B_n) \Psi_\pm = \alpha_k \Psi_\pm \neq 0,$$

and thus  $\mu_\pm(B_n) \neq 0$  or

$$B_n \notin \text{Ker } \mu_\pm \quad (10)$$

for all non-zero simple operators.

It follows from (3) that for each positive operator  $|A| \neq 0$  there exists a Borel set  $\Delta_0$  of  $[0, \infty)$  such that  $f$  assumes only positive values on  $\Delta_0$  and  $E_0(\Delta_0) \neq 0$ . We then find that, since  $f\chi_{\Delta_0}$  belongs to  $\mathcal{B}$  and is non-negative, there exists an increasing sequence of non-negative simple functions  $\{f_n\}$  which converges pointwise to  $f\chi_{\Delta_0}$  [8]. Because  $f\chi_{\Delta_0}$  majorizes  $f_n$  for each  $n$  we find that  $f_n \in \mathcal{B}$  for all  $n$  and that the simple operators  $B_n$  corresponding to  $f_n$  by (3) are positive and bounded. Moreover, each  $f_n$  vanishes outside the Borel set  $\Delta_0$  and for a sufficiently large value of  $n$  ( $n_0$  say)  $f_{n_0} > 0$  on  $\Delta_0$ . It follows that  $B_{n_0} \neq 0$  so that we obtain from (10),

$$\mu_\pm(B_{n_0}) > 0 \quad (11)$$

because  $\mu_\pm$  preserve positivity. Since  $f\chi_{\Delta_0} - f_{n_0} \in \mathcal{B}$  is non-negative we find that the operator  $|A|E_0(\Delta_0) - B_{n_0}$  to which this function corresponds by (3) is positive. We

now again use the fact that  $\mu_{\pm}$  preserve positivity coupled with (11) to show that

$$|A|E_0(\Delta_0) \notin \text{Ker } \mu_{\pm} \quad \text{for } |A| \neq 0,$$

from which  $|A| \notin \text{Ker } \mu_{\pm}$  immediately follows.

Thus, we have shown that  $\text{Ker } \mu_{\pm} \cap \mathcal{A}'_0 = \{0\}$ . Then, since  $\mathcal{A}_0$  has been assumed finite, one finds  $\text{Ker } \mu_{\pm} = \{0\}$  ([6], Corollaire 1 of Proposition 2, p. 256). Hence,  $\mu_{\pm}$  are injective on the finite summand  $(\mathcal{A}_0)_G$ . It follows that the restrictions of  $\mu_{\pm}$  to  $(\mathcal{A}_0)_G$  are direct summand representations in the sense of Fell [9] and are consequently ultraweakly continuous because  $(\mathcal{A}_0)_G$  is of type I [2, 9].

We can now follow AMM and invoke the theorem of Feldman and Fell [2] to deduce that the restrictions of  $\mu_{\pm}$  to the properly infinite summand  $(\mathcal{A}_0)_{I-G}$  are ultraweakly continuous. Finally, we use the linearity of  $\mu_{\pm}$ , the ultraweak continuity of their restrictions to the two summands, and the characterization of ultraweakly continuous homomorphisms in [2] to prove that  $\mu_{\pm}$  are ultraweakly continuous on  $\mathcal{A}_0$ .

W. O. Amrein has recently informed the author that the result of this note has also been proved by V. Georgescu (unpublished) by a different method.

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