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# On a Generalized Moment Problem

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*Abstract.* The generalized moment problem, (1.1) which arises from analysis of the Froissart–Gribov formula for partial amplitudes is considered. The necessary and sufficient conditions of solvability of problem (1.1) by reducing it to the power problem (3.1) are obtained.

## 1. Introduction

Let us consider the following generalized moment problem[1]): for a given sequence of real numbers $\{f_n\}_0^\infty$ it is necessary to find such a function of bounded variation $\psi(x), x \in [x_0, \infty), x_0 > 1$, that for $n = 0, 1, 2, \ldots$

$$f_n = \int_{x_0}^{\infty} Q_n(x)\, d\psi(x), \tag{1.1}$$

where $Q_n(x)$ is the Legendre function of the second kind, and

$$\int_{x_0}^{\infty} Q_0(x)\,|d\psi(x)| = A < \infty. \tag{1.2}$$

Problem (1.1) will be termed definite if there exists a function $\psi(x)$ (a single function with an accuracy up to a constant term) which satisfies equation (1.1) and condition (1.2).

Problem (1.1) arises from the analysis of the constraints on partial scattering amplitudes which result from the general axioms of quantum field theory [2]. Equation (1.1) may be treated as the Froissart–Gribov representation for the partial amplitudes $f_n$ [3]. One of the approaches to the analysis of the properties of this representation is based on using the results of moment theory [4].

In the present paper we obtain the necessary and sufficient conditions of solvability of the problem (1.1), for which the inequality (1.2) is valid. Problem (1.1) is solved by reducing it to the power moment problem (3.1).

In what follows we assume

$$\psi(x_0) = 0. \tag{1.3}$$

Condition (1.3) may be satisfied by fixing the value of the arbitrary constant contained in $\psi(x)$.

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[1]) We use the term 'generalized moment problem' following [1].

## 2. A Criterion of Solvability of the Power Moment Problem

Let $\{a_n\}_0^\infty$ be a sequence of real numbers. In this section we obtain a criterion of solvability of the power moment problem

$$a_n = \int_0^1 u^n \, d\varphi(u), \quad n = 0, 1, 2, \ldots \tag{2.1}$$

in the class of functions $\varphi(u)$ with bounded variation on $[0,1]$, which will be useful in our further discussion.

Let

$$\Delta^k a_n = \sum_{i=0}^{k} (-1)^i \binom{k}{i} a_{n+i} \tag{2.2}$$

for $k, n = 0, 1, 2, \ldots$.

*Lemma 1* [5]. In order to solve problem (2.1) in the class of functions of bounded variation, it is necessary and sufficient that there exist such a number $M < \infty$ that

$$\sum_{k=0}^{n} \binom{n}{k} |\Delta^{n-k} a_k| \leqslant M \tag{2.3}$$

for all $n = 0, 1, 2, \ldots$ and $M$ be independent of $n$.

*Lemma 2* ([6], page 33). Let an analytic function $h(t)$ be regular in a circle $|t| < 1$ and $\operatorname{Im} h(t) = 0$, $\forall t$: $\operatorname{Im} t = 0$. In order that $h(t)$ admits the representation

$$h(t) = \frac{1+t}{1-t} \int_0^{2\pi} \left\{ 1 + \frac{4t}{(1-t)^2} \sin^2 \frac{\tau}{2} \right\}^{-1} d\gamma(\tau), \tag{2.4}$$

where $\gamma(\tau)$ is the function of bounded variation on $[0, 2\pi]$, it is necessary and sufficient that

$$\sup_r \int_0^{2\pi} |\operatorname{Re} h(r\, e^{i\theta})| \, d\theta < \infty. \tag{2.5}$$

Let $Z$ be a complex plane with a cut $(-\infty, -1]$. The transformation

$$t = (\sqrt{1+z} - 1)(\sqrt{1+z} + 1)^{-1}, \quad \sqrt{1} = 1$$

with inverse $z = 4t(1-t)^{-2}$ maps conformally the region $Z$ on to the circle $|t| < 1$.

*Lemma 3.* Let $f(z)$ be a regular analytical function in $Z$ and $\operatorname{Im} f(z) = 0$, $\forall z$: $\operatorname{Im} z = 0$. In order that $f(z)$ admits the representation

$$f(z) = \sqrt{1+z} \int_0^1 \frac{d\varphi(u)}{1+uz}, \tag{2.6}$$

where $\varphi(u)$ is the function of bounded variation on $[0,1]$, it is necessary and sufficient that the function $h(t) = f[4t(1-t)^{-2}]$ satisfies the condition (2.5).

*Proof.* Necessity. Let $f_0(z,u) = \sqrt{1+z}(1+uz)^{-1}$ and $h_0(t,u) = f_0[4t(1-t)^{-2},u]$. At $u \in [0,1]$ $h_0(t,u)$ is regular in the circle $|t|<1$ and $\operatorname{Re} h_0(t,u) \geqslant 0$. Hence ([6], page 30)

$$\operatorname{Re} h_0(t,u) = \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\tau)+r^2}\, d\gamma_0(\tau,u),$$

where $t = re^{i\theta}$ and $\gamma_0(\tau,u)$ is a non-decreasing function at each fixed $u \in [0,1]$. Thus we obtain

$$\int_0^{2\pi} |\operatorname{Re} h(re^{i\theta})|\, d\theta \leqslant \int_0^{2\pi} d\theta \int_0^1 \operatorname{Re} h_0(re^{i\theta},u)\,|d\varphi(u)| = 2\pi \int_0^1 |d\varphi(u)| \int_0^{2\pi} d\gamma_0(\tau,u)$$

$$\leqslant 2\pi \max_{u\in[0,1]} [\gamma_0(2\pi,u)-\gamma_0(0,u)] \int_0^1 |d\varphi(u)| < \infty$$

and consequently the condition (2.5) is satisfied.

Sufficiency. By virtue of Lemma 2 it follows from (2.5) that

$$f(z) = \sqrt{1+z} \int_0^{2\pi} \left(1 + z\sin^2\frac{\tau}{2}\right)^{-1} d\gamma(\tau),$$

where $\gamma(\tau)$ is the function of bounded variation. In the region $|z|<1$ the expansion

$$f(z) = \sqrt{1+z} \sum_{n=0}^{\infty} (-1)^n a_n z^n \tag{2.7}$$

is valid. Here

$$a_n = \int_0^{2\pi} \sin^{2n}\frac{\tau}{2}\, d\gamma(\tau), \quad n = 0, 1, 2, \ldots.$$

The sequence $\{a_n\}_0^\infty$ satisfies the condition of Lemma 1, since

$$\sum_{k=0}^{n} \binom{n}{k} |\Delta^{n-k} a_k| \leqslant \int_0^{2\pi} |d\gamma(\tau)| < \infty.$$

According to this Lemma, there exists a function of bounded variation $\varphi(u)$ such that

$$a_n = \int_0^1 u^n\, d\varphi(u). \tag{2.8}$$

Substituting (2.8) into the series (2.7) and summing, we arrive at the representation (2.6).

*Theorem 1.* In order for problem (2.1) to have a solution in the class of function of bounded variation, it is necessary and sufficient that the function

$$h(t) = \frac{1+t}{1-t} \sum_{n=0}^{\infty} a_n[-4t(1-t)^{-2}]^n \tag{2.9}$$

be regular in the circle $|t| < 1$ and satisfy condition (2.5).

Indeed, the conditions of the theorem, according to Lemma 3, are necessary and sufficient for the function

$$f(z) = h\left(\frac{\sqrt{1+z}-1}{\sqrt{1+z}+1}\right)$$

to be regular in $Z$ and for the representation (2.6), in which $\varphi(u)$ is the function of bounded variation, to be valid. By expanding the integrand in (2.6) in a power series of $z$ and comparing it with the expansion (2.9) we can see that the theorem is valid.

*Remark 1.* The solution of problem (2.1) is unique, since the sequence $\{u_n\}_0^\infty$ is fundamental in the space $C[0,1]$. The solution is defined with the use of function (2.7) as follows ([6], page 35):

$$\varphi(u) - \varphi(v) = -\frac{1}{\pi} \lim_{y\to+0} \int_v^u s^{-1} \operatorname{Im} f\left(-\frac{1}{s} + iy\right) ds \tag{2.10}$$

for $0 \leqslant v < u \leqslant 1$.

## 3. Some Auxiliary Moment Problems

Let $\{f_n\}_0^\infty$ be a sequence of real numbers and let

$$\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} f_n = \int_0^{u_0} u^n \, d\varphi(u) \tag{3.1}$$

for $n = 0, 1, 2, \ldots$. Here $\Gamma(x)$ is the gamma function, $0 < u_0 < 1$ and $\varphi(u)$ is the function of bounded variation on $[0, u_0]$.

*Lemma 4.* If problem (3.1) is definite, then the problem

$$f_n = \int_0^{u_0} u^n \, d\tilde{\varphi}(u) \tag{3.2}$$

is also definite in the class of functions of bounded variation and

$$\tilde{\varphi}(u) = -\frac{2}{\sqrt{\pi}} \int_u^{u_0} \left(1 - \frac{u}{v}\right) d\varphi(v). \tag{3.3}$$

*Proof.* $\{f_n\}_0^\infty$ satisfies the condition of Lemma 1. Really, since

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}=\frac{1}{\sqrt{\pi}}\int\limits_0^1\frac{s^n}{\sqrt{1-s}}\,ds \tag{3.4}$$

for $n=0, 1, 2, \ldots,$

$$\Delta^{n-k}f_k=\frac{1}{\sqrt{\pi}}\int\limits_0^1\frac{ds}{\sqrt{1-s}}\int\limits_0^{u_0}(us)^k(1-us)^{n-k}\,d\varphi(u)).$$

Hence

$$\sum_{k=0}^n\binom{n}{k}|\Delta^{n-k}f_k|\leqslant\Gamma^{-1}(\tfrac{3}{2})\int\limits_0^{u_0}|d\varphi(u)|<\infty.$$

By virtue of Lemma 1 we conclude that problem (3.2) is definite. Substituting (3.3) into (3.2), we find that for $n=0, 1, 2,$

$$f_n=\frac{1}{\sqrt{\pi}}\int\limits_0^{u_0}d\varphi(u)\int\limits_0^u\frac{v^n\,dv}{\sqrt{u(u-v)}}, \tag{3.5}$$

and here, at $n=0$, the value of the internal integral at the point $u=0$ should be regarded as the limit at $u\to 0$. If in (3.5) we replace $v\to s=v/u$ and use equation (3.4), then we obtain the equality (3.1). Thus the function (3.3) solves problem (3.2).

*Lemma 5.* Let problem (1.1) be definite. In the class of the functions of bounded variation the moment problem

$$f_n=\int\limits_0^{u_0}u^n\,d\chi(u),\quad n=0, 1, 2, \ldots \tag{3.6}$$

where $u_0=x_0-\sqrt{x_0^2-1}<1$ has a unique solution

$$\chi(u)=-\int\limits_{x_0}^{1/2(u+u^{-1})}d\psi(x)\int\limits_u^{x-\sqrt{x^2-1}}(1-2vx+v^2)^{-1/2}\,dv, \tag{3.7}$$

in which it is assumed $(1)^{-1/2}=1$.

*Proof.* Using the integral representation for $Q_n(x)$ ([7], formula 8.822.2), from (1.1) we obtain

$$f_n=\int\limits_{x_0}^{\infty}d\psi(x)\int\limits_0^{\infty}(x+\sqrt{x^2-1}\,cht)^{-n-1}\,dt.$$

The replacement $t\to u=(x+\sqrt{x^2-1}\,cht)^{-1}$ yields

$$f_n=\int\limits_{x_0}^{\infty}d\psi(x)\int\limits_0^{x-\sqrt{x^2-1}}u^n(1-2ux+u^2)^{-1/2}\,du. \tag{3.8}$$

According to the definition (2.2).

$$\Delta^{n-k}(u_0^{-k} f_k) = \int_{x_0}^{\infty} d\psi(x) \int_0^{x-\sqrt{x^2-1}} \frac{(u_0^{-1} u)^k (1 - u_0^{-1} u)^{n-k}}{(1 - 2ux + u^2)^{1/2}},$$

so that

$$\sum_{k=0}^{n} \binom{n}{k} |\Delta^{n-k}(u_0^{-k} f_k)| \leqslant \int_{x_0}^{\infty} |d\psi(x)| \int_0^{x-\sqrt{x^2-1}} (1 - 2ux + u^2)^{-1/2} du = A.$$

By virtue of (1.2) and Lemma 1 we conclude that problem (3.6) has a solution. In order to find it we transform the expression (3.8). For $n \geqslant 1$ we have

$$\int_0^{x-\sqrt{x^2-1}} u^n (1 - 2ux + u^2)^{-1/2} du = n \int_0^{x-\sqrt{x^2-1}} u^{n-1} du \int_u^{x-\sqrt{x^2-1}} (1 - 2vx + v^2)^{-1/2} dv$$

and, consequently,

$$f_n = n \int_{x_0}^{\infty} d\psi(x) \int_0^{x-\sqrt{x^2-1}} u^{n-1} du \int_u^{x-\sqrt{x^2-1}} (1 - 2vx + v^2)^{-1/2} dv.$$

In the latter expression we can change the order of the first two integrations, and so

$$f_n = n \int_0^{u_0} u^{n-1} du \int_{x_0}^{1/2(u+u^{-1})} d\psi(x) \int_u^{x-\sqrt{x^2-1}} (1 - 2vx + v^2)^{-1/2} dv$$

$$= - n \int_0^{u_0} u^{n-1} \chi(u) \, du = \int_0^{u_0} u^n \, d\chi(u).$$

In the case $n = 0$, (3.8) gives

$$f_0 = \int_{x_0}^{\infty} d\psi(x) \int_0^{x-\sqrt{x^2-1}} (1 - 2ux + u^2)^{-1/2} du = -\chi(0) = \int_0^{u_0} d\chi(u).$$

Thus, function (3.7) really solves problem (3.6). Lemma 5 is proved.

*Lemma 6.* Let the sequence $\{f_n\}_0^\infty$ be such that for all $n = 0, 1, 2, \dots$ there exist the values

$$\mu_n = \frac{2^{n+1}}{n! \sqrt{\pi}} \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\Gamma(n + \nu + \frac{3}{2})}{\Gamma(\nu + 1)} f_{n+2\nu}. \tag{3.9}$$

In order for problem (1.1) to be definite, it is necessary and sufficient that there exist a function of bounded variation $\psi(x)$ for which

$$\mu_n = \int_{x_0}^{\infty} x^{-n-1} d\psi(x), \quad n = 0, 1, 2, \dots \tag{3.10}$$

and

$$\int_{x_0}^{\infty} x^{-1}|d\psi(x)| = B < \infty. \tag{3.11}$$

The proof is based on the relation

$$x^{-n-1} = \frac{2^{n+1}}{\sqrt{\pi}n!}\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\Gamma(n+\nu+\frac{3}{2})}{\Gamma(\nu+1)}Q_{n+2\nu}(x) \tag{3.12}$$

which is valid for $x \geqslant x_0$. If problem (1.1) has a solution, then for $f_n$ we obtain the representation (3.8). From this representation we find

$$|f_n| \leqslant u_0^n \int_{x_0}^{\infty} Q_0(x)|d\psi(x)| = Au_0^n. \tag{3.13}$$

Thus, the series (3.9), constructed from the moments (1.1), converges for all $n = 0, 1, 2,$ and using (3.12) we find the representation (3.10) for $\mu_n$. The integral in (3.10) exists, since $Q_n(x) \simeq O(x^{-n-1})$ for $x \gg 1$. Condition (3.11) follows from (1.2). Conversely, if the sequence $\{\mu_n\}_0^\infty$ is a system of moments (3.10), then by virtue of condition (3.11)

$$|\mu_n| \leqslant x_0^{-n} \int_{x_0}^{\infty} x^{-1}|d\psi(x)| = Bx_0^{-n} \tag{3.14}$$

and so the values

$$f_n = \frac{\sqrt{\pi}}{2^{n+1}}\frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}\sum_{\mu=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+\mu+1\right)\Gamma\left(\frac{n}{2}+\mu+\frac{1}{2}\right)}{\Gamma(\mu+1)\,\Gamma(n+\mu+\frac{3}{2})}\mu_{n+2\mu} \tag{3.15}$$

are meaningful for $n = 0, 1, 2, \ldots$. Using the relation

$$Q_n(x) = \frac{\sqrt{\pi}}{2^{n+1}}\frac{\Gamma(n+1)\,x^{-n-1}}{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}\sum_{\mu=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+\mu+1\right)\Gamma\left(\frac{n}{2}+\mu+\frac{1}{2}\right)}{\Gamma(\mu+1)\,\Gamma(n+\mu+\frac{3}{2})}x^{-2\mu} \tag{3.16}$$

$x \geqslant x_0$, we can see that the representation (1.1) holds for the values (3.15), and the relations (3.15) are an inversion of the equalities (3.9). So, the function $\psi(x)$ solves the problem (1.1). This solution is unique, and, by virtue of (3.11), it satisfies the condition (1.2).

*Remark 2.* Lemma 6 was discovered by F. Yndurain [4] while considering problem (1.1) in the class of non-decreasing functions. This lemma gives a criterion of solvability of problem (1.1), as expressed in terms of the conditions on the numerical sequence (3.9). In the next section simpler conditions of solvability of the problem (1.1) will be obtained.

## 4. Solution of the Problem (1.1)

*Theorem 2.* In order that problem (1.1) be definite, it is necessary and sufficient that the sequence $\{f_n\}_0^\infty$ admit the representation (3.1) in which $u_0 = x_0 - \sqrt{x_0^2 - 1}$ and $\varphi(u)$ is the function of bounded variation on $[0, u_0]$.

*Proof.* Necessity. Suppose that problem (1.1) has a solution $\psi(x)$ and it satisfies condition (1.2). Then, by virtue of Lemmas 5 and 6, for the numbers $f_n$, $n = 0, 1, 2, \ldots$ we get the representation (3.8) and the estimate (3.13). Thus in the region $|z| < 1$ we can define the function

$$f_1(z) = \sqrt{1+z} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} f_n(-u_0^{-1} z)^n. \tag{4.1}$$

Using the representation (3.8), we find

$$f_1(z) = \Gamma(\tfrac{3}{2}) \sqrt{1+z} \int_{x_0}^{\infty} d\psi(x) \int_0^{x-\sqrt{x^2-1}} \frac{(1+u_0^{-1} uz)^{-3/2}}{(1-2ux+u^2)^{1/2}} du \tag{4.2}$$

and here $(1 + u_0^{-1} uz)^{-3/2} = 1$ at $u = 0$. $f_1(z)$ is regular in the region $Z$. Let $z = 4t(1-t)^{-2}$ and

$$h_1(t) = f_1[4t(1-t)^{-2}]. \tag{4.3}$$

$h_1(t)$ is regular in a unit circle. It is proved in the Appendix that $h_1(t)$ satisfies condition (2.5). By virtue of Theorem 1 there exists a function of bounded variation $\varphi_1(v)$, $0 \leqslant v \leqslant 1$ such that there holds the representation

$$\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} u_0^{-n} f_n = \int_0^1 v^n \, d\varphi_1(v),$$

i.e., the representation (3.1), in which $\varphi(u) = \varphi_1(u_0^{-1} u)$ is the function of bounded variation on $[0, u_0]$.

Sufficiency. Suppose that the sequence $\{f_n\}_0^\infty$ admits the representation (3.1). We can easily see that it satisfies the conditions of Lemma 6. According to Lemma 4, for $f_n$ we have the representation (3.2) and the estimate

$$|f_n| < u_0^n \int_0^{u_0} |d\tilde{\varphi}(u)|,$$

so that the series (3.9) converges and

$$\mu_n = \frac{2}{\pi}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \int_0^{u_0} d\varphi(u) \int_0^u \left(\frac{2v}{1+v^2}\right)^n \frac{dv}{\sqrt{u(u-v)}}. \tag{4.4}$$

Moreover, in this expression the value of the internal integral at $u=0$ should be regarded as the limit for $u \to 0$. From (4.4) we obtain

$$|\mu_n| \leqslant \frac{4}{\sqrt{\pi}}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} x_0^{-n} \int_0^{u_0} |d\varphi(u)|$$

and, consequently, the series

$$f_2(z) = \sqrt{1+z} \sum_{n=0}^{\infty} \mu_n (-x_0 z)^n \tag{4.5}$$

converges uniformly in the circle $|z| < 1$. Substituting (4.4) into (4.5), we get

$$f_2(z) = \frac{1}{\sqrt{\pi}} \int_0^{u_0} d\varphi(u) \int_0^u \left(1 + \frac{2vx_0 z}{1+v^2}\right)^{-3/2} \frac{dv}{\sqrt{u(u-v)}}, \tag{4.6}$$

where it is assumed that

$$\left(1 + \frac{2vx_0 z}{1+v^2}\right)^{-3/2}\Bigg|_{v=0} = 1.$$

$f_2(z)$ is regular in $Z$ and $\operatorname{Im} f_2(z) = 0$, $\forall z: \operatorname{Im} z = 0$. It is proved in the Appendix that the function $h_2(t) = f_2[4t(1-t)^{-2}]$ satisfies condition (2.5). Thus, $f_2(z)$ satisfies the conditions of Lemma 3. The latter implies that there exists a function of bounded variation, $\varphi_2(u)$, $0 \leqslant u \leqslant 1$ such that a representation of the type (2.6) holds for $f_2(z)$. In this representation we set $u = x_0 x^{-1}$ and

$$\psi(x) = -\int_{x_0}^{x} s\,d\varphi_2(x_0 s^{-1}) \tag{4.7}$$

obtaining

$$f_2(z) = \sqrt{1+z} \int_{x_0}^{\infty} \frac{d\psi(x)}{x(1+x_0 x^{-1} z)}. \tag{4.8}$$

The right-hand side of (4.8) can be expanded in a power series of $z$. By comparing it with the expansion (4.5) we see that the numbers $\mu_n$, $n = 0, 1, 2, \ldots$ admit the representation (3.10). Then

$$\int_{x_0}^{\infty} x^{-1} |d\psi(x)| = \int_0^1 |d\varphi_2(u)| < \infty,$$

i.e. condition (3.11) is met. Thus, $\{f_n\}_0^\infty$ satisfies the conditions of Lemma 6. The latter asserts that $\psi(x)$ is a unique solution of (1.1) and condition (1.2) is valid for this function. Theorem 2 is proved. By combining Theorem 2 with Lemma 1 we obtain the criterion of solvability of problem (1.1) in a different form.

*Theorem 3.* In order for problem (1.1) to be definite, it is necessary and sufficient that there exist a constant $N$, independent of $n$, such that

$$\sum_{k=0}^{n}\binom{n}{k}\left|\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}\frac{\Gamma(k+i+\frac{3}{2})}{\Gamma(k+i+1)}u_0^{-(k+i)}f_{k+i}\right|<N \tag{4.9}$$

for all $n=0, 1, 2, \ldots$.

*Theorem 4.* The solutions of problems (1.1) and (3.1) are related by

$$\varphi(u)=\theta(u_0-u)\int_{x_0}^{1/2(u+u^{-1})}\varphi_0(u,x)\,d\psi(x), \tag{4.10}$$

$$\psi(x)=\theta(x-x_0)\int_{x-\sqrt{x^2-1}}^{u_0}\psi_0(x,u)\,d\varphi(u), \tag{4.11}$$

where

$$\varphi_0(u,x)=-\frac{\theta(b-u)}{\sqrt{\pi}}\int_u^b\left\{\left(\frac{s}{u}-1\right)^{-1/2}+\operatorname{arctg}\sqrt{\frac{s}{u}-1}\right\}\frac{ds}{\sqrt{(b-s)(b^{-1}-s)}} \tag{4.12}$$

$$\psi_0(x,u)=\frac{\theta(u-b)}{\pi^{3/2}u^{1/2}}\int_b^u s^{-1}\{(u-s)(s-b)(b^{-1}-s)\}^{-1/2}\,ds \tag{4.13}$$

where $b=x-\sqrt{x^2-1}$ and

$$\theta(y)=\begin{cases}1, & y>0\\ 0, & y\leqslant 0.\end{cases}$$

*Proof.* Suppose that $\{f_n\}_0^\infty$ has the representation (1.1). Let us show that the function (4.10) solves problem (3.1). It is sufficient to show that

$$\int_0^{u_0}\frac{d\varphi(u)}{1-uz}=\int_{x_0}^{\infty}f_0(z,x)\,d\psi(x),$$

where

$$f_0(z,x)=\Gamma(\tfrac{3}{2})\int_0^b\frac{(1-uz)^{-3/2}}{\sqrt{(b-u)(b^{-1}-u)}}.$$

We have

$$\int_0^{u_0} \frac{d\varphi(u)}{1-uz} = -\varphi(0) - \int_0^{u_0} \frac{d}{du}\left(\frac{1}{1-uz}\right) \int_0^{1/2(u+u^{-1})} \varphi_0(u,x)\, d\psi(x)$$

$$= -\int_{x_0}^{\infty} \varphi_0(0,x)\, d\psi(x) - \int_{x_0}^{\infty} d\psi(x) \int_0^b \varphi_0(u,x) \frac{d}{du}\left(\frac{1}{1-uz}\right) du$$

$$= \int_{x_0}^{\infty} d\psi(x) \left\{ \frac{\varphi_0(u,x)}{1-uz}\bigg|_0^b - \int_0^b \varphi_0(u,x) \frac{d}{du}\left(\frac{1}{1-uz}\right) du \right\}$$

$$= \int_{x_0}^{\infty} d\psi(x) \int_0^b \frac{d\varphi_0(u,x)}{1-uz}$$

and here $x$ is fixed in the internal integral. It is still to be shown that

$$\int_0^b \frac{d\varphi_0(u,x)}{1-uz} = f_0(z,x).$$

This equality can be directly verified by using expression (4.12). The second relation of the theorem is checked in a similar manner.

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## Appendix

In this section we prove the properties of the functions $h_1(t)$ and $h_2(t)$ used in proving Theorem 2.

*Lemma 7.* The functions $h_1(t)$ and $h_2(t)$ satisfy the condition (2.5).

*Proof.* According to (4.2) and (4.3), we have

$$h_1(t) = \Gamma(\tfrac{3}{2})\sqrt{1+z} \int_{x_0}^{\infty} d\psi(x) \int_0^b \frac{(1+u_0^{-1}uz)^{-3/2}}{\sqrt{(b-u)(b^{-1}-u)}}, \tag{5.1}$$

where $z = 4t(1-t)^{-2}$, $b = x - \sqrt{x^2-1}$. The internal integral is calculated by parts:

$$\int_0^b \frac{(1+u_0^{-1}uz)^{-3/2}}{\sqrt{(b-u)(b^{-1}-u)}}\,du = \frac{1}{1+bu_0^{-1}z}\left\{2b + \int_0^b \left(\frac{b-u}{1+u_0^{-1}zu}\right)^{1/2} \frac{du}{(b^{-1}-u)^{3/2}}\right\}$$

$$= \frac{1}{1+bu_0^{-1}z}\left\{\frac{2b}{\sqrt{1-b^2}} + \int_0^b \left[\left(\frac{b-u}{1+u_0^{-1}uz}\right)^{1/2} - b^{1/2}\right] \times \frac{du}{(b^{-1}-u)^{3/2}}\right\}.$$

Thus

$$h_1(t) = h_1^{(1)}(t) + h_1^{(2)}(t), \tag{5.2}$$

where

$$h_1^{(1)}(t) = \Gamma(\tfrac{3}{2}) \int_{x_0}^{\infty} \frac{2b}{\sqrt{1-b^2}}\,\frac{\sqrt{1+z}}{1+bu_0^{-1}z}\,d\psi(x), \tag{5.3}$$

$$h_1^{(2)}(t) = -\Gamma(\tfrac{3}{2})\sqrt{1+z}\int_{x_0}^{\infty} d\psi(x) \int_0^b \frac{u(b^{-1}-u)^{-3/2}\,du}{(1+u_0^{-1}uz)^{1/2}(\sqrt{b-u}+\sqrt{b}\sqrt{1+u_0^{-1}uz})}. \tag{5.4}$$

If we denote

$$f_1^{(1)}(z) = h_1^{(1)}\left(\frac{\sqrt{1+z}-1}{\sqrt{1+z}+1}\right),$$

then from (5.3) we obtain that for $f_1^{(1)}(z)$ there holds the representation (2.6) in which

$$\varphi(u) = \int_u^1 \frac{2uu_0}{\sqrt{1-(uu_0)^2}}\,d\psi\left[\tfrac{1}{2}\left(uu_0 + \frac{1}{uu_0}\right)\right]$$

and $\varphi(u)$ is a function of bounded variation, since

$$\int_0^1 |d\varphi(u)| = \int_{x_0}^{\infty} \frac{2b}{\sqrt{1-b^2}}\,|d\psi(x)| < \infty$$

because of condition (1.2). $f_1^{(1)}(z)$ is regular in $Z$. Thus, $f_1^{(1)}(z)$ satisfies the conditions of Lemma 3.

This Lemma implies that $h_1^{(1)}(t)$ meets condition (2.5). Let us now show that

$$\sup_r \int_0^{2\pi} |h_1^{(2)}(re^{i\theta})|\,d\theta < \infty. \tag{5.5}$$

From this it will follow that $h_1^{(2)}(t)$ meets condition (2.5). From (5.4) we obtain, by integrating by parts, that for $z \in Z$

$$h_1^{(2)}(t) = \Gamma(\tfrac{3}{2}) \sqrt{1+z} \int_{x_0}^{\infty} \psi(x) \frac{dI_1(z,x)}{dx} du, \tag{5.6}$$

where $I_1(z,x)$ denotes the internal integral in (5.4). In the region of bounded $|z|$ one can obtain the estimate[2])

$$\left|\frac{dI_1}{dx}\right| \leqslant \frac{C_1}{|1+bu_0^{-1}z|} \sqrt{\frac{b}{x^2-1}} \left\{\sqrt{b} + \int_0^b \frac{du}{(b-u)^{1/2}|1+uu_0^{-1}z|^{1/2}}\right\}. \tag{5.7}$$

For arbitrary $0 < \epsilon < \frac{1}{2}$ we have

$$I_2(z,x) = \int_0^b \frac{(b-u)^{-1/2}\,du}{|1+uu_0^{-1}z|^{1/2}} = \sqrt{b} \int_0^1 \frac{(1-v)^{-1/2}}{|1+vbu_0^{-1}z|^{1/2}} dv$$

$$\leqslant \sqrt{b} \int_0^1 v^{\varepsilon-1}(1-v)^{-1/2-\varepsilon}(1+vbu_0^{-1}z)^{-1/4}(1+vbu_0^{-1}\bar{z})^{-1/4}\,dv.$$

Using equations (3.211) and (9.182.1) of [7], we find that the last integral is equal to the function

$$B(\epsilon_1 \tfrac{1}{2} - \epsilon)(1+bu_0^{-1}z)^{-\varepsilon} F\left[\epsilon, \tfrac{1}{4}; \tfrac{1}{2}; \frac{bu_0^{-1}(z-\bar{z})}{1+bu_0^{-1}z}\right].$$

For sufficiently small $\epsilon$ the hypergeometric function $F(\epsilon, \frac{1}{4}; \frac{1}{2}; y)$ is uniformly bounded over the entire plane $y$, and so

$$I_2(z,x) \leqslant C_2\sqrt{b}|1+bu_0^{-1}z|^{-\varepsilon}. \tag{5.8}$$

With (5.8) taken into account, (5.7) yields

$$\left|\frac{dI_1}{dx}\right| \leqslant \frac{bC_3}{\sqrt{x^2-1}}|1+bu_0^{-1}z|^{-1-\varepsilon}. \tag{5.9}$$

Using the representation (5.6), $h_1^{(2)}(t)$ can now be estimated as follows

$$|h_1^{(2)}(t)| \leqslant C_4|1+z|^{1/2} \int_{x_0}^{\infty} \frac{b}{\sqrt{x^2-1}} |1+bu_0^{-1}z|^{-1-\varepsilon}\,dx = C_4 u_0 |1+z|^{1/2} I_3(z),$$

2) In what follows $C_i$ denote the constants.

where

$$I_3(z) = \int_0^1 |1 + tz|^{-1-\varepsilon}\, dt \leqslant \int_0^1 t^{\varepsilon-1}(1 + tz)^{-(1+\varepsilon)/2}(1 + t\bar{z})^{-(1+\varepsilon)/2}\, dt$$

$$= B(\epsilon, 1)(1 + z)^{-\varepsilon} F\left(\epsilon, \frac{1+\epsilon}{2}; 1 + \epsilon; \frac{z - \bar{z}}{1 + z}\right).$$

For $0 < \epsilon < \frac{1}{2}$

$$F\left(\epsilon, \frac{1+\epsilon}{2}; 1 + \epsilon; y\right)$$

is uniformly bounded, so that

$$I_3(z) \leqslant C_5|1 + z|^{-\varepsilon}.$$

Thus the estimate

$$|h_1^{(2)}(t)| \leqslant C_6|1 + z|^{1/2-\varepsilon} \qquad (5.10)$$

holds for all $z$ of an arbitrary bounded region. One can also obtain from (5.4) that in the region of sufficiently large $|z|$, $|h_1^{(2)}(t)| \leqslant C_7|z|^{-1}$. So, the function $h_1^{(2)}(t)$ is uniformly bounded in a closed unit circle, and therefore it satisfies condition (5.5).

Since both the functions $h_1^{(i)}(t)$, $i = 1, 2$, satisfy condition (2.5), it is also met by $h_1(t)$, according to (5.2).

The fact that condition (2.5) is also satisfied for the function $h_2(t)$ may be proved in a similar way.

Lemma 7 is proved.

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