BCS-model in perturbation theory

Helvetica Physica Acta
Vol. 47, 1974.

Birkhäuser Verlag Basel

BCS-Model in Perturbation Theory

by **Irja Nieminen**[1])

ETH, Hönggerberg, Zürich, Switzerland

(8. X. 74)

Abstract. The BCS-model for superconductivity is investigated in the perturbation theory. The partial sum over ladder graphs in the quantum statistical perturbation expansion for the reduced density matrix (RDM) is examined for the case $n = 2$ (two particles created and two annihilated). We show that this partial sum can be simplified to a sum of three geometrical series. These converge absolutely at high enough temperatures for any coupling strength. At low temperatures the expansion diverges for attractive potentials, when the total momentum of the particle pair approaches zero, however weak the coupling may be. If the coupling is weak enough the expansion also converges at low temperatures when the total momentum of the pair differs sufficiently from zero. We also get Thouless' [1] result that from this theory one can define a critical temperature which for the BCS-potential leads to the familiar equation for the critical temperature in the BCS-theory.

Introduction

To investigate the BCS-model in perturbation theory we examine the quantum statistical perturbation expansion of the following RDM [2]

$$V^2 \langle\!\langle a_V^*(\mathbf{k}_1, \tau_1, \uparrow)\, a_V^*(\mathbf{k}_2, \tau_2, \downarrow)\, a_V(\mathbf{k}_3, \tau_3, \downarrow)\, a_V(\mathbf{k}_4, \tau_4, \uparrow)\rangle\!\rangle_V. \tag{I.1}$$

Here $\langle\!\langle \cdots \rangle\!\rangle_V$ denotes the thermodynamic expectation value in the grand canonical ensemble for a particle system enclosed in a box V.

$$a_V^{\#}(\mathbf{k}_i, \tau_i, \uparrow) = \exp[\tau_i(H_V - \mu N_V)]\, a_V^{\#}(\mathbf{k}_i, \uparrow) \exp[-\tau_i(H_V - \mu N_V)], \tag{I.2}$$

where $a_V^{\#}(\mathbf{k}_i, \uparrow)$ is either a creation (a^*) or an annihilation (a) operator in a volume V for a particle with 'spin up' and momentum $\mathbf{k}_i$ ($\downarrow$ means 'spin down'), H_V is the total Hamiltonian, N_V the particle number operator and μ the chemical potential.

In the perturbation expansion of (I.1) we include only ladder graphs of the type shown in Figure 1.1. We shall be able to simplify this partial expansion to a geometrical series. The convergence conditions for this series are then examined. We use for the two-body potential a Yukawa-like potential. It is found that the series converges absolutely at high enough temperatures for any coupling strength. Then we investigate the expansion at very low temperatures. We carry out calculations under the assumptions that the particle density is low and that the incoming and outgoing particle energies are not too far from the Fermi surface. It is found that the expansion diverges for attractive potentials, when the total momentum of the incoming pair $\mathbf{k}_1 + \mathbf{k}_2$ (= the

---

[1]) Permanent address: Department of Theoretical Physics, University of Helsinki.

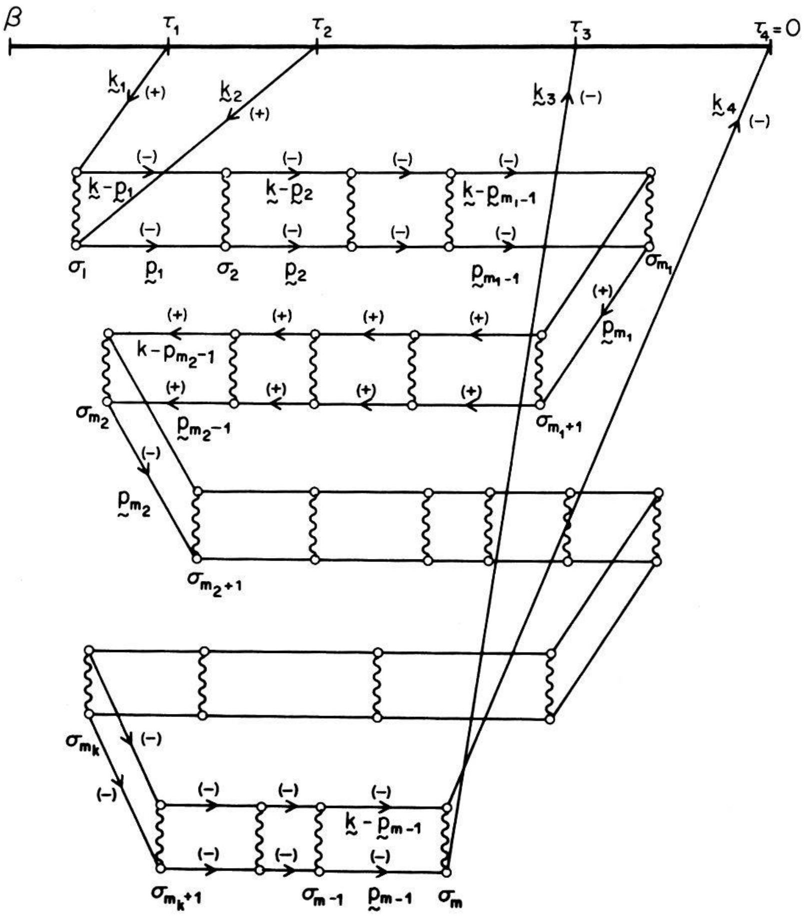

Figure 1.1

total momentum of the outgoing pair $\mathbf{k}_3 + \mathbf{k}_4$) approaches zero. This phenomenon is independent of the energies of the incoming and outgoing particles. Thus the divergence also occurs in a neighborhood above the Fermi surface. When $|\mathbf{k}_1 + \mathbf{k}_2|$ is larger than a certain k_0, the expansion also converges at low temperatures, if the coupling is weak enough (k_0 depends on the coupling constant).

The use of perturbation theory in superconductivity has earlier been investigated by Thouless [1]. He did not find the simple geometrical series which we have below. Instead he found that the sum of our partial expansion satisfies a simple integral equation. From this integral equation he deduces a long list of properties for its solution. His main results is that the condition of convergence of the expansion can be used to define a critical temperature. This turns out to be identical with the critical temperature in the BCS-model [3, 4]. We also find this result from our expansion.

In Section 2 we simplify also the perturbation expansion of the pressure in the ladder approximation to a simple geometrical series. In Section 3 we make use of the results obtained in Section 1 to calculate in ladder approximation the correlation function just above the critical temperature.

1. The Ladder Expansion for the RDM with $n = 2$

The BCS-model is obtained as follows. Consider a state consisting of electron pairs interacting through an attractive two-body potential near the Fermi surface. The electrons in a pair have opposite momenta with equal magnitudes. A physical state consisting of such pairs can be shown to have a lower energy than the state without interaction. This state is then interpreted to correspond to the superconductive state. This would suggest to investigate the RDM for $n=2$ in the ladder approximation. The RDM for $n=2$ is given by (I.1) with $\beta > \tau_1 \geqslant \tau_2 \geqslant \tau_3 \geqslant \tau_4 \geqslant 0$ and $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$. The interaction part of the Hamiltonian is

$$U_V = \frac{1}{V} \sum_{\mathbf{p}_1,\ldots,\mathbf{p}_4} U(\mathbf{p}_1 - \mathbf{p}_4)\, \delta(\mathbf{p}_1 + \mathbf{p}_2, \mathbf{p}_3 + \mathbf{p}_4)\, a_V^*(\mathbf{p}_1, \uparrow)\, a_V^*(\mathbf{p}_2, \downarrow)\, a_V(\mathbf{p}_4, \uparrow)\, a_V(\mathbf{p}_3, \downarrow). \tag{1.1}$$

The ladder approximation of (I.1) includes only graphs of the type shown in Figure 1.1. We use the same Feynman diagrams as in [2] (the particle lines have definite signs). We shall be able to simplify this partial expansion to a sum of three geometrical series. From the convergence condition of these series we shall obtain a definition for the critical temperature which agrees with the critical temperature in the BCS-model. This result was also obtained in Thouless' work [1]. The convergence properties of our expansion are then investigated as described in the introduction.

All essential features of our problem are retained, and a lot of inessential complications are avoided, if we put in (I.1) $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau$. In fact, it is just this simplified function which enters the Thouless thermodynamic potential. From the cyclic invariance of the trace it follows that one can put $\tau_4 = 0$ in (I.1) without restriction. Thus (I.1) with equal τ's becomes equal to

$$V^2 \langle\!\langle a_V^*(\mathbf{k}_1, 0, \uparrow)\, a_V^*(\mathbf{k}_2, 0, \downarrow)\, a_V(\mathbf{k}_3, 0, \downarrow)\, a_V(\mathbf{k}_4, 0, \uparrow) \rangle\!\rangle_V. \tag{1.2}$$

We recall that the $\mathbf{k}_i (i = 1, \ldots, 4)$ satisfy $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 = \mathbf{k}$. Take $\mathbf{k}_2$, $\mathbf{k}_3$ and $\mathbf{k}$ as independent variables among these quantities. Denote the ladder approximation of (1.2) by $V\delta(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_3 + \mathbf{k}_4)\, L_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k}, \tau)$, where we included also the variable τ to remind us that (1.2) equals (I.1) for equal τ's. We get according to [2] and Fig. 1.1

$$\begin{aligned} L_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k}, \tau) = {} & f^-(\mathbf{k}_3) f^-(\mathbf{k} - \mathbf{k}_3) V \delta_{\mathbf{k}_2\mathbf{k}_3} + f^+(\mathbf{k}_2) f^+(\mathbf{k} - \mathbf{k}_2) f^-(\mathbf{k}_3) f^-(\mathbf{k} - \mathbf{k}_3) \\ & \times \sum_{m=1}^{\infty} (-1)^m \sum_{\substack{\text{different} \\ \text{graphs with} \\ \text{fixed } m}} \frac{1}{V^{m-1}} \sum_{\mathbf{p}_1,\ldots,\mathbf{p}_{m-1}} \prod_{j=0}^{m-1} U(\mathbf{p}_j - \mathbf{p}_{j+1}) \\ & \times \prod_{i=1}^{m-1} f^{\pm}(\mathbf{p}_i) f^{\pm}(\mathbf{k} - \mathbf{p}_i)\, I_m, \end{aligned} \tag{1.3}$$

where $\mathbf{p}_0 = \mathbf{k}_2$, $\mathbf{p}_m = \mathbf{k}_3$,

$$f^+(\mathbf{p}) = \frac{1}{1 + \exp[\beta(\mu - \mathbf{p}^2)]}, \quad f^-(\mathbf{p}) = \frac{1}{1 + \exp[\beta(\mathbf{p}^2 - \mu)]}, \tag{1.4}$$

$$\begin{aligned} I_m = {} & \int_0^\beta d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \cdots \int_0^{\sigma_{m_1-1}} d\sigma_{m_1} \int_{\sigma_{m_1}}^{\beta} d\sigma_{m_1+1} \cdots \int_{\sigma_{m_2-1}}^{\beta} d\sigma_{m_2} \int_0^{\sigma_{m_2}} d\sigma_{m_2+1} \cdots \int_{\sigma_{m_k-1}}^{\beta} d\sigma_{m_k} \\ & \times \int_0^{\sigma_{m_k}} d\sigma_{m_k+1} \cdots \int_0^{\sigma_{m-1}} d\sigma_m \exp[\sigma_1(E_1 - E_0)] \exp[\sigma_2(E_2 - E_1)] \cdots \\ & \times \exp[\sigma_{m-1}(E_{m-1} - E_{m-2})] \exp[\sigma_m(E_0' - E_{m-1})], \end{aligned} \tag{1.5}$$

$$E_0 = \mathbf{k}_2^2 + (\mathbf{k} - \mathbf{k}_2)^2, \quad E_0' = \mathbf{k}_3^2 + (\mathbf{k} - \mathbf{k}_3)^2 \tag{1.6}$$

and

$$E_j = \mathbf{p}_j^2 + (\mathbf{k} - \mathbf{p}_j)^2 \quad (j = 1, 2, \ldots, m-1). \tag{1.7}$$

The last equations imply that we consider the unperturbed system to consist of free electrons with a certain effective mass. We use units in which this mass is put equal to 1/2 and $\hbar = 1$. The signs in the propagator product and the integration limits in I_m depend on the graph. We have to sum in (1.3) for fixed m over all possible choices for integers in

$$0 \leqslant m_1 < m_2 < m_3 < \cdots < m_k \leqslant m. \tag{1.8}$$

To do this we make use of the formula

$$f^-(\mathbf{p}) = \exp[\beta(\mu - \mathbf{p}^2)] f^+(\mathbf{p})$$

and write

$$\prod_{i=1}^{m-1} f^\pm(\mathbf{p}_i) f^\pm(\mathbf{k} - \mathbf{p}_i) = \prod_{i=1}^{m-1} f^+(\mathbf{p}_i) f^+(\mathbf{k} - \mathbf{p}_i) \times \prod_{\substack{(-)-\text{lines} \\ \text{only}}} \exp[\beta(2\mu - E_j)] \tag{1.9}$$

Then we can in a simple manner sum over different choices (1.8) to obtain

$$L_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k}, \tau) = f^-(\mathbf{k}_3) f^-(\mathbf{k} - \mathbf{k}_3)\, V\delta_{\mathbf{k}_2\mathbf{k}_3} + f^+(k_2) f^+(\mathbf{k} - \mathbf{k}_2) f^-(\mathbf{k}_3) f^-(\mathbf{k} - \mathbf{k}_3)$$
$$\times \sum_{m=1}^{\infty} (-1)^m \frac{1}{V^{m-1}} \sum_{\mathbf{p}_1, \ldots, \mathbf{p}_{m-1}} \prod_{j=0}^{m-1} U(\mathbf{p}_j - \mathbf{p}_{j+1}) \prod_{i=1}^{m-1} f^+(\mathbf{p}_i) f^+(\mathbf{k} - \mathbf{p}_i)\, K_m, \tag{1.10}$$

where

$$K_m = \int_0^\beta d\sigma_1 \exp[\sigma_1(E_1 - E_0)]$$
$$\times \prod_{j=1}^{m-1} \left[\int_{\sigma_j}^{\beta} d\sigma_{j+1} + \exp[\beta(2\mu - E_j)] \int_0^{\sigma_j} d\sigma_{j+1} \right] \exp[\sigma_{j+1}(E_{j+1} - E_j)] \tag{1.11}$$

with $E_m = E_0'$. To calculate K_m for general m one needs identity (A.1) from Appendix A. The obtained result can be brought into a more convenient form with the aid of identity (A.2). Thus we finally have

$$K_m = \frac{\exp[\beta(2\mu - E_0)] - 1}{E_0' - E_0} \prod_{j=1}^{m-1} \frac{\exp[\beta(2\mu - E_j)] - 1}{E_0' - E_j}$$
$$+ \frac{\exp[\beta(E_0' - E_0)] - \exp[\beta(2\mu - E_0)]}{E_0' - E_0} \prod_{j=1}^{m-1} \frac{\exp[\beta(2\mu - E_j)] - 1}{E_0 - E_j}$$
$$+ \{\exp[\beta(E_0' - 2\mu)] - 1\}\{1 - \exp[-\beta(E_0 - 2\mu)]\}$$
$$\times \sum_{\nu=1}^{m-1} \frac{\exp[\beta(2\mu - E_\nu)]}{(E_0 - E_\nu)(E_0' - E_\nu)} \prod_{\substack{j=1 \\ j \neq \nu}}^{m-1} \frac{\exp[\beta(2\mu - E_j)] - 1}{E_\nu - E_j}. \tag{1.12}$$

Note that the sum of the two first terms in (1.12) approaches

$$\beta \prod_{j=1}^{m-1} \frac{\exp[\beta(2\mu - E_j)] - 1}{E_0 - E_j} \tag{1.13}$$

when $E_0' \to E_0$. In the present problem it is convenient to replace $U(\mathbf{p}_j - \mathbf{p}_{j+1})$ by a separable potential

$$\lambda v^*_{\mathbf{p}_j} v_{\mathbf{p}_{j+1}}. \tag{1.14}$$

This type of potential was also used by Thouless [1]. In (1.10) we go to the limit $V \to \infty$ by replacing

$$\frac{1}{V}\sum_{\mathbf{p}} \to \frac{1}{(2\pi)^3}\int d^3 p \tag{1.15}$$

With the aid of (1.12), (1.14) and (1.15) we get (1.10) in the form

$$L_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k},\tau) = (2\pi)^3\,\delta(\mathbf{k}_2 - \mathbf{k}_3) f^-(\mathbf{k}_3) f^-(\mathbf{k} - \mathbf{k}_3) + L^{(1)}_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k}) + L^{(2)}_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k}) + L^{(3)}_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k}), \tag{1.16}$$

where

$$L^{(1)}_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k}) = f^+(\mathbf{k}_2) f^+(\mathbf{k} - \mathbf{k}_2) f^-(\mathbf{k}_3) f^-(\mathbf{k} - \mathbf{k}_3)\, v^*_{\mathbf{k}_2} v_{\mathbf{k}_3} \frac{\exp[\beta(2\mu - E_0)] - 1}{E_0' - E_0}$$
$$\times \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{(2\pi)^{3(m-1)}} \left(\int d^3 p \frac{|v_{\mathbf{p}}|^2 f^+(\mathbf{p}) f^+(\mathbf{k} - \mathbf{p}) \{\exp[\beta(2\mu - E)] - 1\}}{E_0' - E} \right)^{m-1}. \tag{1.17}$$

The expression for $L^{(2)}_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k})$ is analogous with (1.17).

$$L^{(3)}_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k}) = f^+(\mathbf{k}_2) f^+(\mathbf{k} - \mathbf{k}_2) f^-(\mathbf{k}_3) f^-(\mathbf{k} - \mathbf{k}_3)\, v^*_{\mathbf{k}_2} v_{\mathbf{k}_3} \{\exp[\beta(E_0' - 2\mu)] - 1\}$$
$$\times \{1 - \exp[-\beta(E_0 - 2\mu)]\} \sum_{m=1}^{\infty} \frac{(-\lambda)^m (m-1)}{(2\pi)^{3(m-1)}}$$
$$\times \int d^3 p_1 \frac{|v_{\mathbf{p}_1}|^2 f^-(\mathbf{p}_1) f^-(\mathbf{k} - \mathbf{p}_1)}{(E_0 - E_1)(E_0' - E_1)}$$
$$\times \left(\int d^3 p \frac{|v_{\mathbf{p}}|^2 f^+(\mathbf{p}) f^+(\mathbf{k} - \mathbf{p}) \{\exp[\beta(2\mu - E)] - 1\}}{E_1 - E} \right)^{m-2}. \tag{1.18}$$

Here $E = \mathbf{p}^2 + (\mathbf{k} - \mathbf{p})^2$ and $E_1 = \mathbf{p}_1^2 + (\mathbf{k} - \mathbf{p}_1)^2$. Thus we have been able to simplify the ladder expansion of (1.2) to a sum of three geometrical series. The simultaneous convergence of these series depends on the behaviour of the function

$$F(x, k; \beta) = \int d^3 p \frac{|v_{\mathbf{p}}|^2 f^+(\mathbf{p}) f^+(\mathbf{k} - \mathbf{p}) \{\exp[\beta(2\mu - E)] - 1\}}{x - E}. \tag{1.19}$$

One would expect by inspection that F has a maximum for $x = 2\mu$. It can indeed be proved for the BCS-potential that $F(x,k;\beta)$, with β fixed, has an extremum for $x = 2\mu$, $k = 0$. The proof is identical with that given for Theorem B.1 in Appendix B. If we consider $x = 2\mu$, $k = 0$ as the maximum point for $|F(x,k;\beta)|$, then we get the following sufficient condition for both (1.17) and (1.18) to converge

$$\frac{|\lambda|}{2(2\pi)^3}\int d^3p|v_{\mathbf{p}}|^2\,\frac{tgh[\frac{1}{2}\beta(\mathbf{p}^2-\mu)]}{\mathbf{p}^2-\mu} < 1. \tag{1.20}$$

This is identical with Thouless' condition (18) which was obtained by a different approach. Now one can define a critical temperature by the equation

$$\frac{|\lambda|}{2(2\pi)^3}\int d^3p|v_{\mathbf{p}}|^2\,\frac{tgh[\frac{1}{2}\beta_{cr}(\mathbf{p}^2-\mu)]}{\mathbf{p}^2-\mu} = 1. \tag{1.21}$$

If the potential is specified to the BCS-potential, then this equation becomes identical with the BCS-equation for the critical temperature [1, 3, 4].

We shall now proceed to investigate the convergence properties of (1.17) and (1.18) at high and low temperatures, respectively. We use the Yukawa-like potential

$$v_{\mathbf{p}} = \frac{1}{|\mathbf{p}| + im_0} \tag{1.22}$$

with screening length $1/m_0$. The actual Yukawa potential is non-separable. The calculations given below with the above separable potential can also be carried through with the exact Yukawa potential. The calculations are more elaborate but the qualitative results are the same as obtained with the above potential. This justifies calling (1.14) with (1.22) a Yukawa-like potential.

At high temperatures the case is simple. We assume here without proof that the maximum for $|F(x,k;\beta)|$ occurs when $x = 2\mu$. Then $[|\lambda|/(2\pi)^3]|F(x,k;\,\beta)| < 1$ can always be satisfied for high enough temperatures, since

$$0 < \frac{f^+(\mathbf{p})f^+(\mathbf{k}-\mathbf{p})\{\exp[\beta(2\mu-E)]-1\}}{2\mu-E} < \frac{tgh[\frac{1}{2}\beta(2\mu-E)]}{2\mu-E} < \tfrac{1}{2}\beta.$$

Thus our expansion converges absolutely at high enough temperatures for any coupling strength. Consequently we have at high temperatures

$$L^{(1}_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k}) = \sim \frac{1}{1+\dfrac{\lambda}{(2\pi)^3}F(E_0',k;\beta)} \tag{1.23}$$

and analogous expressions for $L^{(2)}$ and $L^{(3)}$. Note that the geometrical series for $L^{(3)}$ is of the form $[(\partial/\partial F)\,L^{(1)}]_{E_0'=E_1}$. The right-hand side of (1.23) can be used to continue $L^{(1)}$ analytically to values $(\lambda/(2\pi)^3)\,F(E_0',k;\beta) \geqslant 1$ in regions where the geometrical series no longer converges. This happens at low temperatures when $E_0' \to 2\mu$ and $k \to 0$. Similar analytic continuations can be made for $L^{(2)}$ and $L^{(3)}$. There is a singularity at the point $(\lambda/(2\pi)^3)\,F(E_0',k;\beta) = -1$. Thus our expansion diverges for

$$\frac{\lambda}{(2\pi)^3}\,F(E_0',k;\beta) \leqslant -1. \tag{1.24}$$

To investigate our expansion at low temperatures we calculate (1.19) explicitly. We shall be able to give only an approximate calculation for which we estimate the errors. We first bring (1.19) in a more convenient form by introducing in place of the polar variables p, θ new variables ξ, η according to $\xi = \mu - \mathbf{p}^2$, $\eta = \mu - (\mathbf{k}-\mathbf{p})^2$. Then we get

$$F(x,k;\beta) = \frac{2\pi}{4k}\int_{-\infty}^{\mu} d\xi \int_{\eta_1}^{\eta_2} d\eta \frac{\exp[\beta(\xi+\eta)] - 1}{(\mu - \xi + m_0^2)\,(x - 2\mu + \xi + \eta)\,[1 + e^{\beta\xi}]\,[1 + e^{\beta\eta}]}, \tag{1.25}$$

where (1.22) was used and

$$\eta_1 = \xi - k^2 - 2k\sqrt{\mu - \xi}, \quad \eta_2 = \xi - k^2 + 2k\sqrt{\mu - \xi}. \tag{1.26}$$

The integration domain in (1.25) is bounded by a parabola as shown in Figure B.1. For large β (very low temperature) we expect to get a good approximation to (1.25) by the replacement

$$\frac{\exp[\beta(\xi+\eta)] - 1}{[1 + e^{\beta\xi}]\,[1 + e^{\beta\eta}]} \to \Theta(\xi)\,\Theta(\eta) - \Theta(-\xi)\,\Theta(-\eta), \tag{1.27}$$

where $\Theta(\xi) = 1$ for $\xi > 0$ and $\Theta(\xi) = 0$ for $\xi < 0$. We do this and discuss the error in Appendix B. Thus we approximate (1.25) by

$$\bar{F}(x,k) = \frac{\pi}{2k}\int_{-\infty}^{\mu} d\xi \int_{\eta_1}^{\eta_2} d\eta \frac{\Theta(\xi)\,\Theta(\eta) - \Theta(-\xi)\,\Theta(-\eta)}{(\mu - \xi + m_0^2)\,(x - 2\mu + \xi + \eta)}. \tag{1.28}$$

This function is estimated in Appendix C under the assumptions $0 < k < 2\sqrt{\mu}$, $0 \leqslant x \leqslant 4\mu$ and $4\mu \ll \delta \ll m_0^2$. Since the divergence occurs for $k \to 0$, the first restriction covers the interesting region. The second restriction implies for (1.17) that we assume the incoming and outgoing particles to be not too far from the Fermi surface. In (1.18) the $\mathbf{p}_1$-integral contains $f^-(\mathbf{p}_1)f^-(\mathbf{k}-\mathbf{p}_1)$. When we there use the approximation

$$f^-(\mathbf{p}_1)f^-(\mathbf{k}-\mathbf{p}_1) \approx \Theta(\mu - \mathbf{p}_1^2)\,\Theta(\mu - [\mathbf{k}-\mathbf{p}_1]^2) \tag{1.29}$$

in accordance with (1.27), we have in the domain of integration $0 \leqslant E_1 \leqslant 2\mu$. Thus the second restriction covers our need to handle (1.18). The third restriction $4\mu \ll \delta \ll m_0^2$ requires a low particle density. According to Appendix C $F(x,k;\beta)$ is given to a good approximation by

$$F(x,k) = \frac{\pi^2}{m_0} + S \tag{1.30}$$

at low temperatures and low particle densities. S is given by (C.5) and (C.6). We have in (1.17) $x = E_0'$. Then we get for (C.6) $s = \frac{1}{2}|\mathbf{k}_3 - \mathbf{k}_4| \geqslant 0$. In (1.18) we have $x = E_1$. Then $s = (p_1^2 - kp_1 \cos(\mathbf{k},\mathbf{p}_1) + \frac{1}{4}k^2)^{1/2} \geqslant 0$. Thus s is always real and consequently S is real. It can be shown by elementary methods that S satisfies

$$-\frac{10\pi\sqrt{\mu}}{m_0^2} < S < \frac{6\pi}{m_0^2}\left(\frac{\mu}{k} + \sqrt{\mu}\right) \tag{1.31}$$

for any s in the interval $0 \leqslant s \leqslant \infty$ and k has a fixed value satisfying $0 < k < \infty$. Further it can be shown that, when $0 < k < 2\sqrt{\mu}$, S has maximum in the interval

$$\sqrt{\mu} - \tfrac{1}{2}k < s < \sqrt{\mu} + \tfrac{1}{2}k. \tag{1.32}$$

Consider now (1.17). From (1.31) it follows that, when $|\lambda|/m_0$ is small enough and $k > k_0$ (k_0 depends on $|\lambda|/m_0$), the expansion converges also at low temperatures. But this geometrical series converges also for $k \to 0$, if E_0' differs appreciably from 2μ and $|\lambda|/m_0$ is small enough. This follows from

$$S\Big|_{\substack{k=0\\ E_0' \neq 2\mu}} = \frac{2\pi}{m_0^2}\sqrt{\tfrac{1}{2}E_0'}\log\left|\frac{\sqrt{\frac{1}{2}E_0'} + \sqrt{\mu}}{\sqrt{\frac{1}{2}E_0'} - \sqrt{\mu}}\right| - \frac{2\pi\sqrt{\mu}}{m_0^2}. \tag{1.33}$$

Now we shall show that (1.17) diverges, i.e. (1.24) can be satisfied, when $k \to 0$ and $E_0' \to 2\mu$. We first note that according to the left-hand inequality (1.31) we have for low enough particle density $F(E_0', k) > \pi^2/2m_0$. Thus at low temperatures (1.24) can be satisfied only for an attractive potential ($\lambda < 0$). We have from (C.5)

$$S_{|E_0'=2\mu} = \frac{\pi}{m_0^2}\sqrt{\mu - \tfrac{1}{4}k^2}\log\frac{\sqrt{\mu} + \sqrt{\mu - \frac{1}{4}k^2}}{\sqrt{\mu} - \sqrt{\mu - \frac{1}{4}k^2}}. \tag{1.34}$$

When k decreases from $2\sqrt{\mu}$ to 0, this function increases monotonically from 0 to ∞. Consequently (1.24) can be satisfied for small enough k, however small a value we have for $|\lambda|/m_0$. From (1.34) it follows that $L^{(1)}$ diverges like a power series of $(-\lambda\sqrt{\mu}/4\pi^2 m_0^2)\log(4\sqrt{\mu}/k)$, when $E_0' = 2\mu$ and $k \to 0$. $S(E_0', k)$ is continuous everywhere except at the point $E_0' = 2\mu$, $k = 0$. From this and (1.33) it follows that for a given $|\lambda|/m_0$ there exists a neighborhood of $E_0' = 2\mu$ such that our expansion diverges when E_0' is in this neighborhood and $k \to 0$.

Consider now (1.18). With the aid of (1.29) we get

$$L^{(3)}_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k}) = \sim\{\exp[\beta(E_0' - 2\mu)] - 1\}\{1 - \exp[-\beta(E_0 - 2\mu)]\}$$
$$\times \sum_{m=1}^{\infty} \frac{(-\lambda)^m (m-1)}{(2\pi)^{3(m-1)}} \int_{D_I} \frac{d^3 p_1}{(p_1^2 + m_0^2)(E_0 - E_1)(E_0' - E_1)} [F(E_1, k)]^{m-2}, \tag{1.35}$$

where the integration domain D_I is given by

$$0 \leqslant p_1 \leqslant \sqrt{\mu}, \quad 0 \leqslant (\mathbf{k} - \mathbf{p}_1)^2 \leqslant \mu. \tag{1.36}$$

Now we make the assumption that the incoming and outgoing particles are above or at the Fermi surface, i.e.

$$E_0 \geqslant 2\mu, \quad E_0' \geqslant 2\mu. \tag{1.37}$$

Then $(E_0 - E_1)(E_0' - E_1)$ does not change sign in the domain of integration, and we can write

$$L^{(3)} = \sim \int_{D_I} \frac{d^3 p_1}{(p_1^2 + m_0^2)(E_0 - E_1)(E_0' - E_1)} \sum_{m=1}^{\infty} \frac{(-\lambda)^m (m-1)}{(2\pi)^{3(m-1)}} [F(\bar{E}_1^{(m,k)}, k)]^{m-2}, \tag{1.38}$$

where according to (1.32) and (1.36)

$$\bar{E}_1^{(m,k)} = 2\mu - \epsilon^{(m,k)} \quad \text{with} \quad \epsilon^{(m,k)} > 0 \tag{1.39}$$

and

$$\lim_{\substack{m\to\infty \\ k\to 0}} \epsilon^{(m,k)} = 0. \tag{1.39a}$$

Now we can apply to $L^{(3)}$ the above discussion for $L^{(1)}$. We conclude that $L^{(3)}$ diverges for an attractive potential when $k \to 0$. This divergence is independent of the values of E_0 and E_0'. Here it should be noted that from (1.36) and (1.37) it follows that

$$\lim_{\substack{E_0\to 2\mu \\ E_0'\to 2\mu}} \{\exp[\beta(E_0' - 2\mu)] - 1\}$$

$$\times \{1 - \exp[-\beta(E_0 - 2\mu)]\} \int_{D_I} \frac{d^3 p_1}{(p_1^2 + m_0^2)(E_0 - E_1)(E_0' - E_1)}$$

is finite for any k under consideration.

We summarize our results. The perturbation expansion in ladder approximation for (1.2) is given by $L^{(1)} + L^{(2)} + L^{(3)}$, where $L^{(1)}$ is given by (1.17), $L^{(2)}$ by an analogous expression with replacement $E_0' \to E_0$ and $L^{(3)}$ by (1.18). This expansion converges absolutely at high enough temperatures for any coupling strength. If $k = |\mathbf{k}_1 + \mathbf{k}_2|$ differs appreciably from zero and $|\lambda|/m_0$ is small enough, the expansion converges also at low temperatures. When $E_0' \neq 2\mu$ for $L^{(1)}$ and $E_0 \neq 2\mu$ for $L^{(2)}$, the two first expansions may converge also for $k \to 0$. $L^{(1)}$ diverges for an attractive potential when $E_0' \to 2\mu$ and $k \to 0$. The same is true for $L^{(2)}$ when $E_0 \to 2\mu$ and $k \to 0$. $L^{(3)}$ diverges always when $k \to 0$ independently of E_0 and E_0'. Thus the total expansion diverges at low temperatures for an attractive potential when $k \to 0$. These results were obtained under the assumption that the ingoing and outgoing particles are not too far from the Fermi surface. Further we had to make the assumption that the particle density is low in order to carry our explicit calculations at low temperatures. We believe that the above results hold without this assumption.

2. The Pressure

In this section we shall simplify the perturbation expansion for the pressure in the ladder approximation to a geometrical series. The pressure is given by

$$p_V = -\frac{1}{V} R_V = \frac{1}{\beta V} \log Z_V, \tag{2.1}$$

where R_V is the thermodynamic potential and Z_V the grand partition function. According to [2] and analogously with the foregoing section

$$\log \frac{Z_V}{Z_V^0} = \sum_{m=1}^{\infty} (-\lambda)^m \sum_{\substack{\text{different}\\ \text{graphs with}\\ \text{fixed } m}} \frac{1}{V^m} \sum_{\mathbf{k},\mathbf{p}_1,\ldots,\mathbf{p}_m} \prod_{j=1}^{m} |v_{\mathbf{p}_j}|^2 f^{\pm}(\mathbf{p}_j) f^{\pm}(\mathbf{k}-\mathbf{p}_j) I_m, \tag{2.2}$$

where we have used potential (1.14) and I_m is given by an analogous expression with (1.5). Z_V^0 is the grand partition function for the non-interacting system. We refer in (2.2) to a figure similar to Figure 1.1, in which parallel particle lines are denoted by $\mathbf{p}_j$ and $\mathbf{k}-\mathbf{p}_j$. Now it turns out that one can sum over different graphs with fixed m in a simple manner if the calculation is properly symmetrized. In place of each graph we take the average over m(= the number of vertices) identical graphs, in which we take

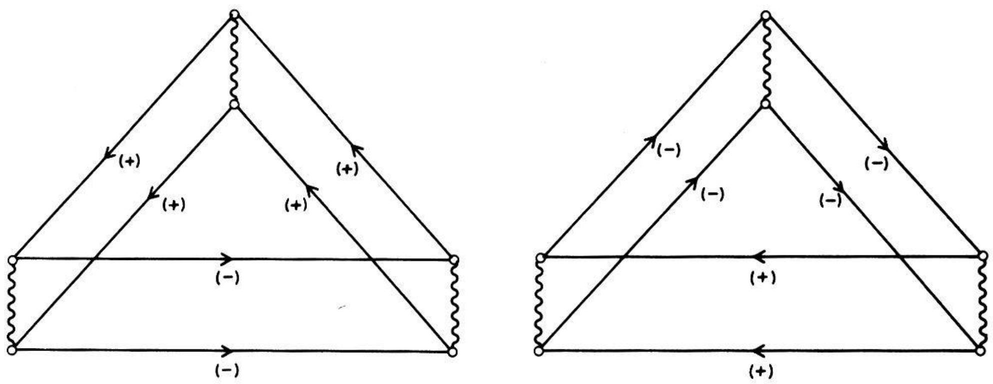

Figure 2.1

each vertex in turn as the distinguished vertex σ_1. This trick we have borrowed from Thouless [1]. Let us first consider the simple case $m = 3$. Then there are only two different graphs shown in Figure 2.1. When we sum over these graphs, we take both three times and divide then by 3. These graphs can be illustrated by particle line sequences

$$(++-), \quad (-++), \quad (+-+); \quad (--+), \quad (+--), \quad (-+-), \tag{2.3}$$

where the first particle line always corresponds to the distinguished vertex σ_1. We use also here (1.9). The time-integral sum in (2.2) is then for $m = 3$

$$\Bigg\{\exp[\beta(2\mu - E_3)] \int_0^\beta d\sigma_1 \int_{\sigma_1}^\beta d\sigma_2 \int_{\sigma_2}^\beta d\sigma_3 + \exp[\beta(2\mu - E_1)] \int_0^\beta d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int_{\sigma_2}^{\sigma_1} d\sigma_3$$

$$+ \exp[\beta(2\mu - E_2)] \int_0^\beta d\sigma_1 \int_{\sigma_1}^\beta d\sigma_2 \int_0^{\sigma_1} d\sigma_3 + \exp[\beta(2\mu - E_1)] \exp[\beta(2\mu - E_2)]$$

$$\times \int_0^\beta d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int_0^{\sigma_2} d\sigma_3 + \exp[\beta(2\mu - E_2)] \exp[\beta(2\mu - E_3)] \int_0^\beta d\sigma_1 \int_{\sigma_1}^\beta d\sigma_2 \int_{\sigma_1}^{\sigma_2} d\sigma_3$$

$$+ \exp[\beta(2\mu - E_1)] \exp[\beta(2\mu - E_3)] \int_0^\beta d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int_{\sigma_1}^\beta d\sigma_3 \Bigg\} \exp[\sigma_1(E_1 - E_3)]$$

$$\times \exp[\sigma_2(E_2 - E_1)] \exp[\sigma_3(E_3 - E_2)]$$

$$= \beta \sum_{\nu=1}^{3} \exp[\beta(2\mu - E_\nu)] \prod_{\substack{j=1 \\ j \neq \nu}}^{3} \frac{\exp[\beta(2\mu - E_j)] - 1}{E_\nu - E_j}, \tag{2.4}$$

where we have made use of identity (A.3). This result can immediately be generalized for any m. Thus we get (2.2) in the form

$$\log \frac{Z_V}{Z_V^0} = \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m} \frac{1}{V^m} \sum_{\mathbf{k}, \mathbf{p}_1, \ldots, \mathbf{p}_m} \prod_{j=1}^{m} |v_{\mathbf{p}_j}|^2 f^+(\mathbf{p}_j) f^+(\mathbf{k} - \mathbf{p}_j) K_m, \tag{2.5}$$

where

$$K_m = \beta \sum_{\nu=1}^{m} \exp[\beta(2\mu - E_\nu)] \prod_{\substack{j=1 \\ j \neq \nu}}^{m} \frac{\exp[\beta(2\mu - E_j)] - 1}{E_\nu - E_j}. \tag{2.6}$$

From (2.1) it now follows after transition $V \to \infty$

$$p_\infty = p_\infty^0 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{(2\pi)^{3(m-1)}} \int d^3 k \int d^3 p_1 |v_{\mathbf{p}_1}|^2 f^-(\mathbf{p}_1) f^-(\mathbf{k} - \mathbf{p}_1)$$

$$\times \left(\int d^3 p |v_{\mathbf{p}}|^2 \frac{f^+(\mathbf{p}) f^+(\mathbf{k} - \mathbf{p}) \{\exp[\beta(2\mu - E)] - 1\}}{E_1 - E} \right)^{m-1}, \tag{2.7}$$

where p_∞^0 is the pressure for non-interacting particles in infinite volume. This is the desired geometrical series. According to Section 1 this series converges absolutely at high temperatures and diverges for an attractive potential at low temperatures (Cf. [1]).

According to Thouless one should get the pressure for infinite volume by setting $\lambda(dp_\infty/d\lambda)$ equal to

$$\frac{-\lambda}{(2\pi)^9}\int d^3 k \int d^3 k_2 \int d^3 k_3 v_{\mathbf{k}_2} v^*_{\mathbf{k}_3} L_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k}),$$

where $L_{\mathbf{k}_2\mathbf{k}_3}(\mathbf{k})$ is given by (1.16). When we calculate p_∞ in this way, we just arrive at (2.7).

3. Correlation Function

We shall now investigate the geometrical series which one can derive for the correlation function with the aid of the results in Secton 1. The correlation function $g(\mathbf{x},\uparrow;\mathbf{x}',\downarrow)$ is defined as the probability per unit volume of finding an electron at $\mathbf{x}$ with spin up and simultaneously another electron at $\mathbf{x}'$ with spin down minus the probability per unit volume of the two electrons appearing without interaction at $\mathbf{x}$ and $\mathbf{x}'$, respectively. Analytically our definition reads

$$g(\mathbf{x},\uparrow;\mathbf{x}',\downarrow) = \frac{1}{V^3}\sum_{\mathbf{k},\mathbf{k}_2,\mathbf{k}_3} V\langle\!\langle a_V^*(\mathbf{k}-\mathbf{k}_2,\tau,\uparrow) a_V^*(\mathbf{k}_2,\tau,\downarrow)\, a_V(\mathbf{k}_3,\tau,\downarrow)\, a_V(\mathbf{k}-\mathbf{k}_3,\tau,\uparrow)\rangle\!\rangle_V$$

$$\times \exp[i(\mathbf{k}_2-\mathbf{k}_3)\cdot(\mathbf{x}-\mathbf{x}')] - [\text{the same thing with } \lambda = 0]. \tag{3.1}$$

With the aid of (1.16) we bring (3.1) into the form

$$g(\mathbf{x},\uparrow;\mathbf{x}',\downarrow) = g^{(1)}(\mathbf{x}-\mathbf{x}') + g^{(2)}(\mathbf{x}-\mathbf{x}') + g^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{3.2}$$

where

$$g^{(1)}(\mathbf{x}-\mathbf{x}') = \frac{1}{(2\pi)^9}\int d^3 k \int d^3 k_2 \int d^3 k_3\, v^*_{\mathbf{k}_2} v_{\mathbf{k}_3} f^+(\mathbf{k}_2) f^+(\mathbf{k}-\mathbf{k}_2) f^-(\mathbf{k}_3) f^-(\mathbf{k}-\mathbf{k}_3)$$

$$\times \frac{\exp[\beta(2\mu - E_{\mathbf{k}_2})] - 1}{E_{\mathbf{k}_3} - E_{\mathbf{k}_2}} \exp[i(\mathbf{k}_2-\mathbf{k}_3)\cdot(\mathbf{x}-\mathbf{x}')] \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{(2\pi)^{3(m-1)}}$$

$$\times \left(\int d^3 p |v_{\mathbf{p}}|^2 \frac{f^+(\mathbf{p}) f^+(\mathbf{k}-\mathbf{p})\{\exp[\beta(2\mu - E)] - 1\}}{E_{\mathbf{k}_3} - E}\right)^{m-1}, \tag{3.3}$$

$$g^{(2)}(\mathbf{x}-\mathbf{x}') = [g^{(1)}(\mathbf{x}-\mathbf{x}')]^* \tag{3.4}$$

and

$$g^{(3)}(\mathbf{x}-\mathbf{x}') = \frac{1}{(2\pi)^9} \int d^3 k \int d^3 k_2 \int d^3 k_3\, v^*_{\mathbf{k}_2} v_{\mathbf{k}_3}\, f^+(\mathbf{k}_2) f^+(\mathbf{k}-\mathbf{k}_2) f^-(\mathbf{k}_3) f^-(\mathbf{k}-\mathbf{k}_3)$$
$$\times \{1 - \exp[\beta(2\mu - E_{\mathbf{k}_2})]\}\{\exp[\beta(E_{\mathbf{k}_3} - 2\mu)] - 1\}$$
$$\times \exp[i(\mathbf{k}_2 - \mathbf{k}_3)\cdot(\mathbf{x}-\mathbf{x}')] \sum_{m=2}^{\infty} \frac{(-\lambda)^m (m-1)}{(2\pi)^{3(m-1)}}$$
$$\times \int d^3 p_1 |v_{\mathbf{p}_1}|^2 \frac{f^+(\mathbf{p}_1) f^+(\mathbf{k}-\mathbf{p}_1)}{(E_{\mathbf{k}_2} - E_1)(E_{\mathbf{k}_3} - E_1)}$$
$$\times \left(\int d^3 p |v_{\mathbf{p}}|^2 \frac{f^+(\mathbf{p}) f^+(\mathbf{k}-\mathbf{p}) \{\exp[\beta(2\mu - E)] - 1\}}{E_1 - E} \right)^{m-2}. \tag{3.5}$$

We shall now make a rough estimate of (3.3) and (3.5) just above the critical temperature. Since the temperature is low we can, at appropriate places, replace $f^{\pm}(\mathbf{p})$ by the step function. However, in the integrals appearing in $(m-1)$th and $(m-2)$th power we must not do this; our expansions have to converge. In (3.3) the contributions to the $\mathbf{k}$- and $\mathbf{k}_3$-integral are strongly dominated by the neighborhoods of $\mathbf{k}=0$ and $|\mathbf{k}_3| = \sqrt{\mu}$. Using potential (1.22) we get a rough estimate

$$g^{(1)}(\mathbf{x}-\mathbf{x}') \approx \frac{1}{(2\pi)^9} \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{(2\pi)^{3(m-1)}} \left(\int d^3 p \frac{[f^+(\mathbf{p})]^2 \{\exp[\beta(2\mu - 2p^2)] - 1\}}{(p^2 + m_0^2)(2\mu - 2p^2)} \right)^{m-1}$$
$$\times \int d^3 k_2 \frac{[f^+(\mathbf{k}_2)]^2 \{\exp[\beta(2\mu - 2k_2^2) - 1\}}{(k_2 - im_0)(2\mu - 2k_2^2)} \exp[i\mathbf{k}_2\cdot(\mathbf{x}-\mathbf{x}')]$$
$$\times \int_{|\mathbf{k}_3| = \sqrt{\mu}} d\Omega_{\mathbf{k}_3} \frac{\exp[-i\mathbf{k}_3\cdot(\mathbf{x}-\mathbf{x}')]}{k_3 + im_0} \int_{|\mathbf{k}_3| = \sqrt{\mu}} d^3 k\, \Theta(\sqrt{\mu} - |\mathbf{k}-\mathbf{k}_3|)$$
$$\times \int_0^{\infty} k_3^2\, dk_3\, \Theta(\sqrt{\mu} - k_3). \tag{3.6}$$

In (3.5) the contributions to $\mathbf{k}$- and $\mathbf{p}_1$-integral are strongly dominated by the neighborhoods of $\mathbf{k}=0$ and $|\mathbf{p}_1| = \sqrt{\mu}$. Using this one can bring (3.5) in an analogous form with (3.6). To make rough estimates of the various integrals in (3.6), we assume $\sqrt{\mu} \ll m_0^2$ and $\mu\beta \gg 1$. The latter condition is justified, since we have experimentally $\mu\beta_{cr} \gtrsim 10^5$. With these assumptions we get

$$\int d^3 p \frac{[f^+(\mathbf{p})]^2 \{\exp[\beta(2\mu - 2p^2)] - 1\}}{(p^2 + m_0^2)(2\mu - 2p^2)} \approx \frac{2\pi\mu}{m_0^2} \int_0^{\infty} dp \frac{tgh[\frac{1}{2}\beta(p^2 - \mu)]}{p^2 - \mu}$$
$$\approx \frac{2\pi\sqrt{\mu}}{m_0^2} \log(2\mu\beta) \tag{3.7}$$

and

$$\mathbf{k}_2\text{-integral} \approx \frac{2\pi \log(2\mu\beta)}{-im_0} \frac{\sin[\sqrt{\mu}|\mathbf{x}-\mathbf{x}'|]}{|\mathbf{x}-\mathbf{x}'|}. \tag{3.8}$$

Since we are close to the critical temperature, we have in the geometrical series

$$\frac{|\lambda|\sqrt{\mu}\log(2\mu\beta)}{4\pi^2 m_0^2} = 1-\epsilon \quad \text{with} \quad 0<\epsilon \ll 1. \tag{3.9}$$

Using these estimates we end up with

$$g(\mathbf{x},\uparrow;\mathbf{x}',\downarrow) \approx \frac{2}{9}\frac{1}{(2\pi)^4}\frac{\mu^3}{1+\dfrac{\lambda\sqrt{\mu}\log(2\mu\beta)}{4\pi^2 m_0^2}} \times\left[-\frac{2\lambda}{|\lambda|}+\frac{1}{1+\dfrac{\lambda\sqrt{\mu}\log(2\mu\beta)}{4\pi^2 m_0^2}}\right]\frac{\sin^2[\sqrt{\mu}|\mathbf{x}-\mathbf{x}'|]}{\mu|\mathbf{x}-\mathbf{x}'|^2}. \tag{3.10}$$

This function goes to zero for $|\mathbf{x}-\mathbf{x}'| \to \infty$ as it should, since at large distances the two coupled electrons should become practically uncoupled. According to (3.10) this happens as soon as the distance between the particles is considerably larger than the average distance between the particles in the system. We note that the correlation function is negative for repulsive potentials, indicating that the probability of finding two electrons near each other is smaller than for the case of free particles. From (3.10) it appears further that, under an attractive potential, when we approach the critical temperature, the probability per unit volume of finding two electrons near each other becomes very large on account of (3.9). Thus the correlation of electrons with opposite spin in the normal state close to the critical temperature is similar to the correlation in the superconductive state in the BCS-model.

Acknowledgment

This work was suggested by Prof. Klaus Hepp. I wish to express my gratitude to Prof. Hepp for valuable discussions and for critical reading of the final manuscript. The work has been supported by a grant from the Herman Rosenberg Foundation in Finland.

APPENDIX A

In this appendix we prove some identities. In the calculation of (1.11) we made use of the identity

$$\prod_{j=1}^{m-1}\frac{1}{E_0'-E_j} = \sum_{\nu=1}^{m-1}\frac{1}{E_0'-E_\nu}\prod_{\substack{j=1\\ j\neq\nu}}^{m-1}\frac{1}{E_\nu-E_j}. \tag{A.1}$$

This formula is proved by considering the integral

$$\int_{a-i\infty}^{a+i\infty} \frac{dz}{z} \prod_{j=1}^{m-1} \frac{1}{-z+E_0'-E_j}.$$

Choose a so that it satisfies $a<0$, $a<E_0'-E_j$ for $j=1,2,\ldots,m-1$. The path of integration can be closed at infinity either in the direction of the positive real axis or in the direction of the negative real axis. The latter yields zero. Hence the former must also vanish. This leads to (A.1). From (A.1) it trivially follows

$$\frac{1}{E_0'-E_0}\left[\prod_{j=1}^{m-1}\frac{1}{E_0'-E_j}-\prod_{j=1}^{m-1}\frac{1}{E_0-E_j}\right]=\sum_{\nu=1}^{m-1}\frac{1}{(E_0'-E_\nu)(E_\nu-E_0)}\prod_{\substack{j=1\\ j\neq\nu}}^{m-1}\frac{1}{E_\nu-E_j}. \tag{A.2}$$

This identity can also be proved directly by considering an above type integral. In Section 2 we used the identity

$$\sum_{\nu=1}^{m}\prod_{\substack{j=1\\ j\neq\nu}}^{m}\frac{1}{E_\nu-E_j}=0, \tag{A.3}$$

which is also easily proved. Evidently other similar identities can be derived.

APPENDIX B

In this appendix we discuss the error made in the replacement (1.27). Let M be a positive constant, such that e^{-M} can be neglected compared with one. Our approximation is good for the domain $|\xi|>M/\beta$, $|\eta|>M/\beta$. The approximation fails in the shaded domains A and B shown in Figure B.1. We shall estimate the error made in regions A and B in the approximation (1.28). We choose β so large that M/β can be neglected compared with μ. Then the contribution to $F(x,k,\beta)$ from region A is given by

$$I_A=\frac{\pi}{2k(\mu+m_0^2)}\int_{-M/\beta}^{M/\beta} d\xi\int_{\eta_1}^{\eta_2} d\eta\,\frac{\exp[\beta(\xi+\eta)]-1}{(x-2\mu+\xi+\eta)[1+e^{\beta\xi}][1+e^{\beta\eta}]}, \tag{B.1}$$

where η_1 and η_2 are given by (1.26). By symmetry

$$I_B=I_A. \tag{B.2}$$

In the approximation (1.28) I_A is replaced by

$$J_A=\frac{\pi}{2k(\mu+m_0^2)}\int_{-M/\beta}^{M/\beta} d\xi\int_{\eta_1}^{\eta_2} d\eta\,\frac{\Theta(\xi)\,\Theta(\eta)-\Theta(-\xi)\,\Theta(-\eta)}{x-2\mu+\xi+\eta}. \tag{B.3}$$

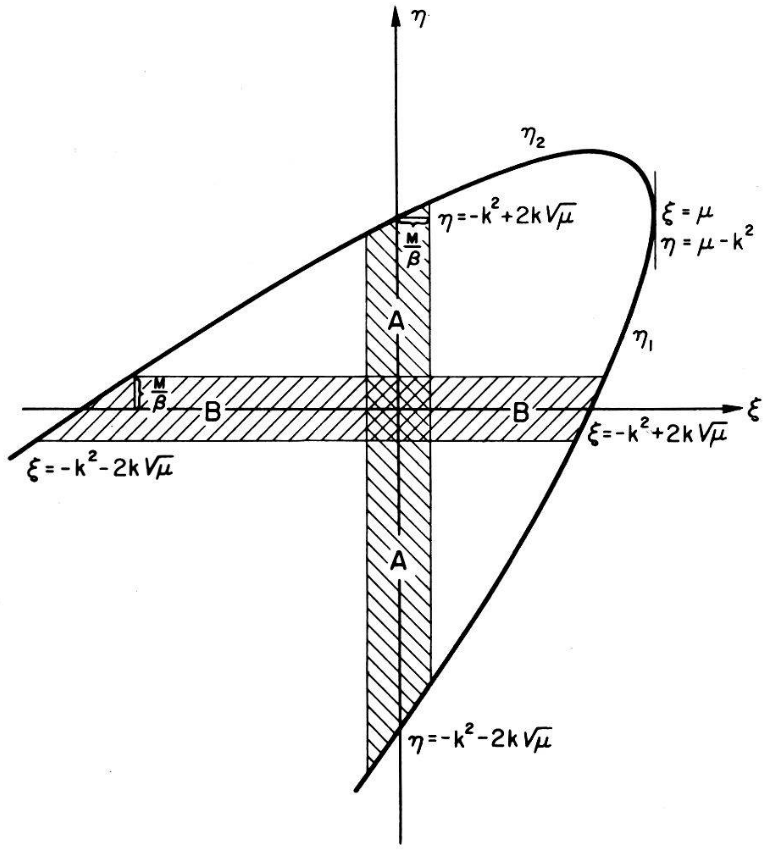

Figure B.1

One would expect by inspection that I_A and J_A have maximum absolute values for $x = 2\mu$. We can prove

Theorem B.1: Functions $I_A(x, k; \beta)$ and $J_A(x, k; \beta)$, with fixed β, have an extremum value at $x = 2\mu$, $k = 0$.

Proof: The proof is the same in both cases. We do it for I_A. From (B.1) it appears that $I_A(x, k) = I_A(x, -k)$. It is easily verified that I_A, as function of k, is continuous at $k = 0$ and has there a continuous derivative. Hence

$$\left.\frac{\partial I_A(x, k)}{\partial k}\right|_{k=0} = 0. \tag{B.4}$$

This equation holds for any x. Now we shall show that the partial derivative with respect to x vanishes at $x = 2\mu$, $k = 0$.

$$\left(\frac{\partial I_A}{\partial x}\right)_{\substack{x=2\mu \\ k=0}} = \lim_{k\to 0}\left\{\frac{-\pi}{2k(\mu + m_0^2)} \int_{-M/\beta}^{M/\beta} d\xi \int_{\xi-2k\sqrt{\mu}}^{\xi+2k\sqrt{\mu}} d\eta \frac{\exp[\beta(\xi+\eta)] - 1}{(\xi+\eta)^2[1 + e^{\beta\xi}][1 + e^{\beta\eta}]}\right\}$$

$$= -\frac{2\pi\sqrt{\mu}}{\mu + m_0^2} \int_{-M/\beta}^{M/\beta} d\xi \frac{tgh(\frac{1}{2}\beta\xi)}{4\xi^2} = 0, \tag{B.5}$$

which completes our proof.

Next we shall find an upper bound for I_A (and J_A) at $x = 2\mu$, $k = 0$. In the same fashion, as in (B.5), we find

$$0 < I_A(x,k)\Big|_{\substack{x=2\mu\\ k=0}} = \frac{2\pi\sqrt{\mu}}{\mu + m_0^2} \int_{-M/\beta}^{M/\beta} d\xi \frac{tgh(\frac{1}{2}\beta\xi)}{2\xi} < \frac{\pi M\sqrt{\mu}}{\mu + m_0^2}. \tag{B.6}$$

One can prove

Theorem B.2: Assume we have a fixed κ satisfying $0 < \kappa < 2\sqrt{\mu}$. Let

$$0 < k < \kappa \quad \text{and} \quad |x - 2\mu| > 2[2\kappa\sqrt{\mu} + \kappa^2] \tag{B.7}$$

or

$$\kappa < k < 2\sqrt{\mu} \tag{B.8}$$

and x has any value. If we choose β sufficiently large that

$$\frac{M}{\beta}\left|\log\frac{M}{\beta}\right| \ll \kappa\sqrt{\mu}, \tag{B.9}$$

then

$$|I_A| \ll \frac{2\pi\sqrt{\mu}}{\mu + m_0^2}. \tag{B.10}$$

This theorem holds also for J_A. We know from Theorem B.1 that I_A and J_A have extremum values at $x = 2\mu$, $k = 0$ in the rectangle $0 < k < \kappa$, $|x - 2\mu| < 2[2\kappa\sqrt{\mu} + \kappa^2]$. It looks plausible that this point is a maximum for $|I_A|$ and $|J_A|$. We proceed as if this were the case. Then we have from (B.2) and (B.6) for $0 < k < 2\sqrt{\mu}$ and β sufficiently large

$$F(x,k;\beta) = \bar{F}(x,k) + \Delta(\beta), \tag{B.11}$$

where $\bar{F}$ is given by (1.28) and the correction term satisfies

$$|\Delta(\beta)| < \frac{4\pi M\sqrt{\mu}}{\mu + m_0^2}. \tag{B.12}$$

APPENDIX C

In this appendix we estimate $\bar{F}(x,k)$ given by (1.28), and then use (B.11) to obtain an estimate for $F(x,k;\beta)$ at low temperatures. As discussed in the main text we can calculate $\bar{F}(x,k)$ only approximately under the assumptions

$$0 < k < 2\sqrt{\mu}, \quad 0 \leqslant x \leqslant 4\mu \quad \text{and} \quad 4\mu \ll \delta \ll m_0^2. \tag{C.1}$$

As we shall see below, under these assumptions our approximation will be good.

According to Fig. B.1 we bring (1.28) into the form

$$\bar{F}(x,k) = \frac{\pi}{2k}\left\{\int_0^{\xi_1} d\xi \int_0^{\eta_2} d\eta + \int_{\xi_1}^{\mu} d\xi \int_{\eta_1}^{\eta_2} d\eta - \int_{-\infty}^{-\xi_2} d\xi \int_{\eta_1}^{\eta_2} d\eta - \int_{-\xi_2}^{0} d\xi \int_{\eta_1}^{0} d\eta\right\}$$

$$\times \frac{1}{(\mu - \xi + m_0^2)(x - 2\mu + \xi + \eta)}, \tag{C.2}$$

where η_1 and η_2 are given by (1.26), $\xi_1 = 2k\sqrt{\mu} - k^2$ and $\xi_2 = 2k\sqrt{\mu} + k^2$. To evaluate (C.2) we split in the third term the ξ-integral into two parts, one to go from $-\delta$ to $-\xi_2$ and the other one from $-\infty$ to $-\delta$. The former integral plus the remaining integrals in (C.2) can be evaluated to a good approximation by using only the first term in the expansion

$$\frac{1}{\mu - \xi + m_0^2} = \frac{1}{m_0^2} - \frac{\mu - \xi}{m_0^4} + \frac{(\mu - \xi)^2}{m_0^6} - \cdots \tag{C.3}$$

In the integral in which ξ varies from $-\infty$ to $-\delta$, we get logarithms from the η-integration, which can be written as rapidly converging power series. We get a good approximation by using only the first terms in these series. Using these approximations we get for (C.2)

$$\bar{F}(x,k) = \frac{\pi^2}{m_0} + S - \frac{2\pi\sqrt{\mu}}{m_0^2} - \frac{\pi x}{m_0^2} S + \Delta_1, \tag{C.4}$$

$$S = \frac{\pi}{k m_0^2}\left\{[(s + \tfrac{1}{2}k)^2 - \mu] \log\left|\frac{\sqrt{\mu} + \frac{1}{2}k + s}{\sqrt{\mu} - \frac{1}{2}k - s}\right| - [(s - \tfrac{1}{2}k)^2 - \mu] \log\left|\frac{\sqrt{\mu} - \frac{1}{2}k + s}{\sqrt{\mu} + \frac{1}{2}k - s}\right|\right\}, \tag{C.5}$$

$$s = \sqrt{\tfrac{1}{2}x - \tfrac{1}{4}k^2} \tag{C.6}$$

and the two last terms are correction terms. The first of these is obtained by evaluating the contribution from the first neglected term in the rapidly converging series (C.3). Δ_1 contains a finite number of terms which all are much smaller than $2\pi\sqrt{\mu}/m_0^2$. Under assumptions (C.1) the three last terms in (C.4) can be neglected compared with the two first ones. Now we combine our result with (B.11). On account of the third assumption (C.1) $4\pi M\sqrt{\mu}/m_0^2 \ll \pi^2/m_0$. Thus we have for low temperatures and low particle density

$$F(x,k;\beta) \approx F(x,k) = \frac{\pi^2}{m_0} + S \tag{C.7}$$

to a good approximation.

REFERENCES

[1] DAVID J. THOULESS, Ann. Phys. *10*, 553 (1960).
[2] IRJA NIEMINEN, *Perturbation expansions in quantum statistical mechanics* (preprint).
[3] J. R. SCHRIEFFER, *Theory of Superconductivity* (W. A. Benjamin, Inc., Massachusetts 1964).
[4] N. N. BOGOLIUBOV, D. N. ZUBAREV and IU. A. TSERKOVNIKOV, Sov. Phys. (Doklady) *2*, 535 (1958).