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Strong Asymptotic Completeness of Wave Operators for Highly Singular Potentials

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Abstract. We prove the existence and strong asymptotic completeness of the wave operators for rotation invariant Schrödinger Hamiltonians with highly singular (repulsive or attractive) potentials.

I. Introduction

This paper is a complement to our recent investigation [1] of the relation between two different definitions of bound states and scattering states for quantum-mechanical N-particle systems. One of these definitions is in terms of spectral subspaces determined by the Hamiltonian of the system: bound states are identified with linear combinations of eigenvectors, and scattering states with vectors belonging to the subspace of continuity of H. The second and physically more transparent definition defines bound states as states in which all particles stay close together at all times and scattering states as states in which the particles separate into at least two clusters moving away from each other as $t \to \pm\infty$. For precise mathematical definitions and additional motivation the reader is referred to [1].

It was shown in [1] that for practically all Hamiltonians of physical interest the two definitions are equivalent, and it was pointed out at the end of Section III that this result was not necessarily to be expected for Schrödinger Hamiltonians with locally highly singular attractive potentials. In order to find out whether the two definitions might still be equivalent for such potentials, we investigated the rather simple case of a single particle moving under the influence of a spherically symmetric such potential, e.g. $V(r) = \alpha r^{-\beta}$ with $\alpha < 0$, $\beta > 1$. If $\beta > 2$, $\hat{H} = \vec{P}^2 + V$ defined on $D(\hat{H}) = C_0^\infty(\mathbb{R}^3/\{0\})$ is not essentially selfadjoint. We stated in [1] without proof that the two definitions of bound states and scattering states were equivalent also for such potentials if H was any spherically symmetric selfadjoint extension of $\hat{H}$. In the present paper we shall give a proof of this result.

The proof is based on Proposition 3 of [1], i.e. one has to verify the existence and asymptotic completeness of the wave operators $\Omega_\pm = s\lim \exp(iHt)\exp(-iH_0 t)$ as

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$t \to \pm\infty$. (An alternative proof is given in the Appendix, cf. Lemma 4.) Asymptotic completeness here means that the range of $\Omega_\pm$ is the entire absolutely continuous subspace of H and that H has no singularly continuous spectrum. Since H and H_0 are both invariant under the rotation group, it suffices to establish existence and asymptotic completeness of the restriction of $\Omega_\pm$ to each partial wave subspace $\mathscr{H}_{lm}$. $\mathscr{H}_{lm}$ is isomorphic to $L^2(\mathbb{R}^+)$, and the restriction of H to $\mathscr{H}_{lm}$ is unitarily equivalent to the ordinary differential operator $-d^2/dr^2 + V(r) + l(l+1)r^{-2}$ in $L^2(\mathbb{R}^+)$ characterized by a boundary condition at the origin ([2], Chapter 11.1.1). Our proof is based on an investigation of the spectral properties of this differential operator. We should also mention here that the first part of the asymptotic completeness property (i.e. $\Omega_\pm \Omega_\pm^* \mathscr{H} = \mathscr{H}_{ac}(H)$) has recently also been proved by a different method for a similar class of potentials by Pearson [3].

Although highly singular attractive potentials are not much used by physicists since H is not bounded below, they give rise to some delicate mathematical problems. We think that from the mathematical point of view it would be interesting to find methods applicable also to non-spherically symmetric potentials. We expect that in various cases, and also for certain spherically symmetric potentials which are rapidly oscillating near $r = 0$, some of the conclusions of the present paper will fail to hold. In particular it can be seen from our proofs that asymptotic completeness will not hold if the spectrum of some selfadjoint extension of the symmetric operator $-d^2/dr^2 + V(r)$ in $L^2(0,1)$ contains an absolutely continuous part (in that case the absolutely continuous spectrum of the S-wave Hamiltonian contains either a negative part or a part having spectral multiplicity two and hence cannot be unitarily equivalent to the free S-wave Hamiltonian which contains no such parts. For an example cf. Pearson [4]).

We conclude this introduction with the statement of our theorems, the proofs of which will then be indicated in Sections II and III. The first three deal with spectral properties of the Hamiltonian, the last one with scattering theory (cf. also Theorem 5 in the Appendix). Definitions can be found in Section II.

Theorem 1: Let $U: (0, \infty) \to \mathbb{R}$ belong to $L^1([a, \infty))$ for each $a > 0$. Let H be one of the selfadjoint operators defined by the differential expressions $l = -d^2/dr^2 + U(r)$ in $L^2(0, \infty), \{E(\lambda)\}$ its spectral family. Then

a) The restriction of H to $E((0, \infty))\,L^2(0, \infty)$ has an absolutely continuous spectrum (in particular the positive singular spectrum if H is void).
b) The negative absolutely continuous spectrum of H coincides with the negative absolutely continuous spectrum of any one of the selfadjoint extensions of $-d^2/dr^2 + U(r)$ in $L^2(0, 1)$.

Remarks: i) We use the definitions of Kato ([5], Chapter X.1.2) for the various parts of the spectrum of a selfadjoint operator.

ii) In the proof of Theorem 1 we shall give even more specific information about the absolutely continuous spectrum of H and its multiplicity.

iii) Suppose U is such that the wave operators $\Omega_\pm = s\lim \exp(iHt)\exp(-iH_{0,0}t)$ as $t \to \pm\infty$ exist ($H_{0,0}$ is the selfadjoint extension of $-d^2/dr^2$ defined by $f(0) = 0$). This is the case, for instance, if $r^2|U(r)|^2 \in L^1(R, \infty)$ for some $R < \infty$, cf. Kupsch and

Sandhas [6]. In general these wave operators will not be asymptotically complete, and the ranges of $\Omega_\pm$ may be strictly smaller than the absolutely continuous subspace $\mathscr{H}_{ac}(H)$. In such a situation the states which belong to $\mathscr{H}_{ac}(H)$ and which are orthogonal to the range of Ω_+ will be absorbed at the origin as $t \to +\infty$; more precisely: if f is such a state, then for every $R > 0$

$$\lim_{t \to +\infty} \|(I - F_R) \exp(-iHt) f\|^2 = 0$$

and similarly for Ω_- and $t \to -\infty$. (F_R is the orthogonal projection onto the subspace of states localized in $(0, R)$, cf. [1].) It is interesting to remark that the vectors in the absolutely continuous subspace of the negative part of H are *bound states* in the sense of the definition of [1] (cf. the addendum for an indication of the proof). This aspect of the completeness problem will be developed in more detail by Pearson [7]. Our aim here is different in that we shall add a condition on the behaviour of U near $r = 0$ which will guarantee that there is no absorption at the origin.

Theorem 2: Let $U:(0, \infty) \to \mathbb{R}$ belong to $L^1([a, \infty))$ for each $a > 0$. Suppose that the essential spectrum of one of the selfadjoint operators H_g^0 defined by the differential expression $-d^2/dr^2 + U(r)$ in $L^2(0, 1)$ is empty. Let H be as in Theorem 1. Then

a) The spectrum of H is simple.
b) The essential spectrum of H is $[0, \infty)$.
c) H has no singularly continuous spectrum.
d) H has no positive eigenvalues.
e) The restriction of H to $E((0, \infty)) L^2(0, \infty)$ is unitarily equivalent to the operator of multiplication by the independent variable in $L^2(0, \infty)$.

Theorem 3: Let U and H be as in Theorem 1. In order for H to have the properties a)–e) of Theorem 2 it is sufficient that one of the following conditions be verified:

1) $U = U_1 + U_2$ where U_1 is an increasing and continuous function of r in $(0, 1)$ and $U_2 \in L^1(0, 1)$.
2) $U = U_1 + U_2$ where

(α) $\quad U_1 \in C^2(0, 1], \qquad U_1(r) < M < \infty \qquad$ for all $r \in (0, 1]$

and

$$\int_0^1 dr\, (M - U_1(r))^{-1/2} < \infty$$

$$\int_0^1 dr \left| \frac{U_1''(r)}{(M - U_1(r))^{3/2}} + \frac{5}{4} \cdot \frac{|U_1'(r)|^2}{(M - U_1(r))^{5/2}} \right| < \infty.$$

(β) $\quad \displaystyle\int_0^1 dr (M - U_1(r))^{-1/2} |U_2(r)| < \infty.$

3) $\lim_{r \to 0} U(r) = \infty.$

4) $\liminf_{r\to 0} r^2 U(r) > -1/4$.

5) $\lim_{r\to 0}[U(r) + (2r)^{-2} + (2r\log r)^{-2}] = \infty$.

Theorem 4: Let l be a non-negative integer, suppose $V:(0,\infty)\to\mathbb{R}$ belongs to $L^1([a,\infty))$ for each $a>0$ and verifies one of the following assumptions:

i) $V(r) + l(l+1)r^{-2}$ satisfies 1) of Theorem 3,

ii) V satisfies 2) of Theorem 3 and

$$(\gamma)\qquad \int_0^1 dr(M - U_1(r))^{-1/2} r^{-2} < \infty,$$

iii) V satisfies one of the conditions 3)–5) of Theorem 3.

Let H_l be one of the selfadjoint operators defined by the differential expression $-d^2/dr^2 + V(r) + l(l+1)r^{-2}$ in $L^2(0,\infty)$. Then H_l has the properties a)–e) of Theorem 2. Furthermore the wave operators $\Omega_{l\pm} = \text{s}\lim \exp(iH_l t)\exp(-iH_{0,l}t)$ as $t\to\pm\infty$ exist and

$$\Omega_{l\pm}\Omega_{l\pm}^*\mathscr{H} = \mathscr{H}_{\text{ac}}(H_l).$$

Remarks: i) $H_{0,l}$ is the restriction to $\mathscr{H}_{lm}$ of the selfadjoint operator $H_0 = \vec{P}^2$ acting in $L^2(\mathbb{R}^3)$.

ii) If V has a *repulsive* singularity at the origin, condition 3) is verified and hence the wave operators are complete. If V has an *attractive* singularity of the form $V(r) = \alpha r^{-\beta}$ with $\alpha<0$ and $\beta>2$, 2) and (γ) are verified with $U_1(r) = V(r)$, $U_2 = M = 0$. Alternatively $V(r) + l(l+1)r^{-2}$ verifies 1) with $U_2 = 0$. For $\beta\leqslant 2$ one may apply 1) or 4) depending on the value of l.

iii) Theorems 1–3 can also be proved for long-range potentials under the assumption that $U = U_S + U_L$ where U_S is as in Theorems 1–3 and U_L belongs to $L^1_{\text{loc}}([0,\infty))$, is of bounded variation near infinity and converges to zero as $r\to\infty$ [8], [13]. (In our proofs we use only the fact that the spectral function of a certain selfadjoint extension of $-d^2/dr^2 + U(r)$ in $L^2(1,\infty)$ is sufficiently regular, and this has been established in [8] also for long-range potentials.)

II. Proof of Theorem 1

We follow the method of Kac [9] and use the terminology of [9] and [10]. The interval $(0,\infty)$ is divided into $(0,1]\cup(1,\infty)$, and one obtains the spectral properties of selfadjoint extensions of $L_0 = -d^2/dx^2 + U(x)$ in $L^2(0,\infty)$ from spectral properties of selfadjoint extensions corresponding to the two subspaces.

For $z\in\mathbb{C}$, one introduces the solutions $u_1(\cdot,z)$ and $u_2(\cdot,z)$ of the differential equation $l(f) = zf$ on $(0,\infty)$ defined by the initial conditions

$$u_1(1,z) = 1,\qquad u_1'(1,z) = 0,\qquad u_2(1,z) = 0,\qquad u_2'(1,z) = -1.$$

Under our assumption on U, l is in the limit-point case at infinity ([10], Theorem 23.3). If $\operatorname{Im} z\neq 0$, there exists precisely one linear combination $\chi_r(x,z) = u_2(x,z)$

$+ \omega_r(z)u_1(x,z)$ such that $\chi_r(\cdot,z) \in L^2(1,\infty)$. At 0, l can be either in the limit-point or in the limit-circle case. In the former case L_0 is selfadjoint and $H = L_0$; in the latter case H is characterized by a boundary condition at 0 of the form

$$f \in D(H) \Leftrightarrow f \in D(L_0^*) \qquad \text{and} \qquad W[f,w] = 0$$

where w is a real function in $D(L_0^*)$ and W denotes the Wronskian ([9], Section 6 or [11], Appendix II.5). In the first case there exists precisely one linear combination $\chi_l(x,z) = u_2(x,z) - \omega_l(z)\,u_1(x,z)$ belonging to $L^2(0,1)$; in the second case $\omega_l(z)$ is determined by the condition that

$$W[u_2(\cdot,z) - \omega_l(z)\,u_1(\cdot,z), w] = 0.$$

We also introduce the selfadjoint operator H_r in $L^2(1,\infty)$ determined by $-d^2/dx^2 + U(x)$ and the boundary condition $f'(1) = 0$ and the selfadjoint operator H_g in $L^2(0,1)$ defined by the boundary condition $f'(1) = 0$ and, if 0 is in the limit-circle case, $W[f,w] = 0$. We first study the spectral function τ_r of H_r which is given by ([11], Appendix II.7)

$$\tau_r(t) = \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_0^t ds\, \mathrm{Im}\, \omega_r(s + i\epsilon). \tag{1}$$

For $k \in \{z \mid \mathrm{Im}\, z \geqslant 0,\, z \neq 0\}$, we define the Jost solution $f(\cdot,k)$ of $l(f) = k^2 f$ by the boundary condition

$$\lim_{x \to \infty} \exp(-ikx) f(x,k) = 1.$$

If $\mathrm{Im}\, z > 0$, $\chi_r(\cdot,z)$ must be proportional to $f(\cdot,z^{1/2})$ (we choose the determination of the square root such that $\mathrm{Im}\, z^{1/2} > 0$), i.e. there exists $c(z) \neq 0$ such that

$$c(z) f(\cdot,z^{1/2}) = u_2(\cdot,z) + \omega_r(z)\,u_1(\cdot,z). \tag{2}$$

Since

$$W(u_2(\cdot,z), u_1(\cdot,z)) \equiv u_2(1,z)\,u_1'(1,z) - u_1(1,z)\,u_2'(1,z) = 1$$

(2) implies that

$$\omega_r(z) = -\frac{W(f(\cdot,z^{1/2}), u_2(\cdot,z))}{W(f(\cdot,z^{1/2}), u_1(\cdot,z))}$$

(notice that the denominator is different from zero, since otherwise $u_1(\cdot,z)$ would belong to $L^2(1,\infty)$, which is impossible since H_r cannot have a non-real eigenvalue).

Since $U \in L^1(1,\infty)$, $f(x,k)$ and $d/dx f(x,k)$ are uniformly continuous functions of k on any compact set not containing the point $k = 0$ in the closed upper half plane (this can be seen for instance from the considerations of [2], Chapter 12.1.1).

Let Λ be a compact interval in $(0,\infty)$. It follows that the limit

$$\lim_{\varepsilon\to+0} W(f(\cdot,\sqrt{\lambda+i\epsilon}),u_k(\cdot,\lambda+i\epsilon))$$

$$=\lim_{\varepsilon\to+0}[f(1,\sqrt{\lambda+i\epsilon})\,u_k'(1,\lambda+i\epsilon)-f'(1,\sqrt{\lambda+i\epsilon})\,u_k(1,\lambda+i\epsilon)]$$

$$=f(1,\lambda^{1/2})\,u_k'(1,\lambda)-f'(1,\lambda^{1/2})\,u_k(1,\lambda)\equiv W(f(\cdot,\lambda^{1/2}),u_k(\cdot,\lambda))$$

exists uniformly in $\lambda\in\Lambda$ and is continuous on Λ.

Since $f(x,\pm\lambda^{1/2})=f(x,\mp\lambda^{1/2})^*$ and $u_k(\cdot,\lambda)$ is real, one has

$$W(f(\cdot,\pm\lambda^{1/2}),u_k(\cdot,\lambda))=W(f(\cdot,\mp\lambda^{1/2}),u_k(\cdot,\lambda))^*. \tag{3}$$

Also

$$f(x,\pm\lambda^{1/2})=-W(f(\cdot,\pm\lambda^{1/2}),u_2(\cdot,\lambda))\,u_1(x,\lambda)+W(f(\cdot,\pm\lambda^{1/2}),u_1(\cdot,\lambda))\,u_2(x,\lambda).$$

By combining the last two equations one gets

$$\begin{aligned}-2i\lambda^{1/2}&=W(f(\cdot,+\lambda^{1/2}),f(\cdot,-\lambda^{1/2}))\\&=-2i\,\mathrm{Im}[W(f(\cdot,+\lambda^{1/2}),u_1(\cdot,\lambda))\,W(f(\cdot,-\lambda^{1/2}),u_2(\cdot,\lambda))]\\&=+2i\,\mathrm{Im}[W(f(\cdot,+\lambda^{1/2}),u_2(\cdot,\lambda))\,W(f(\cdot,-\lambda^{1/2}),u_1(\cdot,\lambda))].\end{aligned} \tag{4}$$

It follows that for $\lambda\in\Lambda$, $W(f(\cdot,\pm\lambda^{1/2}),u_k(\cdot,\lambda))\neq0$. Hence

$$\omega_r(\lambda+io)=\lim_{\varepsilon\to+0}\omega_r(\lambda+i\epsilon)=-\frac{W(f(\cdot,\lambda^{1/2}),u_2(\cdot,\lambda))}{W(f(\cdot,\lambda^{1/2}),u_1(\cdot,\lambda))} \tag{5}$$

exists uniformly in $\lambda\in\Lambda$, in particular it is continuous. A short calculation, using also (4), gives for $\lambda\in\Lambda$

$$\mathrm{Im}\,\omega_r(\lambda+io)=\lambda^{1/2}[|W(f(\cdot,\lambda^{1/2}),\,u_1(\cdot,\lambda)|]^{-2}\neq0. \tag{6}$$

In view of (1), this shows that on $(0,\infty)$ τ_r is absolutely continuous and has a strictly positive and continuous derivative.

By writing the resolvent kernel of H in terms of χ_r and χ_l ([12], p. 1329) and using [10], Section 21.4b, one may calculate the spectral matrix $\{\sigma_{ij}\}$ of H. In particular [9]

$$\sigma(t)\equiv\sigma_{11}(t)+\sigma_{22}(t)=\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_0^t ds\,\mathrm{Im}[\Omega_{11}+\Omega_{22}]\,(s+i\epsilon) \tag{7}$$

with

$$\Omega_{11}(z)=\omega_l(z)\,\omega_r(z)\,[\omega_l(z)+\omega_r(z)]^{-1} \tag{8}$$

$$\Omega_{22}(z)=-[\omega_l(z)+\omega_r(z)]^{-1}. \tag{9}$$

To prove a), it is sufficient to show that σ is absolutely continuous on $(0,\infty)$ (cf. [9], Section 1).

One obtains from (8) and (9)

$$\operatorname{Im}(\Omega_{11}+\Omega_{22}) = |\omega_l + \omega_r|^{-2}[(1+|\omega_r|^2)\operatorname{Im}\omega_l + (1+|\omega_l|^2)\operatorname{Im}\omega_r]. \tag{10}$$

The finite limit of $\omega_l(\lambda+i\epsilon)$ as $\epsilon \to +0$ exists except possibly on a set N of points $\lambda \in \mathbb{R}$ of Lebesgue measure zero ([9], Section 2). By combining this with the properties of $\omega_r(\lambda+io)$, it follows that for $\lambda \in (0,\infty)/N$ the following limit exists, is finite and strictly positive:

$$\lim_{\varepsilon\to+0} \operatorname{Im}(\Omega_{11}+\Omega_{22})(\lambda+i\epsilon) \equiv \operatorname{Im}(\Omega_{11}+\Omega_{22})(\lambda+io)$$

$$= \frac{(1+|\omega_r(\lambda+io)|^2)\operatorname{Im}\omega_l(\lambda+io) + (1+|\omega_l(\lambda+io)|^2)\operatorname{Im}\omega_r(\lambda+io)}{|\omega_l(\lambda+io)+\omega_r(\lambda+io)|^2} \tag{11}$$

By using $\operatorname{Im}\omega_r(\lambda+i\epsilon) > 0$, $\operatorname{Im}\omega_l(\lambda+i\epsilon) \geqslant 0$, one deduces the following inequalities (we omit the argument $\lambda+i\epsilon$)

$$\operatorname{Im}\omega_l|\omega_l+\omega_r|^{-2} \leqslant (\operatorname{Im}\omega_r)^{-1} \tag{12}$$

$$(1+|\omega_l|^2)|\omega_l+\omega_r|^{-2} \leqslant 1 + (\operatorname{Im}\omega_r)^{-2} + (\operatorname{Re}\omega_l)^2|\omega_l+\omega_r|^{-2}. \tag{13}$$

By using the inequality $y^2[(y+b)^2+c^2]^{-1} \leqslant 1+(b/c)^2$ and identifying $y = \operatorname{Re}\omega_l$, $b = \operatorname{Re}\omega_r$, $c = \operatorname{Im}\omega_r$, the last term of (13) can be estimated as follows:

$$(\operatorname{Re}\omega_l)^2|\omega_l+\omega_r|^2 \leqslant 1 + (\operatorname{Re}\omega_r/\operatorname{Im}\omega_r)^2. \tag{14}$$

By inserting (12)–(14) into (10), we get

$$0 < \operatorname{Im}(\Omega_{11}+\Omega_{22})(\lambda+i\epsilon) \leqslant 2\operatorname{Im}\omega_r(\lambda+i\epsilon) + 2\,\frac{1+|\omega_r(\lambda+i\epsilon)|^2}{\operatorname{Im}\omega_r(\lambda+i\epsilon)}.$$

Let $\Lambda = [\lambda_1, \lambda_2] \subset (0,\infty)$ be a closed finite interval. Since ω_r is uniformly continuous and $\operatorname{Im}\omega_r(z) > 0$ on $S = \{z \mid z = \lambda+i\epsilon,\ \lambda\in\Lambda,\ 0\leqslant\epsilon\leqslant\epsilon_0,\ \epsilon_0>0\}$, there exists C such that $0 < \operatorname{Im}(\Omega_{11}+\Omega_{22})(z) < C$ for all z in the interior of S. It then follows from the Lebesgue dominated convergence theorem that

$$\sigma(\lambda) = \sigma(\lambda_1) + \lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{\lambda_1}^{\lambda} d\mu\,\operatorname{Im}(\Omega_{11}+\Omega_{22})(\mu+i\epsilon)$$

$$= \sigma(\lambda_1) + \frac{1}{\pi}\int_{\lambda_1}^{\lambda} d\mu\,\operatorname{Im}(\Omega_{11}+\Omega_{22})(\mu+io).$$

This shows that σ is absolutely continuous on $(0,\infty)$ and that its Radon–Nikodym derivative is almost everywhere on $(0,\infty)$ equal to

$$\sigma'(\lambda) = \frac{1}{\pi}\operatorname{Im}(\Omega_{11}+\Omega_{22})(\lambda+io). \tag{15}$$

If Λ is as above, one can also see that $\sigma'(\lambda) \geqslant m > 0$ for almost every $\lambda \in \Lambda$. Indeed, since ω_r is continuous, there exists $1 < K < \infty$ such that for all $\lambda \in \Lambda$: $|\omega_r(\lambda + io)| < K$. It follows that

$$(1 + |\omega_l|^2)|\omega_l + \omega_r|^{-2} \geqslant \tfrac{1}{2}(1 + |\omega_l|^2)(K^2 + |\omega_l|^2)^{-1} \geqslant \tfrac{1}{2}K^{-2}.$$

From (6) one obtains also that $\operatorname{Im} \omega_r(\lambda + io) \geqslant m_0 > 0$ for all $\lambda \in \Lambda$, and the desired result $\sigma'(\lambda) \geqslant m > 0$ a.e. follows by inserting the above inequalities into (11). These properties of $\sigma'(\lambda)$ are collected in Remark i) at the end of this proof.

We next treat the negative part of the spectrum of H. The proof of b) is based on the following lemma:

Lemma 1: Let $\lambda < 0$. Under the conditions of Theorem 1, the following two statements are equivalent:

1) $\omega_l(\lambda + i\epsilon)$ converges to a finite limit with strictly positive imaginary part as $\epsilon \to +0$,
2) $(\Omega_{11} + \Omega_{22})(\lambda + i\epsilon)$ converges to a finite limit with strictly positive imaginary part as $\epsilon \to +0$.

To proceed with the proof, we need a precise definition of the support Q_{a+} of the absolutely continuous part of a selfadjoint differential operator. We use the definition of Kac [9] which is essentially the same as that given in [14]–[16]. We define $Q_{a+}(H)$ to be the set of points $\lambda \in \mathbb{R}$ for which condition 2) of Lemma 1 is verified. Similarly one introduces $Q_{a+}(H_g)$ and $Q_{a+}(H_r)$ by replacing in condition 2) $\Omega_{11} + \Omega_{22}$ by ω_l resp ω_r.

Lemma 1 states that for $\lambda < 0$ we have

$$\lambda \in Q_{a+}(H_g) \Leftrightarrow \lambda \in Q_{a+}(H).$$

The negative part of the absolutely continuous spectrum of H (resp. H_g) as defined in [5], Chapter X.1.2, coincides with the closure of the set $Q_{a+}(H) \cap (-\infty, 0)$ (resp. $Q_{a+}(H_g) \cap (-\infty, 0)$). (This can easily be seen from the considerations of [14] and [16].) Hence $\sum_{ac}(H) \cap (-\infty, 0) = \sum_{ac}(H_g) \cap (-\infty, 0)$. Part b) of Theorem 1 now follows from the fact that the absolutely continuous spectrum of any selfadjoint extension of $-d^2/dr^2 + U(r)$ in $L^2(0,1)$ is the same as that of H_g (the resolvents at $z = i$ of two selfadjoint extensions differ by an operator of rank two or less, cf. [10], Remark 19.1; the result then follows from [5], Theorem X.4.12). ■

Proof of Lemma 1: Since $\sum_e (H_r) = [0, \infty)$ ([10], Theorem 24.5), ω_r is meromorphic in $\operatorname{Re} z < 0$. The poles of ω_r in $\operatorname{Re} z < 0$ are simple and form a sequence $-\infty < \mu_1 < \mu_2 < \cdots < 0$ which may accumulate at most at $z = 0$. Near μ_k one has

$$\omega_r(\mu_k + i\epsilon) = i\beta_k \epsilon^{-1} + 0(\epsilon) \qquad \text{with} \qquad \beta_k = \tau_r(\{\mu_k\}) > 0. \tag{16}$$

Let

$$A(z) \equiv \Omega_{11}(z) + \Omega_{22}(z) = (\omega_l(z)\,\omega_r(z) - 1)(\omega_l(z) + \omega_r(z))^{-1}. \tag{17}$$

a) Suppose 1) is verified. If $\lambda \neq \mu_k$ for all k, $\lim \omega_r(\lambda + i\epsilon) \equiv \omega_r(\lambda)$ as $\epsilon \to +0$ is real. By hypothesis the denominator of (17) has a strictly positive imaginary part for $z = \lambda + io$. Hence $\lim(\Omega_{11} + \Omega_{22})(\lambda + i\epsilon)$ as $\epsilon \to +0$ exists and is finite. From (11) one sees that its imaginary part is strictly positive.

If $\lambda = \mu_k$, one has from (16) and (17) that

$$\lim_{\varepsilon \to +0} (\Omega_{11} + \Omega_{22})(\lambda + i\epsilon) = \omega_l(\lambda + io).$$

Hence 1) implies 2).

b) Suppose 2) is verified. (17) implies

$$\omega_l(z) = [A(z)\,\omega_r(z) + 1][\omega_r(z) - A(z)]^{-1}. \tag{18}$$

If $\lambda \neq \mu_k$ for all k, $\lim \omega_l(\lambda + i\epsilon)$ as $\epsilon \to +0$ exists and is finite, since $\omega_r(\lambda)$ is real and $\operatorname{Im} A(\lambda + io) > 0$. Also

$$\operatorname{Im} \omega_l(\lambda + io) = \operatorname{Im} A(\lambda + io)[1 + \omega_r(\lambda)^2]|\omega_r(\lambda) - A(\lambda + io)|^{-2} > 0.$$

If $\lambda = \mu_k$:

$$\lim_{\varepsilon \to +0} \omega_l(\lambda + i\epsilon) = A(\lambda + io) = (\Omega_{11} + \Omega_{22})(\lambda + io).$$

Hence 2) implies 1). ∎

Remarks: i) We have proved in particular the following result: Let Λ be a compact set in $(0, \infty)$. There exist two constants m and M depending on Λ such that

$$0 < m < \sigma'(\lambda) < M < \infty \qquad \text{for a.e. } \lambda \in \Lambda.$$

ii) Suppose that the essential spectrum of H_g is contained in $[0, \infty)$. Then the singularly continuous spectrum of H is void and its essential spectrum $\sum_e(H)$ is $[0, \infty)$ (this follows from Theorem 1 because $\sum_e(H) = \sum_e(H_g) \cup \sum_e(H_r)$, cf. [10], Theorem 24.1, and because $\sum_e(H_r) = [0, \infty)$.)

iii) Let $K_+ = Q_{a+}(H_g) \cap Q_{a+}(H_r)$. By using the results of Kac [9], one can easily see that the restriction of H to $E(\mathbb{R}/K_+)\,L^2(0, \infty)$ has simple spectrum and the restriction of H to $E(K_+)\,L^2(0, \infty)$ has a homogeneous spectrum of multiplicity two. Note that, under the assumptions of Theorem 1, one has $K_+ \subset [0, \infty)$.

III. Proof of Theorems 2, 3 and 4

Theorem 2 follows immediately from Theorem 1 and the remarks at the end of Section II if one notices that $\sum_e(H_g) = \sum_e(H_g^0)$ ([11], Theorem 83.1). Theorem 3 follows from Theorem 2 provided that one can show in each case that the essential spectrum of the corresponding differential expression in $(0, 1]$ is void. For (4) and (5) this follows from [12], XIII.10.C25 and XIII.10.C30, respectively, and for (3) from [12], XIII.10.C26 or [10], Theorem 24.2.

For (2), let f be a solution of $(-d^2/dr^2 + U_1(r) - M)f = 0$. Then there exist c_1, $c_2 \in \mathbb{C}$ and two functions $e_k: (0,1) \to \mathbb{C}$ with $\lim e_k(r) = 0$ as $r \to 0$ such that

$$|M - U_1(r)|^{1/4} f(r) = (c_1 + e_1(r)) \exp\left(i \int_0^r |M - U_1(s)|^{1/2} ds \right)$$

$$+ (c_2 + e_2(r)) \exp\left(-i \int_0^r |M - U_1(s)|^{1/2} ds \right)$$

(cf. [12], proof of Theorem XIII.6.20). Hence

$$\int_0^1 |f(r)|^2 |U_2(r)|\, dr = \int_0^1 |f(r)|^2 (M - U_1(r))^{1/2} (M - U_1(r))^{-1/2} |U_2(r)|\, dr$$

$$\leqslant \text{const} \int_0^1 (M - U_1(r))^{-1/2} |U_2(r)| dr < \infty.$$

One may now apply the following lemma

Lemma 2: Let $U_1, U_2 \in L^1_{\text{loc}}(0,1)$. Suppose that every solution f of $-d^2/dr^2 f + U_1(r) f = 0$ is square-integrable on $(0,1)$ and such that $|f(r)|^2 |U_1(r) - U_2(r)| \in L^1(0,1)$. Then every solution g of $-d^2/dr^2 g + U_2(r) g = 0$ belongs to $L^2(0,1)$.

(This follows from Theorem XI.8.1 of [17] where ∞ may be replaced by 0 and the assumption of continuity of U_1 and U_2 can be weakened to $U_1, U_2 \in L^1_{\text{loc}}$.)

It follows that every solution of $-d^2/dr^2 f + (U_1(r) + U_2(r) - M) f = 0$ is square integrable in $(0,1)$. Hence $L_0 = -d^2/dr^2 + U_1(r) + U_2(r)$ acting in $L^2(0,1)$ has deficiency indices (2,2) ([10], Theorem 19.4), and the essential spectrum of any selfadjoint extension of L_0 in $L^2(0,1)$ is void ([10], Remark 19.2).

For (1), the argument is similar. Every solution f of $-d^2/dr^2 f + U_1(r) f = 0$ is uniformly bounded in $(0,1]$ ([12], XIII.6.27). Hence

$$\int_0^1 |f(r)|^2 |U_2(r)| dr \leqslant \text{const} \int_0^1 |U_2(r)| dr < \infty$$

and Lemma 2 implies that $-d^2/dr^2 + U_1(r) + U_2(r)$ has deficiency indices (2,2) in $L^2(0,1)$. This completes the proof of Theorem 3. (Some of the statements of Theorem 3 follow also from the results of [13], in particular Corollary 5.2.) ■

Under the hypotheses of Theorem 4, it follows from Theorem 3 that H_l has the properties a)–e) of Theorem 2. The proof of the existence of the wave operators $\Omega_{l\pm}$ will be given in the Appendix. It remains to show that the range of $\Omega_{l\pm}$ is equal to $\mathscr{H}_{\text{ac}}(H_l)$. For this, suppose $f \in \mathscr{H}_{\text{ac}}(H_l)$ and $f \perp \Omega_l \mathscr{H}_{lm}$. Let $U: \mathscr{H}_{lm} \to L^2(\mathbb{R}^+)$ be the unitary operator such that $U H_l E((0,\infty)) U^{-1}$ is the multiplication operator by the independent variable in $L^2(\mathbb{R}^+)$.[2]) Write $Uf = \{f(\lambda)\}$. For $\epsilon > 0$, let $\Delta_\varepsilon = \{\lambda | \lambda > 0$ and

[2]) $\{E(\lambda)\}$ denotes the spectral family of H_l, $\{E_0(\lambda)\}$ that of $H_{0,l}$.

$|f(\lambda)| > \epsilon\}$, and define $f_\varepsilon = E(\Delta_\varepsilon)f$. Let $\phi \in L^\infty(\mathbb{R}^+)$. From the intertwining property $\exp(iH_l t)\Omega_l = \Omega_l \exp(iH_{0,l}t)$ it follows that for $g \in \mathscr{H}_{lm}$

$$(\phi(H_l)f_\varepsilon, \Omega_l g) = (f, \Omega_l \phi^*(H_{0,l})\, E_0(\Delta_\varepsilon)\, g) = 0.$$

Thus $\phi(H_l)f_\varepsilon \perp \Omega_l\mathscr{H}_{lm}$. But f_ε is cyclic for H_l with respect to $E(\Delta_\varepsilon)\mathscr{H}_{lm}$, i.e. $\{\phi(H_l)f_\varepsilon | \phi \in L^\infty(\mathbb{R}^+)\}$ is dense in $E(\Delta_\varepsilon)\mathscr{H}_{lm}$. Hence $E(\Delta_\varepsilon)\mathscr{H}_{lm} \perp \Omega_l\mathscr{H}_{lm}$. Using again the intertwining property, this gives for all $g, h \in \mathscr{H}_{lm}$

$$0 = (E(\Delta_\varepsilon)\, g, \Omega_l\, h) = (g, \Omega_l\, E_0(\Delta_\varepsilon)\, h).$$

Hence $\Omega_l E_0(\Delta_\varepsilon)\mathscr{H}_{lm} \perp \mathscr{H}_{lm}$, i.e. $\Omega_l E_0(\Delta_\varepsilon)\mathscr{H}_{lm} = 0$. Since Ω_l is isometric, this implies $E_0(\Delta_\varepsilon) = 0$. Since $H_{0,l}$ is unitarily equivalent to the multiplication operator by the independent variable in $L^2(\mathbb{R}^+)$ and $\Delta_\varepsilon \subset \mathbb{R}^+$, it follows that the Lebesgue measure of Δ_ε is zero. Since $\epsilon > 0$ was arbitrary, this means $f(\lambda) = 0$ a.e., i.e. $f = 0$. ∎

Remark: A different but longer proof of asymptotic completeness under the conditions of Theorem 2 will be given in the Appendix in connection with the proof of the existence of the wave operators. Since these are known to exist in many situations [6], the preceding argument demonstrates the role played by the spectral multiplicity in the completeness argument.

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Appendix

Throughout this appendix we shall assume the hypotheses of Theorem 2. It will be proved that they imply the *existence of the wave operators* (cf. Theorem 5 below for the precise statement). We shall use the results of [9], Section 6, especially Theorem 6.

Let (cf. [9] for the notation)

$$u(\cdot, \lambda) = [\operatorname{sgn} \delta_{12}(\lambda)]\sqrt{\delta_{11}(\lambda)}\, u_1(\cdot, \lambda) + \sqrt{\delta_{22}(\lambda)}\, u_2(\cdot, \lambda).$$

By using the definition of δ_{ij} and Ω_{ij} ([9], equations (1.2) and (0.6)) and the fact that ω_l is meromorphic by the hypothesis made on H_g^0, one finds after a short calculation that for $\lambda > 0$ one has

$$u(\cdot, \lambda) = (1 + \omega_l(\lambda)^2)^{-1/2}[u_2(\cdot, \lambda) - \omega_l(\lambda)\, u_1(\cdot, \lambda)]. \tag{19}$$

(If λ is one of the poles of ω_l, it is understood that $u(\cdot, \lambda) = -u_1(\cdot, \lambda)$.)

For $\lambda > 0$ we also define

$$v(\cdot, \lambda) = [\sigma'(\lambda)]^{1/2} u(\cdot, \lambda). \tag{20}$$

Let $P = E((0, \infty))$. According to [9], Theorem 6, one then has:

i) For every $f \in PL^2(0, \infty)$ the integral

$$F(\lambda) \equiv (\mathscr{F} f)(\lambda) = \int_0^\infty v(x, \lambda) f(x)\, dx$$

converges in mean in $L^2(0, \infty)$.

ii) The operator $\mathscr{F}: PL^2(0, \infty) \to L^2(0, \infty)$ is unitary, transforms H into the operator of multiplication by λ, and its inverse is given by

$$(\mathscr{F}^{-1} F)(x) = \int_0^\infty v(x, \lambda)\, F(\lambda)\, d\lambda,$$

(the integral converges in mean in $L^2(0, \infty)$).

Lemma 3: $u(\cdot, \lambda) \in L^2(0,1)$ for every $\lambda > 0$. The function $\lambda \mapsto \|u(\cdot, \lambda)\|_{L^2(0,1)}$ is bounded on every compact set in $(0, \infty)$.

Proof: If 0 is in the limit-circle case, the result is evident. We therefore assume that 0 is in the limit-point case.

Suppose first that λ is an eigenvalue of H_g. The system $L_0(f) = \lambda f$ and $f'(1) = 0$ has a non-trivial solution in $L^2(0,1)$ which is unique up to a multiplicative constant ([10], Theorem 19.4). Since $u_1(\cdot, \lambda)$ is a solution of this system, one must have $u_1(\cdot, \lambda) \in L^2(0,1)$. Since the eigenvalues of H_g coincide with the poles of ω_l, one has $u(\cdot, \lambda) = -u_1(\cdot, \lambda)$ and hence $u(\cdot, \lambda) \in L^2(0,1)$.

Suppose λ is not an eigenvalue of H_g. Then it belongs to the resolvent set $\rho(H_g)$ of H_g. It is easily seen that the kernel of the resolvent of H_g for $0 < y < x \leqslant 1$ is given by ($\mathrm{Im}\, z \neq 0$)

$$G_z(x, y) = [u_2(y, z) - \omega_l(z)\, u_1(y, z)]\, u_1(x, z).$$

For $f \in L^2(0,1)$ one then has

$$[(H_g - z)^{-1} f](1) = \int_0^1 [u_2(y, z) - \omega_l(z)\, u_1(y, z)] f(y)\, dy. \tag{21}$$

By providing the domain of H_g with the graph norm, it can be viewed as a Banach space $D_G(H_g)$. The mapping $D_G(H_g) \ni g \mapsto g(x) \in \mathbb{C}$ is then continuous for each $x \in (0,1]$ (this is a slight generalization of [12], XIII.2.16). On the other hand $z \mapsto (H_g - z)^{-1} f \in D_G(H_g)$ is analytic in some neighbourhood of λ since $\lambda \in \rho(H_g)$. It follows that the limit of the left-hand side of (21) as $z \to \lambda$ is $[(H_g - \lambda)^{-1} f](1)$.

If $f(x) = 0$ in some neighbourhood of $x = 0$, one can take the limit $z \to \lambda$ on the right-hand side of (21) under the integral sign, which gives for such functions f

$$[(H_g - \lambda)^{-1} f](1) = \int_0^1 [u_2(y, \lambda) - \omega_l(\lambda)\, u_1(y, \lambda)] f(y)\, dy. \tag{22}$$

On the other hand

$$|[(H_g - \lambda)^{-1} f](1)| \leqslant c \|(H_g - \lambda)^{-1} f\|_{D_G(H_g)}$$
$$\leqslant c \|(H_g - \lambda)^{-1}\|_{\mathscr{B}(L^2(0,1), D_G(H_g))} \|f\|_{L^2(0,1)} \leqslant C(\lambda) \|f\|_{L^2(0,1)}$$

where C is a continuous function defined on $\rho(H_g)$.

It follows that the right-hand side of (22) defines a bounded linear functional on a dense subset of $L^2(0,1)$. One may now apply the theorem of F. Riesz ([11], Section 17) to conclude that $u_2(\cdot, \lambda) - \omega_l u_1(\cdot, \lambda) \in L^2(0,1)$ and

$$\|u(\cdot, \lambda)\|_{L^2(0,1)} \leqslant C(\lambda) [1 + \omega_l(\lambda)^2]^{-1/2}.$$

This shows that $\|u(\cdot, \lambda)\|_{L^2(0,1)}$ is a bounded function of λ on every compact set Λ in $(0, \infty)$ such that Λ contains no pole of ω_l. On the other hand, if λ_0 is a pole of ω_l, one has the following estimates valid in some neighbourhood of λ_0: $|C(\lambda)| \leqslant c_1 |\lambda - \lambda_0|^{-1}$ and $|\omega_l(\lambda)| \geqslant c_2 |\lambda - \lambda_0|^{-1}$ with $c_2 \neq 0$. This shows that $\|u(\cdot, \lambda)\|_{L^2(0,1)}$ is also bounded in some neighbourhood of each pole of ω_l and proves the lemma. ■

Lemma 4: Let $R < \infty$, $f \in PL^2(0, \infty)$ such that $\mathscr{F} f$ has compact support in $(0, \infty)$. Then

$$\lim_{|t| \to \infty} \|F_R \exp(-iHt) f\| = 0.$$

Proof: Let Λ be a compact set in $(0, \infty)$ containing the support of $\mathscr{F} f$ and $x > 0$. Then

$$[\exp(-iHt) f](x) = \int_\Lambda e^{-i\lambda t} v(x, \lambda) (\mathscr{F} f)(\lambda) \, d\lambda. \tag{23}$$

It follows from (19), (20) and Remark i) at the end of Section II that $v(x, \lambda)$ is essentially bounded on Λ. Thus by the Riemann–Lebesgue lemma, $[\exp(-iHt) f](x)$ converges to zero as $|t| \to \infty$.

By using the Cauchy–Schwarz inequality we find from (23)

$$|[\exp(-iHt) f](x)|^2 \leqslant \|\mathscr{F} f\|^2 \int_\Lambda |v(x, \lambda)|^2 \, d\lambda$$

Also, from Lemma 3

$$\int_0^R dx \int_\Lambda |v(x, \lambda)|^2 \, d\lambda = \int_\Lambda d\lambda \int_0^R dx |v(x, \lambda)|^2$$
$$= \int_\Lambda \|v(\cdot, \lambda)\|^2_{L^2(0,R)} \, d\lambda < \infty.$$

By applying the Lebesgue dominated convergence theorem, one obtains

$$\lim_{|t| \to \infty} \|F_R \exp(-iHt) f\|^2 = \lim_{|t| \to \infty} \int_0^R |[\exp(-iHt) f](x)|^2 \, dx = 0.$$ ■

Lemma 5: There exists a continuous function $\eta:(0,\infty)\to\mathbb{R}$ such that $v(\cdot,\lambda)=\pi^{-1/2}\lambda^{-1/4}\,\mathrm{Im}[\exp(i\eta(\lambda))f(\cdot,\lambda^{1/2})]$.

Proof: Since $v(\cdot,\lambda)$ is a solution of $(-d^2/dr^2+U(r))f=0$, one has

$$v(\cdot,\lambda)=a_+(\lambda)f(\cdot,\lambda^{1/2})+a_-(\lambda)f(\cdot,-\lambda^{1/2}). \tag{24}$$

Since $v(\cdot,\lambda)$ is real:

$$v(\cdot,\lambda)=2\mathrm{Re}[a_+(\lambda)f(\cdot,\lambda^{1/2})]=2\mathrm{Im}[ia_+(\lambda)f(\cdot,\lambda^{1/2})]. \tag{25}$$

It remains to calculate $b(\lambda)\equiv ia_+(\lambda)$. By (4), (24) and (3) it is given by

$$b(\lambda)=\tfrac{1}{2}\lambda^{-1/2}\,W(f(\cdot,-\lambda^{1/2}),v(\cdot,\lambda))=\tfrac{1}{2}\lambda^{-1/2}\,W(f(\cdot,\lambda^{1/2}),v(\cdot,\lambda))^*. \tag{26}$$

b is a continuous function of λ on $(0,\infty)$ since $f(x,\lambda^{1/2})$, $f'(x,\lambda^{1/2})$, $u(x,\lambda)$ and $\sigma'(\lambda)$ are continuous in λ (for σ' this follows from (11) since ω_l is meromorphic). We may write $b(\lambda)=|b(\lambda)|\exp(i\eta(\lambda))$. $\eta(\lambda)$ can be chosen such as to be continuous, and it remains to show that

$$|b(\lambda)|=\tfrac{1}{2}\pi^{-1/2}\lambda^{-1/4}. \tag{27}$$

For this one uses (20), (15), (11), (19) and the properties of ω_l and easily gets that

$$v(\cdot,\lambda)=\pi^{-1/2}|\omega_l(\lambda)+\omega_r(\lambda+io)|^{-1}[\mathrm{Im}\,\omega_r(\lambda+io)]^{1/2}\cdot[u_2(\cdot,\lambda)-\omega_l(\lambda)\,u_1(\cdot,\lambda)].$$

Upon inserting this into (26), and by using (6) to express $\mathrm{Im}\,\omega_r(\lambda+io)$ and the fact that ω_l is real, one obtains

$$|b(\lambda)|=\tfrac{1}{2}\pi^{-1/2}\lambda^{-1/2}\lambda^{1/4}|W(f(\cdot,\lambda^{1/2}),u_1(\cdot,\lambda))|^{-1}|\,\omega_l(\lambda)+\omega_r(\lambda+io)|^{-1}\cdot$$
$$\cdot|W(f(\cdot,\lambda^{1/2}),u_2(\cdot,\lambda))-\omega_l(\lambda)\,W(f(\cdot,\lambda^{1/2}),u_1(\cdot,\lambda))|.$$

In virtue of (5) the last equation reduces to (27). ∎

We now define

$$v_{as}(x,\lambda)=\pi^{-1/2}\lambda^{-1/4}\sin(\sqrt{\lambda}x+\eta(\lambda)).$$

Lemma 6: Let $\Lambda=[\lambda_1,\lambda_2]\subset(0,\infty)$ be a compact interval. There exists a constant $K=K(\Lambda)$ such that for each $R\geqslant 1$ and for all $h\in L^2(0,\infty)$ having support in Λ:

$$\left\|\int_0^\infty[v(\cdot,\lambda)-v_{as}(\cdot,\lambda)]\,h(\lambda)\,d\lambda\right\|_{L^2(R,\infty)}\leqslant K\|h\|_{L^2(0,\infty)}\int_R^\infty|U(r)|dr. \tag{28}$$

Proof: The differential equation for $v(\cdot,\lambda)$ and the condition that $v(\cdot,\lambda)$ behave asymptotically like $v_{as}(\cdot,\lambda)$ can be combined into the following integral equation

$$v(x,\lambda)-v_{as}(x,\lambda)=\int_x^\infty\lambda^{-1/2}\sin[\lambda^{1/2}(y-x)]\,v(y,\lambda)\,U(y)\,dy. \tag{29}$$

By inserting $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ and using the Fourier Sine and Cosine theorems ([12], XIII.5.32–33) one obtains the following estimates valid for $y \geqslant 1$:

$$\left\| \int_0^\infty d\lambda\, \lambda^{-1/2} \sin[\lambda^{1/2}(y-x)]\, v(y,\lambda)\, h(\lambda) \right\|_{L_x^2(0,\infty)} \leqslant \sqrt{\pi} \|\cos(\lambda^{1/2} y)\, v(y,\lambda)$$

$$\times \lambda^{-1/4} h(\lambda)\|_{L_\lambda^2(0,\infty)} + \sqrt{\pi}\|\sin(\lambda^{1/2}y)\, v(y,\lambda)\, \lambda^{-1/4} h(\lambda)\|_{L_\lambda^2(0,\infty)}$$

$$\leqslant \sqrt{\pi}\lambda_1^{-1/4}\|h\|_{L^2(0,\infty)} \sup_{\substack{y \geqslant 1 \\ \lambda \in \Lambda}} |v(y,\lambda)| \equiv K(\Lambda)\|h\|_{L^2(0,\infty)}. \tag{30}$$

The supremum of $|v(y,\lambda)|$ appearing in (30) is finite since v is continuous in both variables and bounded at large values of y according to Lemma 5. It follows with (29) and the triangle inequality for vector-valued functions that

$$\left\| \int_0^\infty [v(\cdot,\lambda) - v_{as}(\cdot,\lambda)]\, h(\lambda)\, d\lambda \right\|_{L^2(R,\infty)} \leqslant \int_R^\infty |U(y)|\, dy \left\| \int_0^\infty d\lambda\, \lambda^{-1/2} \sin[\lambda^{1/2}(y-x)] \right.$$

$$\left. \times\, v(y,\lambda)\, h(\lambda) \right\|_{L_x^2(0,\infty)}.$$

By combining this inequality with (30), one obtains (28). ■

Theorem 5: Let U and $\mathring{U}$ be such that each of them verifies the hypotheses of Theorem 2. Let H be a selfadjoint extension of $-d^2/dr^2 + U(r)$ and $\mathring{H}$ a selfadjoint extension of $-d^2/dr^2 + \mathring{U}(r)$. Then the wave operators $\Omega_\pm = s\lim \exp(iHt)\exp(-i\mathring{H}t)\, E_{ac}(\mathring{H})$ as $t \to \pm\infty$ exist and are asymptotically complete, i.e. $\Omega_\pm \Omega_\pm^* = E_{ac}(H)$.

Remark: It sufficies to set $\mathring{U} = l(l+1)r^{-2}$ in the above result to obtain the existence and asymptotic completeness of the wave operators of Theorem 4.

Proof: All the quantities defined so far in relation with U can also be defined for $\mathring{U}$, in which case they will be distinguished by a superscript °. The definition

$$\Omega_\pm = \mathscr{F}^{-1} \exp[\mp(\eta - \mathring{\eta})]\, \mathring{\mathscr{F}}$$

gives us two unitary operators from $\mathring{P}L^2(0,\infty)$ onto $PL^2(0,\infty)$. We shall show that they are identical with the time-dependent wave operators, i.e. that

$$\lim_{t\to\pm\infty} \|(\Omega_\pm - I)\exp(-i\mathring{H}t) f\| = 0$$

for a dense set $\mathscr{D}$ of vectors f in $\mathring{P}L^2(0,\infty)$.

For $\mathscr{D}$ we choose the set of functions such that the support of $\mathring{\mathscr{F}}f$ is compact in $(0,\infty)$. If $f \in \mathscr{D}$, the support of $\mathscr{F}\Omega_\pm f$ is also compact in $(0,\infty)$. By virtue of Lemma 4, it suffices to show that for all $f \in \mathscr{D}$ (we consider only the case $t \to +\infty$)

$$\lim_{R\to\infty} \sup_{t \geqslant 0} \|(I - F_R)(\Omega_+ - I)\exp(-i\mathring{H}t) f\| = 0. \tag{31}$$

We have

$$\|(I - F_R)(\Omega_+ - I)\exp(-i\mathring{H}t)f\|$$

$$\leqslant \left\| \int_0^\infty [v(\cdot,\lambda) - v_{as}(\cdot,\lambda)]\, e^{-i\eta(\lambda)+i\mathring{\eta}(\lambda)-i\lambda t}(\mathring{\mathscr{F}}f)(\lambda)\, d\lambda \right\|_{L^2(R,\infty)}$$

$$+ \left\| \int_0^\infty [\mathring{v}(\cdot,\lambda) - \mathring{v}_{as}(\cdot,\lambda)]\, e^{-i\lambda t}(\mathring{\mathscr{F}}f)(\lambda)\, d\lambda \right\|_{L^2(R,\infty)}$$

$$+ \left\| \int_0^\infty [v_{as}(\cdot,\lambda)\, e^{-i\eta(\lambda)} - \mathring{v}_{as}(\cdot,\lambda)\, e^{-i\mathring{\eta}(\lambda)}]\, e^{i\mathring{\eta}(\lambda)-i\lambda t}\cdot(\mathring{\mathscr{F}}f)(\lambda)\, d\lambda \right\|_{L^2(R,\infty)}.$$

As a consequence of Lemma 6, the first two terms in the last member of this inequality converge to zero as $R \to \infty$ uniformly in t. It remains to estimate the third term.

We have

$$v_{as}(x,\lambda)\, e^{-i\eta(\lambda)} - \mathring{v}_{as}(x,\lambda)\, e^{-i\mathring{\eta}(\lambda)} = -(2i\pi^{1/2}\lambda^{1/4})^{-1}[e^{-2i\eta(\lambda)} - e^{-2i\mathring{\eta}(\lambda)}]\, e^{-i\sqrt{\lambda}x}.$$

Thus the above third term has the form

$$\left\| \int_0^\infty e^{-i(\sqrt{\lambda}x+\lambda t)} g(\lambda)\, d\lambda \right\|_{L^2_x(R,\infty)} = \left\| \int_0^\infty e^{-i(\mu x+\mu^2 t)} g(\mu^2)\, 2\mu\, d\mu \right\|_{L^2_x(R,\infty)} \tag{32}$$

where g belongs to $L^2(0,\infty)$ and has compact support in $(0,\infty)$. The function $h(\mu) \equiv 2\sqrt{2\pi}\mu g(\mu^2)$ also has these two properties. Given $\epsilon > 0$, we choose a function $h_1 \in C_0^\infty((0,\infty))$ such that $\|h - h_1\| < \epsilon/2$. (32) is equal to

$$\left\| \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i(\mu x+\mu^2 t)} h(\mu)\, d\mu \right\|_{L^2(R,\infty)}$$

$$\leqslant \|h - h_1\| + \frac{\Lambda}{\sqrt{2\pi}} \left\| \int_0^\infty e^{-i(\mu x+\mu^2 t)} h_1(\mu)\, d\mu \right\|_{L^2(R,\infty)} \tag{33}$$

where we have used Parseval's relation for the Fourier transformation to estimate the first term. In the second term on the right-hand side we integrate by parts with respect to the variable μ and obtain for $t \geqslant 0$

$$\left| \int_0^\infty e^{-i(\mu x+\mu^2 t)} h_1(\mu)\, d\mu \right| \leqslant \int_a^b |h_1'(\mu)|(x + 2\mu t)^{-1}\, d\mu + \int_a^b |h_1(\mu)| 2t(x + 2\mu t)^{-2}\, d\mu \tag{34}$$

where $0 < a < b < \infty$ are such that $\operatorname{supp} h_1 \subset [a,b]$. Since $2t(x + 2\mu t)^{-2} \leqslant (4ax)^{-1}$ for all $t \geqslant 0$ and all $\mu \geqslant a$, the right-hand side of (34) is bounded uniformly in $t \geqslant 0$ by $\text{const}\cdot x^{-1}$. It follows that the second term on the right-hand side of (33) is bounded by $\text{const}\cdot R^{-1/2}$ uniformly in $t \geqslant 0$, so that (32) is less than ϵ uniformly in $t \geqslant 0$ provided that R is sufficiently large. ■

Addendum

In the situation of Theorem 1, suppose $f \in \mathscr{H}_{ac}(HE(-\infty,0))$. If $\Omega_\pm$ exist, f is orthogonal to $\Omega_\pm\mathscr{H}$, since $\sum_{ac}(H_0) = [0,\infty)$. Hence, given $\epsilon > 0$ and $R_0 > 0$, there exists T such that $\|(I - F_R)\exp(-iHt)f\|^2 < \epsilon$ for all $|t| > T$ and all $R > R_0$ [7]. On the other hand $t \mapsto \exp(-iHt)f$ is strongly continuous. Since $[-T,T]$ is compact, the set $\{\exp(-iHt)f \,|\, t \in [-T,T]\}$ is compact in $L^2(\mathbb{R}^+)$ and consequently also in $L^2(\mathbb{R})$. By virtue of the Fréchet–Kolmogorov theorem ([18], p. 275) one has

$$\sup_{t\in[-T,T]} \|(I - F_R)\exp(-iHt)f\|^2 < \epsilon$$

provided $R > R_1 = R_1(\epsilon, T)$. Hence

$$\lim_{R\to\infty} \sup_{t\in\mathbb{R}} \|(I - F_R)\exp(-iHt)f\|^2 = 0$$

i.e. f is a bound state in the sense of the definition of [1].

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