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The Connections between the Schrödinger Group and the Conformal Group

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Abstract. The Schrödinger group is generalized to the maximal set of coordinate transformations which leave invariant the Schrödinger equation up to the mass. These generalized Schrödinger transformations do not form a group but contain the Schrödinger group as a subset. It is shown that the non-relativistic limit of the conformal group, which is interpreted as the maximal invariance group up to the mass of the massive Klein–Gordon equation, is a subset of the generalized Schrödinger transformations which, when modified by additional transformations, contains the Schrödinger group.

1. Introduction

The Schrödinger group [1], the maximal kinematical invariance group of the free Schrödinger equation, is a 12-parameter group containing the Galilei group, the dilations and a group of projective transformations. The conformal group [2], originally introduced as invariance group of the Maxwell equations, may also be considered as the maximal kinematical invariance group of the massless Klein–Gordon equation. It consists of the Poincaré group, the dilations and the 4-parameter group of special conformal transformations. The nature of these two groups, and the way they both arise from simple wave equations, naturally lead to the question whether there are any connections between them and, in particular, whether the Schrödinger group is in some sense the non-relativistic limit of the conformal group. The present paper is devoted to an analysis of this question.

The non-relativistic limit of the conformal group has also been studied in a recent paper [3] where it is shown that twelve of the parameters of the conformal group, after some modification, indeed lead to the Schrödinger group while three parameters are not connected with coordinate transformations leaving invariant the Schrödinger equation. It is in the treatment and interpretation of these three parameters that we deviate from [3].

The Schrödinger equation is the non-relativistic limit not of the massless but of the massive Klein–Gordon equation, hence, to compare the two groups, we first have to interpret the conformal group in terms of the massive Klein–Gordon equation.

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This is done by showing that the conformal transformations send the massive Klein–Gordon equation into a similar equation with a possibly different mass. Since we eventually want to analyse the limit of conformal transformations, we may have to enlarge the framework of the Schrödinger group and consider generalized Schrödinger transformations, i.e. transformations which change the mass of the Schrödinger equation.

In Section 2 we determine the full set of generalized Schrödinger transformations which leave invariant the Schrödinger equation up to the mass, and we point out that the set of these transformations does not form a group. In Section 3 the conformal group is interpreted in the context of the massive Klein–Gordon equation and it is shown that, contrary to the Schrödinger group, the conformal group remains maximal, i.e. any transformation leaving invariant the Klein–Gordon equation up to the mass is already contained in the conformal group. Because it is easier to discuss the non-relativistic limit for the generators of transformations rather than for the transformations themselves, we derive the invariance condition for the generators in both cases.

The non-relativistic limit of the conformal transformations is discussed in Section 4. The limit is not simply a Wigner–Inönü contraction [4]: it is taken in such a way that the conformal invariance condition goes into the Schrödinger invariance condition. The result is a set of fifteen generators of generalized Schrödinger transformations. Finally, it is shown that for two of the five types of transformations which change the mass, this change can be compensated by additional generalized Schrödinger transformations, and we thus obtain the Schrödinger group. The remaining three parameters are still generalized Schrödinger transformations but they definitely change the mass and they cannot be combined with the Schrödinger group to form a group.

2. The Generalized Schrödinger Transformations

The Schrödinger group is the largest group of coordinate transformations leaving invariant the Schrödinger equation $(2im\,\partial_t + \Delta)\,\psi(t,\mathbf{x}) = 0$. In addition to these transformations, there may exist coordinate transformations which do not leave invariant the Schrödinger equation but send it into a similar equation with different mass. For reasons of generality, we allow the mass parameter to be a (real) function of $(t,\mathbf{x})$ and we write $\mu(t,\mathbf{x})$ instead of m. Thus a *generalized Schrödinger transformation* is defined as a coordinate transformation

$$g\colon (t,\mathbf{x}) \to g(t,\mathbf{x}), \tag{2.1}$$

which has the property that there exist *companion functions* $f_g(t,\mathbf{x})$ and $F_g(t,\mathbf{x})$ such that the mappings

$$\psi \to T_g\psi, \quad (T_g\psi)\,(t,\mathbf{x}) = f_g[g^{-1}(t,\mathbf{x})]\,\psi[g^{-1}(t,\mathbf{x})], \tag{2.2}$$

$$\mu \to Q_g\mu, \quad (Q_g\mu)\,(t,\mathbf{x}) = F_g[g^{-1}(t,\mathbf{x})]\,\mu[g^{-1}(t,\mathbf{x})] \tag{2.3}$$

send any solution ψ of the equation

$$(2i\mu(t,\mathbf{x})\,\partial_t + \Delta)\,\psi(t,\mathbf{x}) = 0 \tag{2.4}$$

into a solution $T_g\psi$ of the equation

$$(2i(Q_g\mu)(t,\mathbf{x})\,\partial_t + \Delta)(T_g\psi)(t,\mathbf{x}) = 0. \tag{2.5}$$

There may be coordinate transformations which only exist for a certain set of mass functions $\mu(t,\mathbf{x})$ and we restrict ourselves to those transformations which exist for $\mu = $ constant.

The problem of finding all generalized Schrödinger transformations is solved in the appendix. The result is that there are four types of transformations:

I) *Any transformation of time alone*:

$$g(t,\mathbf{x}) = (t',\mathbf{x}), \quad f_g = \text{cst.}, \quad F_g(t) = dt'/dt. \tag{2.6}$$

II) *Rotations, translations and dilations*:

$$g(t,\mathbf{x}) = [\rho^2(t+b), \ \rho(R\mathbf{x}+\mathbf{a})], \quad f_g = \text{cst.}, \quad F_g = 1, \tag{2.7}$$

where $\rho \in \mathbb{R}$, $b \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^3$, $R \in O(3)$.

III) *Special conformal transformations of* $\mathbb{R}^3$:

$$g(t,\mathbf{x}) = \left(t, \frac{\mathbf{x}+\mathbf{c}\mathbf{x}^2}{\sigma}\right), \quad f_g(\mathbf{x}) = \sigma^{1/2}, \quad F_g(\mathbf{x}) = \sigma^2, \tag{2.8}$$

$\sigma \equiv 1 + 2\mathbf{c}\cdot\mathbf{x} + \mathbf{c}^2\mathbf{x}^2, \quad \mathbf{c} \in \mathbb{R}^3.$

IV) *Boosts and projective transformations for* $\mu = cst.$:

$$g(t,\mathbf{x}) = \left(\frac{t}{1+\alpha t}, \frac{\mathbf{x}+\mathbf{v}t}{1+\alpha t}\right), \quad F_g = 1, \tag{2.9}$$

$$f_g(t,\mathbf{x}) = (1+\alpha t)^{3/2}\exp[-\tfrac{1}{2}i\mu(1+\alpha t)^{-1}(\alpha\mathbf{x}^2 - 2\mathbf{v}\cdot\mathbf{x} - \mathbf{v}^2 t)],$$

where $\alpha \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^3$.

Note that there are no conditions on $\mu(t,\mathbf{x})$ for the types I, II, III, but that the transformations of type IV only exist for constant mass. This has the important consequence that the set of all transformations I–IV does not form a group: e.g. boosts and special conformal transformations (III) cannot be combined into a group because after a special conformal transformation the mass function $(Q_g\mu)(t,\mathbf{x})$ is no longer constant. For constant $\mu = m$ the transformations of type II and IV combine to give the Schrödinger group and it can be seen that this is the largest set of transformations which do not affect the mass, i.e. which have $F_g = 1$.

Turning to infinitesimal transformations, we now want to derive the condition that a first-order differential operator $G(t,\mathbf{x})$ generates a generalized Schrödinger transformation g. Let the mappings T_g and Q_g be given infinitesimally by

$$T_g\psi = (1 + i\epsilon G)\,\psi, \quad Q_g\mu = (1 + i\epsilon\hat{G})\,\mu. \tag{2.10}$$

From (2.4), (2.5) we obtain the condition

$$(2i\mu\,\partial_t + \Delta)\,G\psi + 2i(\hat{G}\mu)\,\partial_t\,\psi = 0. \tag{2.11}$$

Subtracting $G(2i\mu\,\partial_t + \Delta)\,\psi = 0$ and using the fact that the only second-order annihilator of an arbitrary solution ψ of (2.4) is a multiple of $2i\mu\,\partial_t + \Delta$ we obtain the ψ-independent *invariance condition*

$$[2i\mu\,\partial_t + \Delta, G] + 2i(\hat{G}\mu)\,\partial_t = i\gamma(2i\mu\,\partial_t + \Delta), \tag{2.12}$$

where $(\hat{G}\mu)$ is the result of applying the operator $\hat{G}$ to the function μ and where $\gamma = \gamma(t, \mathbf{x})$ may be any function.

It is easy to check that all of the previously defined generalized Schrödinger transformations satisfy (2.12); the corresponding generators G, $\hat{G}$ and the functions γ are given as follows:

I)		$G = i\tau(t)\,\partial_t$	$\hat{G} = G - i\dot{\tau}(t)$	$\gamma = 0$
II)	R:	$\mathbf{J} = -i\mathbf{x} \times \nabla$	$\hat{\mathbf{J}} = \mathbf{J}$	$\gamma = 0$
	b:	$P_0 = i\,\partial_t$	$\hat{P}_0 = P_0$	$\gamma = 0$
	$\mathbf{a}$:	$\mathbf{P} = -i\nabla$	$\hat{\mathbf{P}} = \mathbf{P}$	$\gamma = 0$
	$\rho \equiv e^s$:	$S = i(2t\,\partial_t + \mathbf{x}\cdot\nabla + 3/2)$	$\hat{S} = S - (3/2)\,i$	$\gamma = 2$
III)	$\mathbf{c}$:	$\mathbf{C} = -i(2\mathbf{x}\mathbf{x}\cdot\nabla - \mathbf{x}^2\nabla + \mathbf{x})$	$\hat{\mathbf{C}} = \mathbf{C} - 3i\mathbf{x}$	$\gamma = -4\mathbf{x}$
IV)	α:	$A = -i(t^2\,\partial_t + t\mathbf{x}\cdot\nabla + \frac{3}{2}\,t) - \frac{1}{2}\mu\mathbf{x}^2$	$\hat{A} = A + \frac{3}{2}\,it + \frac{1}{2}\mu\mathbf{x}^2$	$\gamma = -2t$
	$\mathbf{v}$:	$\mathbf{K} = it\nabla + \mu\mathbf{x}$	$\hat{\mathbf{K}} = \mathbf{K} - \mu\mathbf{x}$	$\gamma = 0$

(2.13)

where $\tau(t)$ is an arbitrary function. Note that the generators in (2.13) do not form a Lie algebra: a simple counter-example is provided by $[\mathbf{K}, i\tau(t)\,\partial_t] = \tau(t)\nabla$ for $\dot{\tau} \neq 0$. This is the Lie algebra analogon of the fact that the generalized Schrödinger transformations do not form a group.

3. The Massive Interpretation of the Conformal Group

The conformal group is the set of coordinate transformations

$$g\colon\; x^\mu \to g(x)^\mu = \rho\left(\Lambda^\mu_\nu\,\frac{x^\nu + c^\nu x^2}{u(x)} + a^\mu\right), \qquad u(x) \equiv 1 + 2c\cdot x + c^2 x^2, \tag{3.1}$$

where $\rho \in \mathbb{R}$, a^μ, $c^\mu \in \mathbb{R}^4$, $\Lambda^\mu_\nu \in SO(3,\,1)$, $c\cdot x = c^0 x^0 - \mathbf{c}\cdot\mathbf{x}$. Together with the transformations

$$\varphi(x) \to (T_g\,\varphi)\,(x) = f_g[g^{-1}(x)]\,\varphi[g^{-1}(x)], \quad f_g(x) = \frac{1}{\rho}\,u(x), \tag{3.2}$$

they form the maximal kinematical invariance group of the massless Klein–Gordon equation, i.e.

$$\Box\varphi = 0 \Rightarrow \Box(T_g\varphi) = 0. \tag{3.3}$$

We now wish to interpret this group in the context of the massive Klein–Gordon equation $(\Box + c^2\mu^2(x))\varphi(x) = 0$ $(\hbar = 1)$, where, for generality, we have allowed the mass μ to depend on x. Transforming the mass function in the same way as the solutions φ in (3.2), one can show that the transformations (3.1) send the massive Klein–Gordon equation into another Klein–Gordon equation with a different mass, i.e.

$$(\Box + c^2\mu^2(x))\varphi(x) = 0 \Rightarrow (\Box + c^2(T_g\mu)^2(x))(T_g\varphi)(x) = 0. \tag{3.4}$$

The proof is simple if one uses the relation

$$\Box' = \frac{1}{\rho^2}u^2\left[\Box - \frac{4}{u}(c^\mu + c^2x^\mu)\,\partial_\mu\right] \tag{3.5}$$

for $x' \equiv g(x)$. (3.4) holds independent of the special form of the mass function $\mu(x)$, hence, unlike the generalized Schrödinger transformations of Section 2, the conformal transformations do form a group. We refer to this group as the *massive conformal group* but it should be noted that it is the same group as the conformal group but looked upon from a different interpretation.

Next we want to find the conditions satisfied by the generators of the massive conformal group. With the notation

$$T_g = 1 + i\epsilon G \tag{3.6}$$

for infinitesimal transformations we obtain from (3.4), using the same argumentation as in Section 2, the φ-independent *invariance condition*

$$[\Box + c^2\mu^2, G] + 2c^2\mu(G\mu) = i\gamma(\Box + c^2\mu^2), \tag{3.7}$$

where $\gamma = \gamma(x)$ is an arbitrary function. The generators of the massive conformal group, together with the functions γ, are given by

$$\begin{array}{lll} a^\mu: & P_\mu = i\,\partial_\mu & \gamma = 0 \\ \Lambda^\mu_\nu: & M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) & \gamma = 0 \\ \rho \equiv e^s: & S = i(x\cdot\partial + 1) & \gamma = 2 \\ c^\mu: & C_\mu = -i(2x_\mu x\cdot\partial - x^2\partial_\mu + 2x_\mu) & \gamma = -4x_\mu. \end{array} \tag{3.8}$$

Finally, we want to show that the massive conformal group already exhausts all coordinate transformations which send the massive Klein–Gordon equation into a similar equation with different mass. The condition for such a transformation is (3.4) except that we now replace $T_g\mu$ by the more general

$$(Q_g\mu)(x) = F_g[g^{-1}(x)]\,\mu[g^{-1}(x)], \tag{3.9}$$

where F_g need not be equal to f_g as in (3.4). The invariance condition similar to (3.7) is then

$$[\Box + c^2 \mu^2, G] + 2c^2 \mu(\hat{G}\mu) = i\gamma(\Box + c^2 \mu^2), \tag{3.10}$$

where $Q_g = 1 + i \in \hat{G}$. We now show that (3.10) is equivalent to the conditions

$$[\Box, G] = i\gamma\Box, \quad \hat{G}_0 = \tfrac{1}{2}i\gamma, \tag{3.11}$$

where $\hat{G}_0$ is the non-derivative part of $\hat{G}$. Since the first of the equations (3.11) is the condition that G generates a symmetry of the massless Klein–Gordon equation and hence belongs to the Lie algebra of the conformal group, we have then proved the assertion that the massive conformal group is *maximal*.

To prove the equivalence of (3.10) and (3.11) we first write (3.10) in the form

$$[\Box, G] + 2c^2 \mu^2 \hat{G}_0 = i\gamma(\Box + c^2 \mu^2), \tag{3.12}$$

where we used the fact that G is a first-order differential operator and that the derivative parts of G and $\hat{G}$ coincide. G and $\hat{G}$ are of the form

$$G = i(a^\mu(x)\, \partial_\mu + b(x)), \quad \hat{G} = i(a^\mu(x)\, \partial_\mu + e(x)), \tag{3.13}$$

and the condition that (3.11) and (3.12) be equivalent is clearly $\gamma = 2e$. Now, inserting (3.13) into (3.12), we obtain the independent conditions

$$a_{\mu,\,\nu} + a_{\nu,\,\mu} = g_{\mu\nu}\gamma: \quad 2b_{,\,\mu} + \Box a_\mu = 0; \quad \Box b + 2c^2 \mu^2 e = c^2 \mu^2 \gamma. \tag{3.14}$$

We first prove $\Box\gamma = 0$; From (3.14) we obtain

$$a_{\mu,\,\nu\rho} + a_{\nu,\,\mu\rho} = g_{\mu\nu}\gamma_{,\,\rho}.$$

Subtracting this equation from the two cyclically permuted equations we have

$$a_{\rho,\,\mu\nu} = \tfrac{1}{2}(g_{\rho\mu}\gamma_{,\,\nu} + g_{\rho\nu}\gamma_{,\,\mu} - g_{\mu\nu}\gamma_{,\,\rho}),$$

and inserting this into the relation $a_{\rho,\,\mu\nu}{}^{\mu} = a_{\rho,\,\mu}{}^{\mu}{}_{\nu}$ we obtain

$$g_{\nu\rho}\Box\gamma = -2\gamma_{,\,\nu\rho}$$

and hence $\Box\gamma = 0$. It then follows from (3.14) that

$$\Box b = -\tfrac{1}{2}\Box a_{\mu,}{}^{\mu} = -\Box\gamma = 0,$$

hence $2e = \gamma$, and we have proved that (3.12) implies (3.11).

4. The Non-relativistic Limit of the Conformal Group

As we have seen, the main difference between the transformations of Sections 2 and 3 is that the transformations of the massive conformal group form a group and the generalized Schrödinger transformations do not form a group. In the latter case, this is due to the appearance of the mass in the generators **K** and A which, in turn, is responsible for the mass dependence of the corresponding companion functions f_g. In the conformal group, the generators and the companion functions are independent of the mass. The ultimate reason for this behaviour is the fact that in the non-relativistic case we are dealing with a projective representation of the Galilei group and the projectivity is governed by the mass.

In this section we want to derive the non-relativistic limit of the conformal transformations. Since we wish the boosts to be included in the non-relativistic transformations, and since boosts are only permissible for constant mass, we confine ourselves to constant mass throughout this section, writing m instead of μ.

In going to the limit $c \to \infty$ we have to put

$$x^0 = ct, \quad \partial_0 = c^{-1}\,\partial_t - imc. \tag{4.1}$$

The second of these equations is motivated by the fact that the relativistic energy contains the rest energy mc^2 which is not counted in the non-relativistic energy. From (4.1) we obtain

$$\square + m^2 c^2 = -(2im\,\partial_t + \Delta) + c^{-2}\,\partial_t^2, \tag{4.2}$$

thus the massive Klein–Gordon equation goes into the Schrödinger equation. Before we take the limit of the generators (3.8) we are free to multiply them by powers of c; this merely amounts to a redefinition of the corresponding group parameters and is well known under the name of contraction [4]. However, it should be noted that what we do here is not equivalent to a Wigner–Inönü contraction and it cannot be equivalent because a contraction of the conformal group would again be a group while the set of generalized Schrödinger transformations is not a group. We are not so much interested in the limit of the generators themselves, but in the *limit of the invariance condition* (3.7), and it is in this sense that the term 'non-relativistic limit' is used. The hope is that the power of c a generator is multiplied with can be chosen in such a way that not only does the limit of (3.7) exist but that it is of the form of the invariance condition (2.12). If this is the case, we may then extract from this limit a generator of a generalized Schrödinger transformation.

The limiting procedure for the Poincaré group is well known [5] and the resulting Galilei generators are the quantities **J**, P_0, **P**, **K** of (2.13) which we know to satisfy (2.12). The remaining five generators of the conformal group present a more interesting case and we now treat them in turn.

S: The generator

$$S = i(t\,\partial_t + \mathbf{x}\cdot\nabla + 3/2) + mc^2 t - \tfrac{1}{2}i$$

satisfies (3.7) in the form

$$[\square + m^2 c^2, S] + 2im^2 c^2 = 2i(\square + m^2 c^2).$$

Hence with (4.2) we obtain

$$[2im\,\partial_t + \Delta, i(t\,\partial_t + \mathbf{x}\cdot\nabla + 3/2)] - 2m\,\partial_t = 2i(2im\,\partial_t + \Delta) + o(c^{-2}).$$

Note that the terms $\sim c^2$ have cancelled each other. In the limit $c \to \infty$ we now obtain equation (2.12), satisfied by the generator

$$S' = i(t\,\partial_t + \mathbf{x}\cdot\nabla + 3/2), \quad \hat{S}' = i, \quad \gamma = 2, \tag{4.3}$$

where the derivative part of $\hat{S}'$ is dropped because the mass is assumed to be constant. The constant term in S' is not determined by the limit procedure and has been chosen for convenience. Note that, if trivial non-derivative generators are excluded, a non-relativistic limit of the generator S itself does not exist.

C^0: We have

$$C^0 = -ic(t^2\,\partial_t + 2t\mathbf{x}\cdot\nabla + 3t - im\mathbf{x}^2) + ict - mc^3 t^2 - ic^{-1}\mathbf{x}^2\,\partial_t$$

and

$$[\Box + m^2 c^2, C^0] - 4ictm^2 c^2 = -4ict(\Box + m^2 c^2).$$

With (4.2) we obtain the equation

$$c[2im\,\partial_t + \Delta, -i(t^2\,\partial_t + 2t\mathbf{x}\cdot\nabla + 3t - im\mathbf{x}^2)] + 4mct\,\partial_t = -4ict(2im\,\partial_t + \Delta) + o(c^{-1}).$$

Hence to define a limit $c \to \infty$ we have to multiply C^0 by c^{-1} and then again the invariance condition (2.12) is satisfied for the generator

$$A' = -i(t^2\,\partial_t + 2t\mathbf{x}\cdot\nabla + 3t - im\mathbf{x}^2), \quad \hat{A}' = -2it, \quad \gamma = -4t. \tag{4.4}$$

$\mathbf{C}$: This is the most interesting case and it is here where we deviate from the procedure of [3]. Consider the generators

$$\mathbf{C} = (C^1, C^2, C^3) = -i(2\mathbf{x}\mathbf{x}\cdot\nabla - \mathbf{x}^2\nabla + \mathbf{x}) - i\mathbf{x}(2t\,\partial_t + 1) + c^2(-it^2\nabla - 2m\mathbf{x}t). \tag{4.5}$$

Unlike the cases of S and C^0 there exists a non-trivial limit of the generators $\mathbf{C}$, namely the limit $c^{-2}\mathbf{C} \to -it^2\nabla - 2m\mathbf{x}t$ and it is these operators which are chosen in [3]; however, it is easy to check that they do not satisfy (2.12). The generators $\mathbf{C}$ satisfy (3.7) in the form

$$[\Box + m^2 c^2, \mathbf{C}] - 4i\mathbf{x}m^2 c^2 = 4i\mathbf{x}(\Box + m^2 c^2), \tag{4.6}$$

and using (4.2) the terms $\sim c^2$ cancel and we obtain

$$[2im\,\partial_t + \Delta, -i(2\mathbf{x}\mathbf{x}\cdot\nabla - \mathbf{x}^2\nabla + \mathbf{x})] + 8m\mathbf{x}\,\partial_t = -4i\mathbf{x}(2im\,\partial_t + \Delta) + o(c^{-2}), \tag{4.7}$$

which, in the limit, is (2.12) with the generators

$$\mathbf{C}' = -i(2\mathbf{x}\mathbf{x}\cdot\nabla - \mathbf{x}^2\nabla + \mathbf{x}), \quad \hat{\mathbf{C}}' = -4i\mathbf{x}, \quad \gamma = -4\mathbf{x}. \tag{4.8}$$

We stress once more that the generators S', A', $\mathbf{C}'$ are not directly the limits of the relativistic generators S, C^0, $\mathbf{C}$ but they are generators obtained by the requirement that the limit of the conformal invariance condition (3.7) is the Schrödinger invariance condition (2.12).

Of the fifteen generators resulting from the limit procedure, thirteen are generators included in the set (2.13). The remaining two generalized Schrödinger generators, S' of (4.3) and A' of (4.4), are not directly contained in (2.13). However, since (2.12) also admits the generators of pure time transformations (type I in (2.13)), and since (2.12) is linear in $(G, \hat{G}, \gamma)$, we may modify the generators S', A' by adding type I generators which, in particular, can be chosen in such a way that the resulting generators do not change the mass, i.e. have vanishing $\hat{G}$. We thus define

$$S = S' + X = i(2t\,\partial_t + \mathbf{x}\cdot\nabla + 3/2), \quad \hat{S} = 0, \quad \gamma = 2,$$
$$A = \tfrac{1}{2}(A' + Y) = -i(t^2\,\partial_t + t\mathbf{x}\cdot\nabla + \tfrac{3}{2}t) - \tfrac{1}{2}m\mathbf{x}^2, \quad \hat{A} = 0, \quad \gamma = -2t, \tag{4.9}$$

where

$$X = it\,\partial_t, \quad \hat{X} = -i, \quad \gamma = 0$$
$$Y = -it^2\,\partial_t, \quad \hat{Y} = 2it, \quad \gamma = 0 \tag{4.10}$$

are type I generators. By this manipulation the transformation of the mass has been transferred to a transformation of the time-coordinate. The two generators (4.9) are precisely the generators S, A of (2.13) and we have now recovered the twelve generators of the Schrödinger group and the three generators $\mathbf{C}$ of the special conformal transformations. However, as already mentioned in Section 2, the special conformal transformations do not combine with the Schrödinger group to form a group. Also note that the generators $\mathbf{C}$ cannot be modified in the same way as the generators S, A in (4.9) because they multiply the mass by a function of $\mathbf{x}$ whereas type I generators multiply the mass by functions of t alone.

To summarize then: *The non-relativistic limit of the conformal group is a 15-parameter set of generalized Schrödinger transformations which in itself does not form a group but which, upon combination with additional generalized Schrödinger transformations, can be made to contain the Schrödinger group.*

APPENDIX

To find all generalized Schrödinger transformations g we have to solve the equation (2.5) or, equivalently, the equation

$$[2iF_g(t,\mathbf{x})\,\mu(t,\mathbf{x})\,\partial'_t + \Delta'][f_g(t,\mathbf{x})\,\psi(t,\mathbf{x})] = 0, \quad (t',\mathbf{x}') \equiv g(t,\mathbf{x}), \tag{A.1}$$

for the unknowns g, f_g, F_g. The analysis of equation (A.1) proceeds partly along the same lines as a similar analysis in [1] and it is only briefly sketched here. Defining

$$a(t,\mathbf{x}) = \partial t/\partial t', \quad c_i(t,\mathbf{x}) = \partial t/\partial x'_i,$$
$$b_i(t,\mathbf{x}) = \partial x_i/\partial t', \quad d_{ik}(t,\mathbf{x}) = \partial x_i/\partial x'_k, \tag{A.2}$$

we write (A.1) as a differential equation in ∂_t, ∂_i and, using that ψ is a solution of (2.4), we obtain the equations

$$c_i = 0, \tag{A.3}$$

$$d_{ir}\, d_{kr} = aF_g\, \delta_{ik}, \tag{A.4}$$

$$2aF_g \partial_k f_g + (d_{ri}\,\partial_r d_{ki} + 2iF_g\,\mu b_k) f_g = 0, \tag{A.5}$$

$$aF_g \Delta f_g + (d_{ri}\,\partial_r d_{ki} + 2iF_g\,\mu b_k)\,\partial_k f_g + 2iaF_g\,\mu \dot{f}_g = 0. \tag{A.6}$$

There exists a set of obvious solutions, namely

$$t' \text{ arbitrary}, \quad \mathbf{x}' = \mathbf{x}, \quad f_g = \text{cst.}, \quad F_g = dt'/dt. \tag{A.7}$$

That (A.7) is a solution is already seen from the relation

$$2i\mu\,\partial_t + \Delta = 2i\mu \frac{dt'}{dt} \partial'_t + \Delta. \tag{A.8}$$

We now turn to the equation (A.4). This equation is quite different from the corresponding equation in [1] because, due to the appearance of the unknown function $F_g(t, \mathbf{x})$, the right-hand side may now depend on $\mathbf{x}$. With the definition

$$\alpha_{ik} = \partial x'_i/\partial x_k = (aF_g)^{-1} d_{ki} \equiv \zeta(t, \mathbf{x})\, d_{ki} \tag{A.9}$$

equation (A.4) takes the form

$$\alpha_{ri}\,\alpha_{rk} = \zeta\,\delta_{ik}. \tag{A.10}$$

Equation (A.10) is the equation occurring in the determination of the conformal group in $\mathbb{R}^3$ and its general solutions [6] are

$$\mathbf{x}' = \kappa\left(R\frac{\mathbf{x} + \mathbf{c}\mathbf{x}^2}{\sigma} + \mathbf{y}\right), \quad \zeta = \kappa^2 \frac{1}{\sigma^2}, \tag{A.11}$$

where $\sigma \equiv 1 + 2\mathbf{c}\cdot\mathbf{x} + \mathbf{c}^2\mathbf{x}^2$ and $\kappa \in \mathbb{R}$, $\mathbf{c}$, $\mathbf{y} \in \mathbb{R}^3$, $R \in O(3)$. Thus (A.11) solves (A.4) and determines the $\mathbf{x}$-dependence of the coordinate transformations. However, the parameters κ, $\mathbf{c}$, $\mathbf{y}$, R may yet depend on t and to determine the t-dependence, we have to solve equations (A.5) and (A.6).

We already know that the transformations of the Schrödinger group [1] are solutions for $\mu = m = \text{cst.}$ and we now have to check whether they are also solutions for arbitrary μ. It is easy to see that the rotations, translations and dilations are solutions for arbitrary μ. For the remaining Schrödinger transformations,

$$(t', \mathbf{x}') = \left(\frac{t}{1 + \alpha t}, \frac{\mathbf{x} + \mathbf{v}t}{1 + \alpha t}\right), \tag{A.12}$$

we obtain from (A.5) and (A.6) the equations

$$\begin{aligned} \boldsymbol{\nabla}(\ln f_g) &= i\mu \frac{1}{1 + \alpha t}(\mathbf{v} - \alpha\mathbf{x}), \\ \partial_t(\ln f_g) &= \frac{1}{2}\frac{1}{1 + \alpha t}[3\alpha - \boldsymbol{\nabla}(\ln\mu)\cdot(\mathbf{v} - \alpha\mathbf{x})] + \tfrac{1}{2}i\mu \frac{1}{(1 + \alpha t)^2}(\mathbf{v} - \alpha\mathbf{x})^2. \end{aligned} \tag{A.13}$$

If the integrability conditions for (A.13) are analysed, it is seen that they can only be satisfied for constant μ.

We now turn back to the transformations (A.11). The function F_g which is responsible for a transformation of the mass is given by $F_g = (a\zeta)^{-1} = \sigma^2(a\kappa^2)^{-1}$. Here we may assume $F_g = \sigma^2$ because $a\kappa^2 = 1$ may always be achieved through an additional transformation of the type (A.7). In this case, the parameters κ, $\mathbf{y}$, R do not lead to a change of the mass, hence they belong to the Schrödinger group and have already been dealt with. We are left with the transformations

$$\mathbf{x}' = (\mathbf{x} + \mathbf{c}\mathbf{x}^2)\,\sigma^{-1}, \quad F_g(t, \mathbf{x}) = \sigma^2, \tag{A.14}$$

and we have to determine the time-dependence of $\mathbf{c}$. The equations (A.5) and (A.6) now become

$$\begin{aligned} \nabla(\ln f_g) &= \tfrac{1}{2}\nabla(\ln\sigma) + i\mu(\dot{\mathbf{c}}\mathbf{x}^2 - 2\dot{\mathbf{c}}\cdot\mathbf{x}\mathbf{x}), \\ \partial_t(\ln f_g) &= \tfrac{1}{2}\partial_t(\ln\sigma) + 2\dot{\mathbf{c}}\cdot\mathbf{x} - \tfrac{1}{2}\nabla(\ln\mu)\cdot(\dot{\mathbf{c}}\mathbf{x}^2 - 2\dot{\mathbf{c}}\cdot\mathbf{x}\mathbf{x}) + \tfrac{1}{2}i\mu\dot{\mathbf{c}}^2\,\mathbf{x}^4. \end{aligned} \tag{A.15}$$

For $\dot{\mathbf{c}} = 0$ we obtain the solutions

$$f_g = f_g(\mathbf{x}) = \sigma^{1/2}(\mathbf{x}). \tag{A.16}$$

There may be other solutions for $\dot{\mathbf{c}} \neq 0$, but we are only interested in transformations which are possible for $\mu = \text{cst.}$ and in this case the integrability condition for the first of the equations (A.15) demands $\dot{\mathbf{c}} = 0$. Thus, the solutions with $\dot{\mathbf{c}} \neq 0$ are incompatible with constant mass. (For completeness we write down these additional solutions; they are $\mathbf{c}(t) = K(t)\,\mathbf{k}$, $K(t)$ any real function, $\mathbf{k} \in \mathbb{R}^3$, and they demand $\mu(t,\mathbf{x}) = \dot{K}(t)\,\mathbf{x}^{-4}$.)

The transformations (A.7), the Schrödinger transformations, and the special conformal transformations (A.14) are thus the desired set of generalized Schrödinger transformations. All of them are compatible with a constant mass, but whereas the transformations (A.12) demand $\mu = \text{cst.}$ the others are compatible with any function $\mu(t, \mathbf{x})$.

Finally, let us remark that there is a singular solution of (A.10), namely the inversion at the unit sphere,

$$\mathbf{x}' = \frac{\mathbf{x}}{\mathbf{x}^2}, \quad \zeta = \frac{1}{\mathbf{x}^4}. \tag{A.17}$$

It has not been treated separately because it can be obtained as a limit of other transformations.

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