Zeitschrift: Helvetica Physica Acta
Band: 46 (1973)
Heft: 3

Artikel: On factors of type II and quantum mechanics
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DOI: https://doi.org/10.5169/seals-114490

Helvetica Physica Acta
Vol. 46, 1973. Birkhäuser Verlag Basel

On Factors of Type *II* and Quantum Mechanics

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(22. III. 73)

Abstract. It is shown that a self-adjoint operator in a separable Hilbert space can be affiliated to a factor of type *II* if and only if 'its spectrum is of infinite multiplicity'. This result is then put on a form suitable for application, and an example is given.

Introduction

Von Neumann algebras have proved themselves a powerful tool in the study of quantum systems of a large number of degrees of freedom. Due to the generality of the notion of a von Neumann algebra, it has mainly been employed at the fundamental, structural, level. In this paper we propose to look for possibilities of using this tool for actual calculations.

N. M. Hugenholtz is one of those who have suggested that in the physics of quantum systems of a large number of degrees of freedom, the system be considered infinite from the beginning, instead of only in the end, by way of a limit process. In [1], p. 244, we find this passage: 'In standard statistical mechanics a state of the system is determined by a density operator ρ which in turn determines a positive linear form $\phi(A) = \mathrm{Tr}(\rho A)$. The density operator ρ has no limit for an infinite system but the positive linear form $\phi(A)$ is well defined in that limit. Therefore, to determine a state of an infinite system one cannot use a density matrix ρ but must simply give the positive linear form $\phi(A)$; ...' We shall here try and follow up Hugenholtz's program (as we construe it) in relation to matters technical: Would it be possible, by changing the notions of trace function and density operator, to retain ϕ in a tractable form in the case of an infinite system? This line of inquiry is suggested by the observation that *factors of type II* do have a trace function: if $A = \int \lambda E(d\lambda)$ belongs to such a factor $\mathscr{A}$, then $\mathrm{tr}_{\mathscr{A}}(A) = \int \lambda \dim_{\mathscr{A}}(E(d\lambda))$, where $\dim_{\mathscr{A}}$ is a dimension function (unique to within a constant factor) on $\mathscr{A}$. As a first step in our inquiry we shall ask what kind of self-adjoint operators ('observables') can belong to (more generally: can be affiliated to) factors of type *II*.

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[1]) This paper forms the substance of a thesis presented to the University of Geneva in partial fulfilment of the requirements for the degree of docteur ès sciences physiques (Thèse n° 1582). This work was performed under the supervision of Professor C. Piron, and was supported by the Norwegian Research Council for Science and the Humanities and by the Swiss National Foundation for Scientific Research.

1. Preliminary

Our tool in dealing with the first step of our inquiry (as outlined in the Introduction) will be the theory of unitary invariants (of normal operators) due to A. I. Plesner and V. A. Rohlin [2]. With regard to our aim, we restrict this summary to the case of self-adjoint operators in a separable Hilbert space.

Let $\mathfrak{H}$ be a separable Hilbert space, A a self-adjoint operator in $\mathfrak{H}$. By 'subspace (of $\mathfrak{H}$)' we shall mean 'closed linear subspace (of $\mathfrak{H}$)'.

Let $u \in \mathfrak{H}$. The smallest subspace of $\mathfrak{H}$ which contains u and reduces A is called the *cyclic subspace* (*with respect to* A) *determined by* u and is written $\mathfrak{H}(u)$. A subspace $\mathfrak{X}$ of $\mathfrak{H}$ is called *cyclic* (*with respect to* A) if a $u \in \mathfrak{H}$ exists such that $\mathfrak{X} = \mathfrak{H}(u)$. The operator A is termed *cyclic* if $\mathfrak{H}$ itself is a cyclic subspace (with respect to A).

We denote by $\mathscr{B}(\mathbb{R})$ the Borel field of the real line, by $\mathfrak{M}$ the set of finite positive measures on $\mathscr{B}(\mathbb{R})$. If $\mu, \nu \in \mathfrak{M}$ and μ is absolutely continuous with respect to ν (see, e.g., [3], p. 190), then we write $\mu \prec \nu$. The relation $\prec$ is a quasi-ordering on $\mathfrak{M}$, defining in the usual manner an equivalence relation $\sim$ on $\mathfrak{M}$, a quotient set $\mathfrak{S} =_{\text{def}} \mathfrak{M}/\sim$, and a partial ordering $\leqslant$ on $\mathfrak{S}$. In fact, $\mathfrak{S}$ is a σ-complete lattice. The members of $\mathfrak{S}$ are called *Hellinger types*; we shall denote by μ' the Hellinger type to which $\mu \in \mathfrak{M}$ belongs. We have $\mu_1' \vee \mu_2' = (\mu_1 + \mu_2)'$ (with a generalization to the countable case).

Consider $A = \int \lambda E(d\lambda)$. For $v \in \mathfrak{H}$ we denote by $\mu_{v,A}$ the measure $\delta \mapsto (E(\delta) v | v)$ on $\mathscr{B}(\mathbb{R})$. We get a subset $\mathfrak{S}_A =_{\text{def}} \{\mu_{v,A}' : v \in \mathfrak{H}\}$ of $\mathfrak{S}$, also a σ-complete lattice. $\mathfrak{S}_A$ may have a largest member, say $\mu_{u,A}'$: in that case A is said to be of *maximal Hellinger type* $\mu_A' =_{\text{def}} \mu_{u,A}'$ (since $\mathfrak{S}_A$ is a lattice, a maximal member is necessarily its (unique) largest member). In fact, in a separable Hilbert space (which is the case of interest to us), every self-adjoint operator is of maximal Hellinger type.

If a subspace $\mathfrak{X}$ of $\mathfrak{H}$ reduces A, then $A|\mathfrak{X}$ is called the *part* of A in $\mathfrak{X}$. Parts of A in mutually orthogonal subspaces of $\mathfrak{H}$ are called *orthogonal parts* of A. An *orthogonal sequence of Hellinger type* μ' (*with respect to* A) is a sequence $(A_i)_{i \in I}$ of pairwise orthogonal *cyclic* parts of A, all of maximal Hellinger type μ'. The set of such sequences can be partially ordered: $(A_i)_{i \in I}$ will be considered smaller than $(A_k)_{k \in K}$ if the latter (map) is an extension of the former. If $(A_i)_{i \in I}$ is a maximal member of this partially ordered set (maximal members do exist), then the cardinal $\mathfrak{m}(\mu', A)$ of I is uniquely determined by A; $\mathfrak{m}(\mu', A)$ is called the *multiplicity of* μ' (*with respect to* A).

A non-zero member μ' of $\mathfrak{S}_A$ is called *homogeneous* (*with respect to* A) if for each non-zero $\nu' \leqslant \mu'$, $\nu' \in \mathfrak{S}$, one has $\mathfrak{m}(\nu', A) = \mathfrak{m}(\mu', A)$ (necessarily, $\mathfrak{m}(\mu', A) \leqslant \mathfrak{m}(\nu', A)$). Any self-adjoint operator A of homogeneous maximal Hellinger type μ_A' of multiplicity $\mathfrak{m}(\mu_A', A)$ can be decomposed into a direct sum of $\mathfrak{m}(\mu_A', A)$ cyclic parts, each of maximal Hellinger type μ_A'.

The ordered pair $(\mu_A', \mathfrak{m}(\mu_A', A))$ is called the *spectral type* of the operator A of homogeneous maximal Hellinger type. Two self-adjoint operators of homogeneous maximal Hellinger type are unitarily equivalent if and only if they have equal spectral types. (Operators not of homogeneous maximal Hellinger type will not concern us here.)

2. Affiliation of a Self-Adjoint Operator to Factors of Type II_1 in a Separable Hilbert Space

Let $\mathfrak{H}$ be a separable Hilbert space. If $\mathfrak{X}$ is a subspace of $\mathfrak{H}$, we shall sometimes write $P_{\mathfrak{X}}$ for the (orthogonal) projection onto $\mathfrak{X}$; more often, however, we shall use the letters $\mathfrak{X}$ and E, equipped with identical markers, for a subspace and the projection onto it, respectively.

If $A = \int \lambda E(d\lambda)$ is a self-adjoint operator in $\mathfrak{H}$, then A is said to be *affiliated to* a von Neumann algebra $\mathscr{A}$, written $A\eta\mathscr{A}$, if each $E(\lambda) \in \mathscr{A}$.

If $\mathscr{A}$ is a factor of type II_1 in $\mathfrak{H}$, then $\dim_{\mathscr{A}}$ will denote the normalized dimension function on the set of projections from $\mathscr{A}$.

2.1. *The maximal Hellinger type of a self-adjoint operator affiliated to a factor of type II_1*

(2.1) Lemma. *Let $\mathscr{A}$ be a factor of type II_1 in a separable Hilbert space $\mathfrak{H}$. A self-adjoint operator $A = \int \lambda E(d\lambda)$ exists such that $A\eta\mathscr{A}$ and* $\dim_{\mathscr{A}}(E(\lambda)) = \lambda$, $\lambda \in [0,1]$.

We shall omit the proof of this result. However, the reader would have no difficulty in reconstructing it on the basis of the following remarks and suggestions:

1° If $E \in \mathscr{A}$ is a projection, then E is the sum of two projections $E', E'' \in \mathscr{A}$ such that $E'E'' = 0$ and $E' \sim E''$.

2° Consider the numbers 0, 1, and furthermore all numbers $(2j-1)2^{-k}$, where j, k are integers > 0, and where $1 \leqslant j \leqslant 2^{k-1}$. Arrange these numbers in succession: 0, 1, $(2\cdot 1-1)2^{-1}, \ldots, (2\cdot 1-1)2^{-k}, \ldots, (2\cdot 2^{k-1}-1)2^{-k}, \ldots$; let (r_n) denote the sequence so obtained, with $r_0 =_{\text{def}} 0$ (the range of (r_n) consists of all numbers in $[0,1]$ which are of the form $j\cdot 2^{-k}$, where j, k are integers $\geqslant 0$ and $j \leqslant 2^k$). By the principle of transfinite induction one obtains, making repeated use of 1°, by recursive definition a sequence $(E(r_n))$ of projections from $\mathscr{A}$ such that $\dim_{\mathscr{A}}(E(r_n)) = r_n$ for $n = 0, 1, \ldots$ and such that $r_{n_1} < r_{n_2}$ entails $E(r_{n_1}) < E(r_{n_2})$.

3° For any number $\lambda \in [0,1]$ there exists a sequence (c_n^λ), where c_n^λ equals 0 or 1, such that $\lambda = \sum_n c_n^\lambda 2^{-n}$. If λ is different from all $j\cdot 2^{-k}$, where j, k are integers, $j > 0$, $k \geqslant 0$, then the sequence (c_n^λ) is unique; in the opposite case there are precisely two sequences of the kind mentioned: one of them has no non-zero members past a certain index n, whereas the other one has non-zero members beyond any index n; it seems preferable to argue consistently in terms of the second possibility. For each $\lambda \in [0,1]$ one has $\lambda = \sum_{n=0}^{\infty} c_n^\lambda 2^{-n}$. The sequence (e_p^λ), $e_p^\lambda =_{\text{def}} \sum_{n=0}^{p} c_n^\lambda 2^{-n}$ for $p = 0, 1, \ldots$, is an increasing sequence of members of (r_n), converging to λ. To each p, therefore, a projection $E(e_p^\lambda)$ from $\mathscr{A}$ exists such that $\dim_{\mathscr{A}}(E(e_p^\lambda)) = e_p^\lambda$. One obtains a projection $E(\lambda) \in \mathscr{A}$, the strong limit of $(E(e_p^\lambda))_p$, with the property $\dim_{\mathscr{A}}(E(\lambda)) = \lambda$ ([4], Cor. II.1.23, gives that result). In fact, the family $E(\cdot)$ of projections from $\mathscr{A}$, extended to $\mathbb{R}$ by $E(\lambda) = 0$ for $\lambda < 0$, $E(\lambda) = \mathbb{1}$ for $\lambda > 1$, is a spectral family, and $A =_{\text{def}} \int \lambda E(d\lambda)$ is an operator fulfilling the requirements of Lemma (2.1).

We obtain readily the following more general result:

(2.2) Lemma. *Let $\mathscr{A}$ be a factor of type II_1 in a separable Hilbert space $\mathfrak{H}$. Let f be a real-valued function on $\mathbb{R}$ which is increasing (not necessarily strictly so), continuous on the right, and which satisfies $f(-\infty) = 0$, $f(\infty) = 1$. A self-adjoint operator $A = \int \lambda E(d\lambda)$ exists such that $A\eta\mathscr{A}$ and* $\dim_{\mathscr{A}}(E(\lambda)) = f(\lambda)$, $\lambda \in \mathbb{R}$.

The proof is not difficult: Let $\int \lambda F(d\lambda)$ be the operator of Lemma (2.1). We define

$$E(\lambda) =_{\text{def}} F(f(\lambda)), \quad \lambda \in \mathbb{R}.$$

$E(\cdot)$ is a spectral family, and $A =_{\text{def}} \int \lambda E(d\lambda)$ satisfies Lemma 2.2).

We are now in a position to make a connexion with the subject of Section 1:

Reference to [5], pp. 141–42, shows that any function f of the kind mentioned in Lemma (2.2) determines a positive measure μ_f on $\mathscr{B}(\mathbb{R})$, (called) the Borel–Stieltjes measure in $\mathbb{R}$ determined by f. To any such f, then, there corresponds a member of $\mathfrak{S}$.

On the other hand, let $\mu' \in \mathfrak{S}$ be given, and let $\mu \in \mu'$ be so chosen that $\mu(\mathbb{R}) = 1$. We set

$$f_\mu(\lambda) =_{\text{def}} \mu([-\infty, \lambda]).$$

The (real-valued) function f_μ is increasing, it satisfies $f_\mu(-\infty) = 0$, $f_\mu(\infty) = 1$, and it is continuous on the right (at every point of $\mathbb{R}$): Remembering that in a metric space, like $\mathbb{R}$, sequential convergence implies convergence proper, we select a strictly decreasing sequence (r_n) which converges to (an arbitrary point) $\lambda \in \mathbb{R}$. We find that

$$1 - f_\mu(\lambda) = \mu(]\lambda, \infty]) = \mu\left(\bigcup_{n=0}^{\infty}]r_n, \infty]\right) = \mu(]r_0, \infty] \cup]r_1, r_0] \cup]r_2, r_1] \cup \ldots)$$

$$= \mu(]r_0, \infty]) + \mu(]r_1, r_0]) + \mu(]r_2, r_1]) + \ldots = \lim_n \mu(]r_n, \infty]),$$

whence

$$f_\mu(\lambda) = \lim_n \mu([-\infty, r_n]) = \lim_n f_\mu(r_n),$$

so f_μ is continuous on the right.

These considerations allow us to put Lemma (2.2) on this form:

(2.2)′ Lemma. *Let $\mathscr{A}$ be a factor of type II_1 in a separable Hilbert space $\mathfrak{H}$. Let μ' be any member of $\mathfrak{S}$, μ a representative of μ' such that $\mu(\mathbb{R}) = 1$. A self-adjoint operator $A = \int \lambda E(d\lambda)$ exists such that $A \eta \mathscr{A}$ and $\dim_{\mathscr{A}}(E(\delta)) = \mu(\delta)$, $\delta \in \mathscr{B}(\mathbb{R})$.*

The following is the main result of this subsection:

(2.3) Theorem. *Let $\mathscr{A}$ be a factor of type II_1 in a separable Hilbert space $\mathfrak{H}$. Let μ' be any member of $\mathfrak{S}$. A self-adjoint operator exists which is affiliated to $\mathscr{A}$ and whose maximal Hellinger type is μ'.*

Proof. For the proof we shall rely on certain properties of Hellinger types, as well as on a certain fact concerning the trace function $\operatorname{tr}_{\mathscr{A}}$ on $\mathscr{A}$. Before stating the latter, we shall take note of some facts regarding the dimension functions on $\mathscr{A}$ and $\mathscr{A}'$ (see [4], Th. II.2.7):

Let $\mathfrak{D}$ and $\mathfrak{D}'$ be the ranges of $\dim_{\mathscr{A}}$ and $\dim_{\mathscr{A}'}$, respectively, and let $\mathfrak{D}_0$ and $\mathfrak{D}_0'$ be those of the functions $E_x^{\mathscr{A}'} \mapsto \dim_{\mathscr{A}}(E_x^{\mathscr{A}'})$ and $E_x^{\mathscr{A}} \mapsto \dim_{\mathscr{A}'}(E_x^{\mathscr{A}})$, respectively, x ranging over $\mathfrak{H}$ (where $E_x^{\mathscr{A}'}$ is the projection onto $\mathfrak{X}_x^{\mathscr{A}'}$, the subspace determined by $\{A'x \colon A' \in \mathscr{A}'\}$; similarly for $E_x^{\mathscr{A}}$). The dimension functions $\dim_{\mathscr{A}}$ and $\dim_{\mathscr{A}'}$ should be normalized, when possible. Now let

$$\varphi(\dim_{\mathscr{A}}(E_x^{\mathscr{A}'})) =_{\text{def}} \dim_{\mathscr{A}'}(E_x^{\mathscr{A}}).$$

In fact,

$$\varphi(\lambda) = c\lambda, \quad \lambda \in \mathfrak{D}_0,$$

where c is a strictly positive constant, the *linking constant* of $\mathscr{A}$ (see [4], Def. II.2.8). Moreover, $\mathfrak{D}_0 = [0, \lambda_0]$, $\mathfrak{D}'_0 = [0, c\lambda_0]$, for some $\lambda_0 \in [0,1]$, and $\mathfrak{D}_0 = \mathfrak{D}$ or $\mathfrak{D}'_0 = \mathfrak{D}'$ (or both). If $\mathscr{A}'$ is a factor of type II_∞, then $\dim_{\mathscr{A}'}$ cannot be normalized. In that case we may, and shall, choose $c = 1$, and we have $\mathfrak{D}_0 = \mathfrak{D}$ $(=[0,1])$. If on the other hand $\mathscr{A}'$ is a factor of type II_1, then $\mathfrak{D}_0 = \mathfrak{D}$, corresponding to $c \leqslant 1$, or $\mathfrak{D}'_0 = \mathfrak{D}'$, corresponding to $c \geqslant 1$.

We start by assuming that $c \leqslant 1$. In that case there exists a member x of $\mathfrak{H}$ such that for all $T \in \mathscr{A}$ we have

$$\operatorname{tr}_{\mathscr{A}}(T) = (Tx|x)$$

(see [4], Cor. II.3.4).

Now let us choose $\mu \in \mu'$ such that $\mu(\mathbb{R}) = 1$, and let us denote by $A = \int \lambda E(d\lambda)$ the operator of Lemma (2.2)$'$. We have

$$\mu_{x,A}(\delta) = (E(\delta)x|x) = \operatorname{tr}_{\mathscr{A}}(E(\delta)) = \dim_{\mathscr{A}}(E(\delta)) = \mu(\delta), \quad \delta \in \mathscr{B}(\mathbb{R}),$$

whence $\mu_{x,A}{}' = \mu'$. So we know that $\mu' \in \mathfrak{S}_A$. We shall show that μ' is the largest member of $\mathfrak{S}_A$, i.e., the maximal Hellinger type of A. To that effect we compare an arbitrarily chosen $\mu_{y,A}{}'$, $y \in \mathfrak{H}$, with μ'. This we can do because of the following circumstance: by the formula $\operatorname{tr}_{\mathscr{A}}(T) = (Tx|x)$ we find

$$\operatorname{tr}_{\mathscr{A}}(E_x^{\mathscr{A}'}) = (E_x^{\mathscr{A}'}x|x) = (x|x) = (\mathbb{1}x|x) = \operatorname{tr}_{\mathscr{A}}(\mathbb{1}) = 1,$$

i.e., $E_x^{\mathscr{A}'} = \mathbb{1}$.

Let T be any member of $\mathscr{A}'$. The operator T can be written as

$$T = \alpha_1 U_1 + \alpha_1 U_1^* + i\alpha_2 U_2 + i\alpha_2 U_2^*,$$

where U_1, U_2 are unitary operators from $\mathscr{A}'$, and where α_1, α_2 are real numbers (see [6], Ch. I, Sec. 1, Prop. 3). Thus

$$\mu_{Tx,A}(\delta) = (E(\delta)\,Tx|Tx) = \sum_{i,j=1}^{4} \alpha_{ij}(E(\delta)\,V_i x|V_j x), \quad \delta \in \mathscr{B}(\mathbb{R}),$$

where the V_i are unitary operators from $\mathscr{A}'$ and the α_{ij} are numbers. Let us consider the (arbitrary) term

$$\alpha_{ij}(E(\delta)\,V_i x|V_j x) = \alpha_{ij}(E(\delta)\,x|V_i^* V_j x).$$

Now,

$$\mu(\delta) = 0 \;.\supset.\; \|E(\delta)\,x\| = 0 \;:\supset:\; E(\delta)\,x = 0 \;.:\supset.\; (E(\delta)\,x|V_i^* V_j x) = 0.$$

The join $\bigvee_i \mu_i{}'$ of members $\mu_i{}'$ of $\mathfrak{S}$ being $(\sum_i \mu_i)'$ (provided $\sum_i \mu_i(\mathbb{R}) < \infty$, which we can always obtain by suitably choosing the μ_i) (see [2], Th. 10.2.1), we see that $\mu_{Tx,A}{}'$ is the join of a certain number of members of $\mathfrak{S}_A$, all of which are smaller than $\mu_{x,A}{}' = \mu'$. This means that all Tx, $T \in \mathscr{A}'$, are members of $\mathfrak{H}_{\mu'} =_{\text{def}} \{v\colon\ v \in \mathfrak{H}.\mu_{v,A}{}' \leqslant \mu\}$. According to [2], Th. 10.3.3, $\mathfrak{H}_{\mu'}$ is a subspace of $\mathfrak{H}$; hence linear combinations of members Tx of $\mathfrak{H}$, $T \in \mathscr{A}'$, as well as limits of sequences of such, all belong to $\mathfrak{H}_{\mu'}$, so $\mathfrak{H}_{\mu'} = X_x^{\mathscr{A}'} = \mathfrak{H}$. That is, μ' is the maximal Hellinger type of A.

We go on to consider the case $c > 1$. Let m be a positive integer such that $c/m < 1$. Let us choose m projections $E_1, \ldots, E_m$ from $\mathscr{A}$ with the properties $\dim_{\mathscr{A}}(E_i) = 1/m$ and $E_i E_j = \delta_{ij} E_i$ (then $\sum_i E_i = 1$). That can be done as follows: Let us first look for projections F_i, $i = 1, 2, \ldots, m-1$, from $\mathscr{A}$ such that $\dim_{\mathscr{A}}(F_i) = i/m$ and $F_1 \leqslant F_2 \leqslant \cdots \leqslant F_{m-1}$. If $\lambda_i \in \mathbb{R}$ satisfies $\dim \mathscr{A}(E(\lambda_i)) = i/m$, then we set $F_i =_{\text{def}} E(\lambda_i)$. If no such λ_i exists, then we go on to consider

$$F_i' =_{\text{def}} \sup\{E(\lambda) : \dim_{\mathscr{A}}(E(\lambda)) < i/m\},$$
$$F_i'' =_{\text{def}} \inf\{E(\lambda) : \dim_{\mathscr{A}}(E(\lambda)) > i/m\}.$$

We find quite easily that $\dim_{\mathscr{A}}(F_i') \leqslant i/m$, $\dim_{\mathscr{A}}(F_i'') > i/m$ (using, e.g., [6], App. II, and Ch. I, Sec. 3, Cor. of Prop. 1). In case $\dim_{\mathscr{A}}(F_i') = i/m$, we obviously choose F_i' to be the projection F_i. In the opposite case a trivial argument gives us a projection F_i from $\mathscr{A}$ satisfying $\dim_{\mathscr{A}}(F_i) = i/m$ and $F' < F_i < F''$. We define $E_1 =_{\text{def}} F_1$, $E_2 =_{\text{def}} F_2 - E_1, \ldots, E_m =_{\text{def}} \mathbb{1} - E_{m-1}$. Actually, we have obtained somewhat more than we were seeking for: the E_j commute with A; this fact will be of avail in the proof of Theorem (2.4).

We remind the reader of the following (see [7], Sec. 11.3):

Let $\mathfrak{X} \neq \{0\}$ be a subspace $\mathfrak{H}$, $\mathfrak{X}\eta\mathscr{A}$, and let $E =_{\text{def}} P_{\mathfrak{X}}$. Consider those members T of $\mathscr{A}$ for which $TE = ET = T$, and denote by T_E their restrictions to $\mathfrak{X}$. Let $\mathscr{A}_E$ be the set of all such T_E. Like $\mathscr{A}$, $\mathscr{A}_E$ is a factor of type II_1. The map $T \mapsto T_E$ is an algebraic isomorphism, mapping onto $\mathscr{A}_E$ the set $E\mathscr{A}E$ of all $T \in \mathscr{A}$ which satisfy $TE = ET = T$. Similarly, $T \mapsto T_E$ is an algebraic isomorphism of all of $\mathscr{A}'$ onto $\mathscr{A}_E' =_{\text{def}} (\mathscr{A}_E)' = (\mathscr{A}')_E$.

We write $\mathscr{A}_i =_{\text{def}} \mathscr{A}_{E_i}$, $\mathscr{A}_i' =_{\text{def}} \mathscr{A}_{E_i}'$. We define further $\dim_{\mathscr{A}_i}(F_{E_i}) =_{\text{def}} \dim_{\mathscr{A}}(F)$ for projections F from $\mathscr{A}$ such that $F \leqslant E_i$, and $\dim_{\mathscr{A}_i'}(F_{E_i}') =_{\text{def}} \dim_{\mathscr{A}'}(F')$ for projections F' from $\mathscr{A}'$. The functions $\dim_{\mathscr{A}_i}$ and $\dim_{\mathscr{A}_i'}$ are dimension functions on $\mathscr{A}_i$ and $\mathscr{A}_i'$, respectively, with respective ranges $[0, \dim_{\mathscr{A}}(E_i)]$ and $[0,1]$. If we normalize $\dim_{\mathscr{A}_i}$ we get a linking constant of $\mathscr{A}_i$ which is $c_i =_{\text{def}} (\dim_{\mathscr{A}}(E_i)/\dim_{\mathscr{A}'}(\mathbb{1}))\,c = c/m \leqslant 1$ ([7], Sec. 11.4).

Thus the linking constant of $\mathscr{A}_i$ is smaller than unity and so we can apply what we found above. We let $P_{\mathfrak{X}_i} =_{\text{def}} E_i$. For each i there is a member x_i' of $\mathfrak{X}_i$ such that $\operatorname{tr}_{\mathscr{A}_i}(T_{E_i}) = (T_{E_i}x_i'|x_i')$, $T_{E_i} \in \mathscr{A}_i$. From the last paragraph it is clear that. $\operatorname{tr}_{\mathscr{A}} = (1/m)\operatorname{tr}_{\mathscr{A}_i}$, in the following sense: If $T \in E_i\mathscr{A}E_i$, then

$$\operatorname{tr}_{\mathscr{A}}(T) = (1/m)\operatorname{tr}_{\mathscr{A}_i}(T_{E_i}) = (1/m)\,(T_{E_i}x_i'|x_i') = (1/m)\,(Tx_i'|x_i') = (Tx_i|x_i),$$

where $x_i =_{\text{def}} (1/\sqrt{m})\,x_i$. Due to [8], Lemma 3.3.5, there results for $\operatorname{tr}_{\mathscr{A}}(T)$ the following expression:

$$\operatorname{tr}_{\mathscr{A}}(T) = \sum_{i=1}^{m} (Tx_i|x_i), \quad T \in \mathscr{A}.$$

Using this formula we find that

$$\mu(\delta) = \dim_{\mathscr{A}}(E(\delta)) = \sum_{i=1}^{m} (E(\delta)\,x_i|x_i) = \sum_{i=1}^{m} \mu_{x_i,A}(\delta), \quad \delta \in \mathscr{B}(\mathbb{R}).$$

Hence $\mu' = \bigvee_{i=1}^{m} \mu_{x_i,A}'$. Since $\mathfrak{S}_A$ is a lattice, $\mu' \in \mathfrak{S}_A$. Now, for each i, $E_{x_i}^{A_i'} = E_i$; therefore $\mathfrak{X}_i = \mathfrak{H}_{\mu x_i,A'}$ according to the first part of this proof; moreover

$$\mathfrak{X}_i = \mathfrak{H}_{\mu_{x_i},A'} \subseteq \mathfrak{H}_{\mu'} \subseteq \mathfrak{H}, \text{ so } \mathfrak{H} = \mathfrak{X}_1 \oplus \cdots \oplus \mathfrak{X}_m \subseteq \mathfrak{H}_{\mu'} \subseteq \mathfrak{H}.$$

Again, then, μ' is the maximal Hellinger type of A.

Hence the operator A fulfils the requirements of Theorem (2.3).

2.2. *The spectral type of a self-adjoint operator affiliated to a factor of type II_1*

(2.4) Theorem. *Let $\mathscr{A}$ be a factor of type II_1 in a separable Hilbert space $\mathfrak{H}$, and let A be a self-adjoint operator in $\mathfrak{H}$ affiliated to $\mathscr{A}$. The maximal Hellinger type μ'_A of A is homogeneous and* $\mathfrak{m}(\mu'_A, A) = \aleph_0$.

Proof. We shall prove that $\mathfrak{m}(\mu'_A, A)$ is larger than any integer $n \geqslant 1$. The homogeneity of μ'_A then follows from the separability of $\mathfrak{H}$ and the fact that if $\mu', \nu' \in \mathfrak{S}_A$, $\mu' \leqslant \nu'$, then $\mathfrak{m}(\nu' A) \leqslant \mathfrak{m}(\mu', A)$.

In the first place, let $c \leqslant 1$. In that case there is a member x of $\mathfrak{H}$ such that $\mathrm{tr}_{\mathscr{A}} = (\cdot x|x)$; moreover, $E_x^{\mathscr{A}'} = \mathbb{1}$. These facts were mentioned in the proof of Theorem (2.3); there it was further shown that x is a member of $\mathfrak{H}$ whose Hellinger type is maximal with respect to $A: \mu_{x,A}' = \mu'_A$.

We write $A = \int \lambda E(d\lambda)$. We denote by $\mathscr{B}$ the von Neumann algebra determined by $A: \mathscr{B} = \{A\}''$. The symbol $\smile$ will denote the relation of commutability (of operators) The proof:

Let us consider first the case $n = 1$. We proceed by the method of indirect proof:

$\vdash. A$ is a cyclic operator (1)

$\vdash. (1) . \supset \vdash. \mathscr{B} = \mathscr{B}'$ (2)

$\vdash. \mathscr{A}$ is a factor of type II_1 (3)

$\vdash. \mathscr{B} \subset \mathscr{A} . \mathscr{A}' \subset \mathscr{B}'$ (4)

$\vdash. (2) . (4) . \supset \vdash. \mathscr{A}' \subset \mathscr{B}' = \mathscr{B} \subset \mathscr{A}$ (5)

$\vdash. (5) . \supset \vdash. \mathscr{A} \cap \mathscr{A}' = \mathscr{A}'$ (6)

$\vdash. (3) . (6) . \supset \vdash. (3) . \sim(3)$

We conclude that A is not a cyclic operator, so $\mathfrak{m}(\mu'_A, A) > 1$ (see [2], Ch. X, Secs. 4.1 and 4.2). (As for the implication in (2) above, a proof can be found in [9].) (In this proof, $\vdash$ is the sign of assertion, $\sim$ the sign of negation.)

Next we consider the case $n > 1$. We want to prove

$\mathfrak{m}(\mu'_A, A) > n$ (Prop).

$\vdash. \mathfrak{m}(\mu'_A, A) > 1$ (1)

$\vdash: (m): 1 \leqslant m < n . \supset \vdash. \mathfrak{m}(\mu'_A, A) > m$ (2)

We use again the method of indirect proof:

$\vdash. (1) . (2) . \sim$(Prop) (3)

$\vdash. (3) . \supset \vdash. \mathfrak{m}(\mu'_A, A) = n$ (4)

[[6], Ch. I, Sec. 1, Prop. 4]

$\vdash: (y). y \in \mathfrak{H} . \supset . E_y^{\mathscr{B}} \in \mathscr{B}'$ (5)

[(4). (5). [4], Cor. II.3.6]

$\vdash. (\exists\, U_1, \ldots, U_n). U_1, \ldots, U_n$ are unitary members of $\mathscr{A}'$.

$\mathfrak{H} = \mathfrak{X}_{U_1 x}^{\mathscr{B}} \oplus \cdots \oplus \mathfrak{X}_{U_n x}^{\mathscr{B}} . (i). A | \mathfrak{X}_{U_i x}^{\mathscr{B}}$ is cyclic (6)

[[10], Sec. 75]

$$\vdash: (6) \,.\, T \in \mathscr{B} \,.\supset \vdash.\; T = \varphi(A) = \int \varphi(\lambda)\, E(d\lambda)$$
$$= \int \varphi(\lambda)\, E(d\lambda)\, (E^{\mathscr{B}}_{U_1 x} + \cdots + E^{\mathscr{B}}_{U_n x})$$
$$= \int \varphi(\lambda)\, E^{\mathscr{B}}_{U_1 x}\, E(d\lambda)\, E^{\mathscr{B}}_{U_1 x} | \mathfrak{X}^{\mathscr{B}}_{U_1 x} \oplus \cdots$$
$$\oplus \int \varphi(\lambda)\, E^{\mathscr{B}}_{U_n x}\, E(d\lambda)\, E^{\mathscr{B}}_{U_n x} | \mathfrak{X}^{\mathscr{B}}_{U_n x} \qquad (7)$$

$$\vdash: \mathfrak{X}_1,\ \mathfrak{X}_2 \text{ are subspaces of } \mathfrak{H} \text{ reducing } A.\ T_1 \in \mathscr{L}(\mathfrak{X}_1),\ T_2 \in \mathscr{L}(\mathfrak{X}_2).$$
$$T_1 \smile A|\mathfrak{X}_1,\ T_2 \smile A|\mathfrak{X}_2 \,.\supset.\; T_1 \oplus T_2 \smile A|\mathfrak{X}_1 \oplus A|\mathfrak{X}_2 \qquad (8)$$

$$\vdash.\ (6)\,.\,(7)\,.\,(8)\,.\supset \vdash.\ \mathscr{B}' = \{A\}''' = [\{A|\mathfrak{X}^{\mathscr{B}}_{U_1 x} \oplus \cdots$$
$$\oplus A|\mathfrak{X}^{\mathscr{B}}_{U_n x}\}'']' \subseteq [\{A|\mathfrak{X}^{\mathscr{B}}_{U_1 x}\}'' \oplus \cdots \oplus \{A|\mathfrak{X}^{\mathscr{B}}_{U_n x}\}'']'$$
$$= [\{A|\mathfrak{X}^{\mathscr{B}}_{U_1 x}\}' \oplus \cdots \oplus \{A|\mathfrak{X}^{\mathscr{B}}_{U_n x}\}']'$$
$$\subseteq \{A|\mathfrak{X}^{\mathscr{B}}_{U_1 x} \oplus \cdots \oplus A|\mathfrak{X}^{\mathscr{B}}_{U_n x}\}'' = \{A\}'' = \mathscr{B} \qquad (9)$$

$$\vdash.\ (9)\,.\supset \vdash.\ (1)\,.\sim(1)$$
$$\vdash.\ (10)\,.\supset \vdash.\ (\text{Prop}) \qquad (10)$$

Thus $\mathfrak{m}(\mu'_A, A) = \aleph_0$, by the principle of transfinite induction.

Let us explain how we arrived at the statement (6) above. That $\mathfrak{m}(\mu'_A, A) = n$ means that A can be decomposed into a direct sum of n pairwise orthogonal cyclic parts each of spectral type $(\mu'_A, 1)$, or, what is the same, that $\mathfrak{H}$ can be decomposed into a direct sum of n pairwise orthogonal subspaces $\mathfrak{H}(x_i)$ such that $\mu_{x_i,A}{}' = \mu'_A$ (see Sec. 1). Now $\mu'_A = \mu_{x,A}{}'$, and [4], Cor. II.3.6, says that each of the x_i can be attained from x by means of some unitary operator from $\mathscr{A}'$.

In the second place, let $c > 1$. As in the proof of Theorem (2.3), from the sixth paragraph onwards, we start by choosing an integer $m > 0$ such that $c/m \leqslant 1$. We then let $E_1, \ldots, E_m$ be m projections from $\mathscr{A}$, all commuting with A, satisfying $\dim_{\mathscr{A}}(E_i) = 1/m$ and $E_i E_j = \delta_{ij} E_i$. For each i, $A|\mathfrak{X}_i$ is a self-adjoint operator affiliated to the factor $\mathscr{A}_{E_i}$; the latter is a factor of type II_1 whose linking constant is smaller than 1. The maximal Hellinger type $\mu_{x_i,A_i}{}'$ of $A_i =_{\text{def}} A|\mathfrak{X}_i$ is homogeneous and $\mathfrak{m}(\mu_{x_i,A_i}{}', A_i) = \aleph_0$. Notice that $\mu_{x_i,A_i}{}' = \mu_{x_i,A}{}'$ (since E_i reduces A), and that $\mathfrak{m}(\mu_{x_i,A}{}', A) = \aleph_0$ (since $\mathfrak{m}(\mu_{x_i,A_i}{}', A_i) = \aleph_0$) (the remarks in Sec. 1 may convince the reader of that; otherwise he may consult [2], Ch. X, Sec. 4.2). $\mathfrak{S}_A$, as well as $\mathfrak{S}_{A_i}$, is a so-called *admissible* (sub-)lattice (of $\mathfrak{S}$), i.e., it is σ-closed and contains with any member μ' also $\mathfrak{S}_{\mu'} =_{\text{def}} \{\nu' :\ \nu' \in \mathfrak{S}.\ \nu' \leqslant \mu'\}$ ([2], Ch. X, Sec. 2.3). Hence $\mathfrak{S}_{\mu_{x_i,A_i}{}'} = \{\mu_{u_i,A_i}{}' :\ u_i \in \mathfrak{X}_i\}$, for each i. Let now $i \neq j$, and let $\rho' \leqslant \mu_{x_i,A_i}{}'$, $\rho' \leqslant \mu_{x_j,A_j}{}'$. We have $\rho' = \mu_{v_i,A_i}{}'$, $\rho' = \mu_{jv,A_j}{}'$, for some $v_i \in \mathfrak{X}_i$, $v_j \in \mathfrak{X}_j$. Thus $v_i = v_j = 0$, so $\rho' = 0'$. This we express by saying that $\mu_{x_i,A_i}{}'$, $\mu_{x_j,A_j}{}'$ are *independent* Hellinger types (2], Ch. X, Sec. 1.4). From [2], Ch. X, Sec. 4.2, we get: if $(\mu_i{}')$ is a finite or countable sequence of pairwise independent Hellinger types, and $\mu' = \bigvee_i \mu_i{}'$, then $\mathfrak{m}(\mu', A) = \min_i \mathfrak{m}(\mu_i{}', A)$. In our case, therefore, $\mathfrak{m}(\mu'_A, A) = \aleph_0$, so μ'_A is again homogeneous.

(2.5) Theorem. *Let A be a self-adjoint operator in a separable Hilbert space $\mathfrak{H}$, of spectral type $(\mu'_A, \aleph_0)$. There is in $\mathfrak{H}$ a factor $\mathscr{A}$ of type II_1 such that $A \eta \mathscr{A}$.*

Proof. This follows from Theorems (2.3) and (2.4): Choose any factor $\mathscr{A}_1$ of type II_1 in $\mathfrak{H}$. Affiliated to $\mathscr{A}_1$ is a self-adjoint operator A_1 whose maximal Hellinger type is μ'_A, according to Theorem (2.3). Using Theorem (2.4) we infer that μ'_A is homogeneous and that $\mathfrak{m}(\mu'_A, A_1) = \aleph_0$. Hence (see Sec. 1) A and A_1 are unitarily equivalent: $A = UA_1U^*$ for some unitary operator U, and $\mathscr{A} =_{\text{def}} U\mathscr{A}_1U^*$ is a factor of type II_1 with $A\eta\mathscr{A}$.

3. Affiliation of Self-Adjoint Operators to Factors of Type II_∞ in a Separable Hilbert Space

3.1. *The maximal Hellinger type of a self-adjoint operator affiliated to a factor of type II_∞*

(3.1) Theorem. *Let $\mathscr{A}$ be a factor of type II_∞ in a separable Hilbert space $\mathfrak{H}$. Let μ' be a member of $\mathfrak{S}$. A self-adjoint operator exists which is affiliated to $\mathscr{A}$ and whose maximal Hellinger type is μ'.*

Proof. From [4], Lemma II.1.8, we see that the identity operator $\mathbb{1}$ in $\mathfrak{H}$, being an infinite projection from $\mathscr{A}$, can be written as

$$\mathbb{1} = \sum_{i=1}^{\infty} E_i,$$

where the projections E_i from $\mathscr{A}$ are finite, pairwise orthogonal, and pairwise equivalent.

For each i, consider $\mathfrak{X}_i =_{\text{def}} E_i\mathfrak{H}$. On referring to [4], Lemma II.2.23, we learn that $\mathscr{A}_i =_{\text{def}} \mathscr{A}_{E_i}$ is a factor of type II_1 in $\mathfrak{X}_i$.

Let μ be any member of μ'. We represent $\mathbb{R}$ as the union of closed intervals I_i:

$$\mathbb{R} = \bigcup_{i=1}^{\infty} I_i,$$

and write μ as

$$\mu = \sum_{i=1}^{\infty} \mu_i,$$

where

$$\mu_i(\delta) =_{\text{def}} \begin{cases} \mu(\delta) & \text{if } \delta \subseteq I_i \\ 0 & \text{if } \delta \cap I_i = \phi \end{cases}, \qquad \delta \in \mathscr{B}(\mathbb{R}).$$

From Theorem (2.3) we know that for each i a self-adjoint operator A_i exists which is affiliated to $\mathscr{A}_i$ and whose maximal Hellinger type is μ_i'. From [2], Ch. X, Sec. 3.2, we get: The direct sum A of finitely or countably many self-adjoint operators A_i of maximal Hellinger types μ_i' is again an operator of maximal Hellinger type, and $\mu'_A = \bigvee_i \mu_i'$. Thus in our case

$$\sum_{i=1}^{\infty}{}^{\oplus} A_i$$

is a self-adjoint operator, affiliated to $\mathscr{A}$, of maximal Hellinger type

$$\mu' = \bigvee_{i=1}^{\infty} \mu_i' \left(= \left(\sum_{i=1}^{\infty} \mu_i \right)' \right).$$

3.2. The spectral type of a self-adjoint operator affiliated to a factor fo type II_∞

(3.2) Theorem. *Let $\mathscr{A}$ be a factor of type II_∞ in a separable Hilbert space $\mathfrak{H}$, and let A be a self-adjoint operator in $\mathfrak{H}$ affiliated to $\mathscr{A}$. The maximal Hellinger type μ'_A of A is homogeneous and $\mathfrak{m}(\mu'_A, A) = \aleph_0$.*

Proof. We write again $\mathbb{1}$ in the form

$$\mathbb{1} = \sum_{i=1}^{\infty} E_i,$$

the projections E_i from $\mathscr{A}$ being finite, pairwise orthogonal, and pairwise equivalent. We further require that the E_i should reduce A. That can always be obtained: cf. the proof of Theorem (2.3). Writing $A_i =_{\text{def}} A|\mathfrak{X}_i$ we have the decomposition

$$A = \sum_{i=1}^{\infty} \oplus A_i$$

of A, where for each i A_i is affiliated to the factor $\mathscr{A}_i =_{\text{def}} \mathscr{A}_{E_i}$ of type II_1.

To the above decomposition of A there corresponds the decomposition

$$\mu'_A = \bigvee_{i=1}^{\infty} \mu'_{A_i}$$

of μ'_A, where for each i μ'_{A_i} is the maximal Hellinger type of A_i. Referring to the proof of Theorem (2.4), we realize that the μ'_{A_i} are pairwise independent.

From Theorem (2.4) we learn that each μ'_{A_i} is homogeneous (with respect to A_i), and that $\mathfrak{m}(\mu'_{A_i}, A_i) = \aleph_0$. From the last paragraph of the proof of Theorem (2.4) we see that $\mathfrak{m}(\mu'_A, A) = \aleph_0$, and hence that μ'_A is homogeneous (with respect to A).

(3.3) Theorem. *Let A be a self-adjoint operator in a separable Hilbert space $\mathfrak{H}$, of spectral type $(\mu_A, \aleph_0)$. There is in $\mathfrak{H}$ a factor $\mathscr{A}$ of type II_∞ such that $A \eta \mathscr{A}$.*

Proof. Cf. the proof of Theorem (2.5).

4. With a View to Applications

We have solved above the problem of affiliation of self-adjoint operators to factors of type II in a separable Hilbert space. However, we had something more in mind, namely, by means of the trace-functions $\text{tr}_{\mathscr{A}}$, on the trace classes $\mathscr{C}_1(\mathscr{A})$ of suitably chosen factors of type II, to extend the formal apparatus of quantum mechanics to situations formerly inaccessible to direct numerical treatment. It is well known how, traditionally, states (of physical systems) are connected with the trace function on the trace class of $\mathscr{L}(\mathfrak{H})$. It would seem desirable to be able to cope with situations falling outside the traditional framework, such as when the Hamiltonian of a large thermodynamic system does not have a pure point spectrum (see also the Introduction). Now if $\mathscr{A}$ is a factor of type II, and $A = \int \lambda E(d\lambda) \in \mathscr{C}_1(\mathscr{A})$, then

$$\text{tr}_{\mathscr{A}}(A) = \int \lambda \dim_{\mathscr{A}}(E(d\lambda)).$$

Clearly, in order to use this formula for explicit calculations, we should need to know the function $\lambda \mapsto \dim_{\mathscr{A}}(E(\lambda))$. Let us consider the following problem:

Let a self-adjoint operator B in $\mathfrak{H}$ be given. The maximal Hellinger type μ_B of B should have the multiplicity $\aleph_0$. Can we construct a function f on $\mathbb{R}$ such that $A = \int \lambda E(d\lambda)$, with

$$\dim_{\mathscr{A}}(E(\lambda)) = f(\lambda), \quad \lambda \in \mathbb{R},$$

is unitarily equivalent to B?

Theorems (2.3) and (2.4) give a positive answer to our question in the case II_1: we choose any function f_μ, where $\mu \in \mu'_B$ and $\mu(\mathbb{R}) = 1$. Theorems (3.1) and (3.2) imply that the answer would be 'yes' also in the case II_∞.

In this section we shall formulate our results (on affiliation) in terms of functions rather than in terms of (maximal) Hellinger types. What freedom do we have regarding the choice of the function f (see above)? We shall be particularly interested in the possibilities of choosing f to be unbounded (i.e., in the case II_∞).

4.1. Preliminary results

(4.1) Lemma. *Let $\mathscr{A}$ be a factor of type II_∞ in a separable Hilbert space $\mathfrak{H}$. A self-adjoint operator $A = \int \lambda E(d\lambda)$ exists such that $A \eta \mathscr{A}$ and*

$$E(\lambda) = 0, \quad \lambda < 0,$$

$$\dim_{\mathscr{A}}(E(\lambda)) = \lambda, \quad \lambda \geqslant 0.$$

Proof. We set down again the decomposition of the identity operator $\mathbb{1}$ in $\mathfrak{H}$ which appeared in the proof of Theorem (3.1):

$$\mathbb{1} = \sum_{i=1}^{\infty} E_i,$$

and we impose the further condition $\dim_{\mathscr{A}}(E_i) = 1$, where $\dim_{\mathscr{A}}$ is (for the moment) any dimension function on the set of projections from $\mathscr{A}$. Moreover, we represent $\mathbb{R}^+$ as the union of closed intervals $I_i =_{\text{def}} [i-1, i]$:

$$\mathbb{R}^+ = \bigcup_{i=1}^{\infty} I_i.$$

In the remainder of this proof we shall make use of the notation introduced in the proof of Theorem (2.3). We see that the factors $\mathscr{A}_i$ are of type II_1; due to the convention $\dim_{\mathscr{A}}(E_i) = 1$, made above, the dimension functions $\dim_{\mathscr{A}_i}$ are all normalized.

According to Lemma (2.1) there exists a self-adjoint operator $\int_0^1 \lambda E_{E_i}^{(i)}(d\lambda)$, affiliated to $\mathscr{A}_i$, determined by the requirement

$$\dim_{\mathscr{A}_i}(E_{E_i}^{(i)}(\lambda)) = \lambda, \quad \lambda \in [0, 1].$$

Here the $E^{(i)}(\lambda)$ should be thought of as projections from $\mathscr{A}$, with $E^{(i)}(1) = E_i$.

Let

$$E(\lambda) =_{\text{def}} \begin{cases} E^{(1)}(\lambda), & \lambda \in I_1, \\ \sum_{i=1}^{n-1} E_i + E^{(n)}(\lambda - n + 1), & \lambda \in I_n,\ n = 2, 3, \dots \end{cases}$$

Then

$$\begin{aligned}\dim_{\mathscr{A}}(E(\lambda)) &= n-1+\dim_{\mathscr{A}}(E^{(n)}(\lambda-n+1))\\ &= n-1+\dim_{\mathscr{A}_n}(E^{(n)}_{E_n}(\lambda-n+1))\\ &= n-1+\lambda-n+1\\ &= \lambda.\end{aligned}$$

For $\lambda < 0$ let $E(\lambda) =_{\text{def}} 0$.

It is not difficult to verify that $E(\cdot)$ is a spectral family. The operator

$$A =_{\text{def}} \int \lambda E(d\lambda)$$

satisfies our lemma.

As Lemma (4.1) is an analogue of Lemma (2.1) for the case II_∞, so we propose the following as an analogue of Lemma (2.2):

(4.2) Lemma. *Let $\mathscr{A}$ be a factor of type II_∞ in a separable Hilbert space $\mathfrak{H}$. Let f be a real-valued function on $\mathbb{R}$ which is increasing (not necessarily strictly so), continuous on the right, which satisfies $f(-\infty) = 0$, and is finite everywhere. A self-adjoint operator $A = \int \lambda E(d\lambda)$ exists such that $A\eta\mathscr{A}$ and $\dim_{\mathscr{A}}(E(\lambda)) = f(\lambda)$, $\lambda \in \mathbb{R}$.*

Proof. Cf. the proof of Lemma (2.2).

4.2. Absolute continuity as a relation between functions

Our aim in this section is to ascertain what freedom we have with respect to the choice of the function f, given that the maximal Hellinger type of the operator $A = \int \lambda E(d\lambda)$ (with $\dim_{\mathscr{A}}(E(\lambda)) = f(\lambda)$, $\lambda \in \mathbb{R}$) should be equal to μ'_B (see the introductory paragraphs of this section). We shall solve that problem in terms of a concept of absolute continuity between functions, applied to the functions f and f_μ ($\mu \in \mu'_B$). We start by defining the concept:

(4.3) *Definition.* Let f and g be two real-valued functions on $\mathbb{R}$. We shall say that g is *absolutely continuous with respect to f on $\mathbb{R}$* if, given $\epsilon > 0$, there is a $\delta > 0$ such that for any finite sequence $([a_i, b_i])$ of disjoint intervals of $\mathbb{R}$ with $\Sigma_i |f(b_i) - f(a_i)| < \delta$, one has $\sum_i |g(b_i) - g(a_i)| < \epsilon$. Similarly if '$\mathbb{R}$' is replaced by 'a closed interval of $\mathbb{R}$'.

This is an obvious generalization of the (Vitali) concept of an absolutely continuous function (see [3], p. 192). We have the following lemma, easily obtained on the model of [3], pp. 192–93:

(4.4) Lemma. *Let f and g be two real-valued functions on $\mathbb{R}$ which are increasing (not necessarily strictly so), continuous on the right, and finite everywhere. Let μ_f and μ_g be the corresponding Borel–Stieltjes measures. Then μ_g is absolutely continuous with respect to μ_f if and only if g is absolutely continuous with respect to f.*

4.3. Affiliation of self-adjoint operators to factors of type II in a separable Hilbert space: another formulation

(4.5) Theorem. *Let A be the self-adjoint operator of Lemma (2.2) (in the case II_1) or Lemma (4.2) (in the case II_∞), and let B be a self-adjoint operator in $\mathfrak{H}$ of spectral type*

$(\mu'_B, \aleph_0)$. *Let* μ *be a member of* μ'_B. *A and B are unitarily equivalent if and only if* f *and* f_μ *are absolutely continuous with respect to each other on every closed interval of* $\mathbb{R}$.

Proof. We shall use the notation of Subsection 4.1. For simplicity we shall assume that f is strictly increasing. Moreover, we shall carry out the proof only for the case that f is unbounded. Let us write

$$J_i =_{\text{def}} [f^{-1}(i-1), f^{-1}(i)] \quad (f^{-1}(0) \text{ meaning } -\infty).$$

We let n be any integer > 0. If $F'(\cdot)$ is the spectral family of the operator of Lemma (4.1), then the spectral family $E(\cdot)$ of A is given by

$$E(\lambda) = F'(f(\lambda)) = \begin{cases} F'^{(1)}(\lambda), & \lambda \in J_1, \\ \sum_{i=1}^{n-1} E_i + F'^{(n)}(f(\lambda) - n + 1), & \lambda \in J_n, n = 2, 3, \ldots \end{cases}$$

Consider

$$A_{E_n} = \left(\int \lambda E(d\lambda)\right)_{E_n} = \int \lambda E_{E_n}(d\lambda) = \int \lambda F'^{(n)}_{E_n}(f(d\lambda) - n + 1)$$

$$= \int_{J_n} \lambda F'^{(n)}_{E_n}(f(d\lambda) - n + 1 = \int \lambda F^{(n)}_{E_n}(d\lambda),$$

where

$$F^{(n)}_{E_n}(\lambda) =_{\text{def}} F'^{(n)}_{E_n}(f(\lambda) - n + 1), \quad \lambda \in J_n.$$

Clearly,

$$\dim_{\mathscr{A}_n}(F^{(n)}_{E_n}(\lambda)) = \dim_{\mathscr{A}_n}(F'^{(n)}_{E_n}(f(\lambda) - n + 1) = f(\lambda) - n + 1, \quad \lambda \in J_n,$$

where $\dim_{\mathscr{A}_n}(F^{(n)}_{E_n}(\lambda))$ increases from 0 to 1 as λ runs through J_n. The reader will understand that $\dim_{\mathscr{A}_n}(F^{(n)}_{E_n}(\cdot))$ determines the maximal Hellinger type of A_{E_n} (cf. the proofs of Lemma (2.2)$'$ and Theorem (2.3)).

Consider next the restriction $f^{(n)}_\mu$ of f_μ to J_n. The function $f^{(n)}_\mu - f_\mu(f^{-1}(n-1))$ determines in an obvious way the maximal Hellinger type of $B_{F(J_n)}$.

With the results of Sec. 2 in mind we realize that A_{E_n} and $B_{F(J_n)}$ are unitarily equivalent if and only if the functions $\dim_{\mathscr{A}_n}(F^{(n)}_{E_n}(\cdot))$ and $f^{(n)}_\mu$ are absolutely continuous with respect to each other. Only a trivial step now remains in proving this theorem, and we omit it.

We have assumed everywhere spectral families to be defined on all of $\mathbb{R}$; accordingly, in Lemmas (2.2) and (4.2) we have brought in functions f defined on all of $\mathbb{R}$. Sometimes that is unpractical; the reader will understand how to modify the wording of these lemmas in such cases.

5. An Example

We give here an example of affiliation of a self-adjoint operator A in a separable Hilbert space $\mathfrak{H}$ to a factor of type II_∞ in $\mathfrak{H}$.

Let $\mathfrak{H} = L_2(\mathbb{R}^3)$, and let A be the maximal multiplication operator defined by

$$(Af)(\lambda) =_{\text{def}} |\lambda|^2 f(\lambda)$$

(A is (proportional to) the Fourier transform of the Hamiltonian of a free particle; see [11], Ch. V, Sec. 5, N° 2). Due to the spherical symmetry of the situation we introduce polar coordinates (r, θ, φ): by A we shall mean the maximal multiplication operator in $\mathfrak{H}$ defined by

$$(Af)(r, \theta, \varphi) = r^2 f(r, \theta, \varphi).$$

Let us write $\mathfrak{H}$ as

$$L_2(\mathbb{R}^+, r^2\, dr) \otimes L_2([0, \pi] \times [0, 2\pi], d\Omega),$$

or, for short,

$$\mathfrak{H} = \mathfrak{H}[r] \otimes \mathfrak{H}[\theta, \varphi].$$

Clearly,

$$A = A_{\mathfrak{H}[r]} \otimes \mathbb{1}_{\mathfrak{H}[\theta, \varphi]}.$$

With

$$A_{\mathfrak{H}[r]} = \int_0^\infty \lambda^2\, F(d\lambda), \quad (F(\lambda)\, u)\, (x) = \chi_{[0, \lambda]}(x)\, u(x),$$

we have

$$A = \int_0^\infty \lambda^2 (F(d\lambda) \otimes \mathbb{1}_{\mathfrak{H}[\theta, \varphi]}).$$

We shall show that the multiplicity of the maximal Hellinger type μ'_A of A is $\aleph_0$. To that end we write $\mathfrak{H}[\theta, \varphi]$ as

$$\mathfrak{H}[\theta, \varphi] = \sum_{i=1}^{\infty}{}^{\oplus}\, \mathfrak{H}[\theta, \varphi]_i,$$

where the $\mathfrak{H}[\theta, \varphi]_i$ are all one-dimensional; we shall take them to be determined by the members of a basis (e_i) of $\mathfrak{H}[\theta, \varphi]$. Then A can be written as

$$A = \int_0^\infty \lambda^2 (F(d\lambda) \otimes \sum_{i=1}^{\infty}{}^{\oplus}\, \mathbb{1}_{\mathfrak{H}[\theta, \varphi]_i})$$

$$= \sum_{i=1}^{\infty}{}^{\oplus} \int_0^\infty \lambda^2 (F(d\lambda) \otimes \mathbb{1}_{\mathfrak{H}[\theta, \varphi]_i}).$$

We write

$$A_i =_{\text{def}} \int_0^\infty \lambda^2 (F(d\lambda) \otimes \mathbb{1}_{\mathfrak{H}[\theta, \varphi]_i}).$$

First, it is trivial that the A_i are all of the same maximal Hellinger type, and that the latter is equal to μ'_A.

Secondly, for each i, A_i is a cyclic operator: that we shall prove by exhibiting a member u_i of $\mathfrak{H}[r] \otimes \mathfrak{H}[\theta, \varphi]_i$, the subspace in which A_i acts, such that

$\mathfrak{X}_{u_i}^{\mathscr{B}_i} = \mathfrak{H}[r] \otimes \mathfrak{H}[\theta, \varphi]_i$; here, of course, $\mathscr{B}_i = \{A_i\}''$. We consider

$$u_i =_{\text{def}} v \otimes e_i,$$

where

$$v(r) =_{\text{def}} \alpha_i > 0, \quad r \in [i-1, i],$$

and

$$(u_i|u_i) = (v \otimes e_i|v \otimes e_i) = (v|v) \cdot 1 = \int_0^\infty v(r)^2 r^2 \, dr < \infty.$$

That u_i fulfils what is required of it comes from the fact that the characteristic functions form a dense set in $\mathfrak{H}(r) = L_2(\mathbb{R}^+, r^2 dr)$ (see [10], Sec. 69). In fact we can take for v any function on $\mathbb{R}^+$ which differs from 0 almost everywhere.

Thirdly, the A_i are seen to be pairwise orthogonal parts of A.

It follows that (A_i) is an orthogonal family of Hellinger type μ'_A (with respect to A). Clearly, it is a maximal family, and we have $\mathfrak{m}(\mu'_A, A) = \aleph_0$.

We can now invoke Theorem (4.5). Is there any obvious choice of the function f? Clearly not *a priori*; the choice has to be dictated by physical considerations. We shall not go into these matters here, but merely make the choice

$$f(\lambda) = \lambda^3$$

(in the context of the elementary statistical mechanics of the ideal gas it would be possible to relate $df(\lambda)$ to the number of one-particle states for which the norm of the momentum has a value in the interval $d\lambda$ around λ).

We now have to see whether f and f_μ, with $\mu \in \mu'_A$, are absolutely continuous with respect to each other. (The relation of absolute continuity is a transitive one; hence it is immaterial which $\mu \in \mu'_A$ we answer the question for.) The reader will understand from the proof of Theorem (4.5) that if we can exhibit a sequence of pairwise disjoint closed intervals J_i of $\mathbb{R}$, with $\bigcup_{i=1}^{\infty} J_i = \mathbb{R}^+$, such that the restrictions $f^{(i)}$ and $f_\mu^{(i)}$ of f and f_μ, respectively, to each J_i are absolutely continuous with respect to each other, then our question would thereby be answered in the affirmative. To that we now turn.

We let the $J_i = [a_{i-1}, a_i]$, $i = 1, 2, \ldots$, be intervals of $\mathbb{R}^+$ with $\bigcup_{i=1}^{\infty} J_i = \mathbb{R}^+$. Clearly, if for each i the derivatives of $f^{(i)}$ and $f_\mu^{(i)}$ were proportional functions, then $f^{(i)}$ and $f_\mu^{(i)}$ would be absolutely continuous with respect to each other. Tentatively, therefore, we set down

$$f_\mu^{(i)}(\lambda) = \beta_i \lambda^3 + \gamma_i,$$

where β_i and γ_i are (real) constants. This entails

$$\beta_i a_{i-1}^3 + \gamma_i = b_{i-1},$$

$$\beta_i a_i + \gamma_i = b_i,$$

where we further require that $b_{i-1} < b_i$ and that $b_i \xrightarrow[i\to\infty]{} 1$. Let us choose $a_i =_{\text{def}} \sqrt[3]{i}$, $b_i =_{\text{def}} i/(i+1)$. It follows that $\beta_i = 1/i(i+1)$, $\gamma_i = (i-1)/(i+1)$.

Let us now show that there exists a function $u(r, \theta, \varphi) = v(r) \cdot 1$ compatible with f_μ:

$$f_\mu(\lambda) = \mu([0, \lambda]) = (F(\lambda)\, u|u) = \int_0^\lambda |v(r)|^2 r^2\, dr \cdot 4\pi.$$

It is not difficult to see that $v^{(i)} =_{\text{def}} v|J_i$ has to be constant. Thus, for $\lambda \in J_i$,

$$f_\mu(\lambda) = \frac{1}{i(i+1)}\lambda^3 + \frac{i-1}{i+1} = f_\mu(\sqrt[3]{i-1}) + \int_{\sqrt[3]{i-1}}^{\lambda} |v^{(i)}|^2 r^2\, dr \cdot 4\pi;$$

$$\frac{1}{i(i+1)}\lambda^3 + \frac{i-1}{i+1} = \frac{i-1}{i} + 4\pi|v^{(i)}|^2\left(\frac{\lambda^3}{3} - \frac{i-1}{3}\right);$$

$$\frac{1}{i(i+1)}[\lambda^3 - (i-1)] = \frac{4\pi}{3}|v^{(i)}|^2[\lambda^3 - (i-1)];$$

$$|v^{(i)}|^2 = \frac{3}{4\pi i(i+1)}.$$

For definiteness we let $v^{(i)} =_{\text{def}} \sqrt{[3/4\pi i(i+1)]}$. Evidently, v is different from 0 almost everywhere. This ensures that $\mathfrak{H}(v) = \mathfrak{H}[r]$. From [2], Ch. X, Sec. 1.4, we learn that for a *cyclic* operator T, $\mathfrak{H}(v_1) = \mathfrak{H}(v_2)$ if and only if $\mu_{v_1,T}' = \mu_{v_2,T}'$. Now $A_{\mathfrak{H}[r]}$ is a cyclic operator, so $\mu_{v,A_{\mathfrak{H}[r]}}' = \mu_{A_{\mathfrak{H}[r]}}'$. But $\mu_{v,A_{\mathfrak{H}[r]}}' = \mu_{u,A}'$, and $\mu_{A_{\mathfrak{H}[r]}}' = \mu_A'$, so $\mu_{u,A}' = \mu_A'$, and we are through: $\mu \in \mu_A'$.

Let us choose any factor $\mathscr{A}_1$ of type II_∞ in $\mathfrak{H}$. Affiliated to $\mathscr{A}_1$ there is a self-adjoint operator $A_1 = \int \lambda E_1(d\lambda)$, unitarily equivalent to A, characterized by $\dim_{\mathscr{A}}(E_1(\lambda)) = f(\lambda)$, $\lambda \in \mathbb{R}^+$. We have $A = UA_1U^*$ for some unitary operator U. It follows that $\mathscr{A} =_{\text{def}} U\mathscr{A}_1U^*$ is a factor of type II_∞ with $A\eta\mathscr{A}$. The spectral family of A is $E(\cdot) =_{\text{def}} UE_1(\cdot)U^*$. From [7], Lemma 8.5.1, we see that we are justified in writing $\dim_{\mathscr{A}}(E(\lambda)) = \dim_{\mathscr{A}_1}(E_1(\lambda))$, $\lambda \in \mathbb{R}^+$. (With our choice of the function f, we have, e.g., $e^{-A} \in \mathscr{C}_1(\mathscr{A})$.)

6. Concluding Remarks

We have here laid the groundwork for a certain line of application of factors of type II in quantum mechanics. There are unsatisfactory points connected with our program, such as the existence of (uncountably many) non-isomorphic factors of type II_1 and of type II_∞. We should not know how to justify a particular choice. On the other hand, that problem may not be a serious one: after all, ours is a technical problem, not a fundamental theoretical one, and a pragmatic approach may be admissible; e.g., we might concentrate on hyperfinite factors, with their convenient properties (see, e.g., [4], Ch. II, Sec. 6; [6], Ch. III, Sec. 7; and [12]).

The physical interpretation of the trace function on factors of type II presents some difficulties. It seems to be a question of exploiting the following circumstance: According to Murray and von Neumann the trace function on II_1 factors generalizes those on I_n factors (n finite); similarly, the trace function on II_∞ factors generalizes the one on I_∞ factors: if $\sigma(A)$ is the spectrum of an operator A affiliated to a factor of type II, then it makes sense to talk of 'eigenvalue n° λ of A', for all $\lambda \in \sigma(A)$ (cf. [7], the Introduction and Ch. XV).

Apart from the program sketched in the Introduction, another program is suggested by recent work on translation invariance in the one-electron theory of solids; it is found that the algebra of constants of motion is a factor of type *II* in some interesting cases [13].

We expect to come back to applications in a sequel to this study.

Acknowledgments

It is a pleasure to acknowledge the hospitality of Professor J. M. Jauch at the Department of Theoretical Physics, Geneva. I would equally like to thank Professor C. Piron for suggesting the topic of this study, and Drs. J. P. Antoine, F. Jenč, and K. B. Sinha for valuable comments.

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