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Autor: Bell, J.S.
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On Wave Packet Reduction in the Coleman–Hepp Model

by **J. S. Bell**

CERN, Geneva

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Abstract. The quantum mechanical measurement problem is considered in a model due to Hepp and Coleman. Whereas Hepp emphasized a 'rigorous "reduction of the wave packet"', in a certain mathematical limit, it is emphasized here that no such reduction ever actually occurs. Some general remarks are made on the advantages of the Heisenberg picture for such considerations, especially in connection with extension to relativistic theories. The non-reduction of the wave packet is directly related to the deterministic character of Heisenberg equations of motion.

1. Introduction

In a very elegant and rigorous paper [1], K. Hepp has discussed quantum measurement theory. He uses the C^* algebra description of infinite quantum systems. Here an attempt is made to give a more popular account of some of his reasoning. Such an attempt seems worth while because many people not familiar with the C^* algebra approach, and even somewhat intimidated by it, have been intrigued by the following statement in Hepp's abstract:

> 'In several explicitly soluble models, the measurement leads to macroscopically different "pointer positions" and to a rigorous "reduction of the wave packet" with respect to all local observables.'

This could look like a clean solution at last to the infamous measurement problem[1]). But it is not so, nor thought by Hepp to be so. Here we will take one[2]) of his models and analyse it in elementary text-book terms. It will be insisted that the 'rigorous reduction' does not occur in physical time but only in an unattainable mathematical limit. It will be argued that the distinction is an important one.

We will work at first in the Schrödinger picture, but later, with the extension to relativistic systems in mind, it will be argued that such considerations become particularly clear in the Heisenberg picture.

2. Model

The model is the following. The 'apparatus' is a semi-infinite linear array of spin-$\frac{1}{2}$ particles, fixed at positions $x = 1, 2, \ldots$. The 'system' is a moving spin-$\frac{1}{2}$ particle, with

[1]) For a general survey, see, for example, d'Espagnat [2].
[2]) Note that Hepp considers several other models, making points not presented here, in particular concerning the possibility of 'catastrophic' time evolutions.

position co-ordinate x and spin operators $\vec{\sigma}_0(\equiv \sigma_0^1, \sigma_0^2, \sigma_0^3)$; it is the third component σ_0^3 which is to be 'measured'. The combined system is described by a wave function, where all σ_n take values ± 1,

$$\psi(t, x, \sigma_0, \sigma_1, \sigma_2, \ldots)$$

in a representation where all σ_n^3 are diagonal:

$$\sigma_n^3 \psi(t, x, \sigma_0, \sigma_1, \sigma_2, \ldots) = \sigma_n \psi(t, x, \sigma_0, \sigma_1, \sigma_2, \ldots). \tag{1}$$

The Hamiltonian is taken to be

$$H = \frac{1}{i}\frac{\partial}{\partial x} + \sum_{n=1}^{\infty} V(x-n)\,\sigma_n^1(\tfrac{1}{2} - \tfrac{1}{2}\sigma_0^3). \tag{2}$$

Note that the 'kinetic energy' here is linear rather than quadratic in the particle momentum $p = (1/i)(\partial/\partial x)$. This has the convenience that free particle wave packets do not diffuse; they just move without change of form, and with unit velocity, in the positive x-direction. The interaction V is supposed to have 'compact support' – i.e., to be zero beyond some range r:

$$V(x) = 0 \quad \text{for } |x| > r. \tag{3}$$

It is also supposed, for reasons that will appear, that

$$\int_{-\infty}^{\infty} dx\, V(x) = \frac{\pi}{2}. \tag{4}$$

The Schrödinger equation

$$\frac{\partial \psi}{\partial t} = -iH\psi$$

is readily solved

$$\psi(t, x, \sigma_0, \ldots) = \prod_{n=1}^{\infty} \exp[-iF(x-n)\,\sigma_n^1(\tfrac{1}{2} - \tfrac{1}{2}\sigma_0^3)]\,\phi(x-t, \sigma_0, \ldots) \tag{5}$$

where ϕ is arbitrary and

$$F(x) = \int_{-\infty}^{x} dy\, V(y). \tag{6}$$

Note that

$$\left.\begin{aligned} F(x) &= 0 \quad \text{for } x < -r \\ F(x) &= \frac{\pi}{2} \quad \text{for } x > +r. \end{aligned}\right\} \tag{7}$$

Consider in particular states in which initially the lattice spins are all up and the moving spin is either up or down:

$$\left.\begin{aligned} \psi_+(t,x,\ldots) &= \chi(x-t)\,\psi_+(\sigma_0)\prod_{n=1}^{\infty}\psi_+(\sigma_n) \\ \psi_-(t,x,\ldots) &= \chi(x-t)\,\psi_-(\sigma_0)\prod_{n=1}^{\infty}\psi'_+(\sigma_n, x-n) \end{aligned}\right\} \tag{8}$$

where

$$\left.\begin{aligned} \psi_\pm(\sigma) &= \delta_{\sigma\mp 1} \\ \psi'_+(\sigma_n, x-n) &= \exp[-iF(x-n)\,\sigma_n^1]\,\psi_+(\sigma_n). \end{aligned}\right\} \tag{9}$$

Note that in virtue of (7)

$$\left.\begin{aligned} \psi'_+(\sigma_n, x-n) &= \psi_+(\sigma_n) && \text{for } x-n<-r \\ &= -i\psi_-(\sigma_n) && \text{for } x-n>+r. \end{aligned}\right\} \tag{10}$$

Let us suppose that the wave packet χ has compact support:

$$\chi(x)=0 \quad \text{for } |x|>w. \tag{11}$$

Then, from (10) we can use in (8)

$$\left.\begin{aligned} \psi'_+(\sigma_n, x-n) &= \psi_+(\sigma_n) && \text{for } n>t+r+w \\ \psi'_+(\sigma_n, x-n) &= -i\psi_-(\sigma_n) && \text{for } n<t-r-w. \end{aligned}\right\} \tag{12}$$

Thus (8) has the interpretation that when the system spin is up nothing happens to the apparatus spins, but when the system spin is down each apparatus sspin in turn is flipped from up to down.

Hepp's 'macroscopic pointer position' can be defined here by considering the limit $M\to\infty$ of

$$C_M=\frac{1}{M}\sum_{n=1}^{M}\sigma_n^3. \tag{13}$$

Clearly

$$\operatorname*{Lim}_{M\to\infty}\left(\operatorname*{Lim}_{t\to\infty}(\psi_\pm, C_M\psi_\pm)\right)=\pm 1. \tag{14}$$

So we have his 'macroscopically different pointer positions'. From the fact that the two states have different values here (for what Hepp calls a 'classical observable', involving infinitely many of the basic operators $\vec{\sigma}$) Hepp infers that

$$\operatorname*{Lim}_{t\to\infty}(\psi_\pm, Q\psi_\mp)=0 \tag{15}$$

for any 'local observable' Q – i.e., one constructed from a *finite* number of $\vec{\sigma}$'s. This is plausible in general because such a difference means, loosely speaking, that the two states differ significantly at infinitely many lattice points, and so remain mutually orthogonal after any operation involving only finitely many lattice points. In this particular case, we see explicitly from (12) that if a particular Q involves only $(\vec{\sigma}_0, \vec{\sigma}_1 \ldots \vec{\sigma}_N)$ then

$$(\psi_{\pm}, Q\psi_{\mp}) = 0 \quad \text{for } t > 1 + N + r + w \tag{16}$$

which includes (15).

The result (15) is the 'rigorous reduction of the wave packet'. If the 'local observables' Q (as distinct in particular from the 'classical observables') are thought of as those which can in principle actually be observed, then the vanishing of their matrix elements between the two states means that coherent superpositions of ψ_+ and ψ_- cannot be distinguished from incoherent mixtures thereof. In quantum measurement theory such elimination of coherence is the philosopher's stone. For with an incoherent mixture specialization to one of its components can be regarded as a purely mental act, the innocent selection of a particular sub-ensemble, from some total statistical ensemble, for particular further study.

We insist, however, that $t = \infty$ never comes, so that the wave packet reduction never happens. The mathematical limit $t \to \infty$ is of physical relevance only in so far as it suggests what might be true, or nearly so, for large t. The result (15) [and more sharply, in this particular case, (16)] shows that any *fixed* observable Q will eventually give a very poor (zero, in this case) measure of the persisting coherence. But nothing forbids the use of different observables as time goes on. Consider for example the unitary operator

$$z = \sigma_0^1 \prod_{n=1}^{N(t-r-w)} \sigma_n^2 \tag{17}$$

where $N(t)$ is the largest integer smaller than t. The increasing string of factors here serves to unflip the flipped spins, so that

$$(\psi_+, z\psi_-) = \int dx |\chi(x-t)|^2 \prod_{N(t-r-w)}^{N(t+r+w)} (\psi_+(\sigma_n), \psi'_+(\sigma_n, x-n)) \tag{18}$$

becomes a periodic function of t. Trivially,

$$(\psi_+, z\psi_+) = (\psi_-, z\psi_-) = 0. \tag{19}$$

Thus in the Hermitean operators z we have a sequence of local observables whose matrix elements

$$(\psi_{\mp}, z\psi_{\pm}) \tag{20}$$

do *not* approach zero. So long as nothing, in principle, forbids consideration of such arbitrarily complicated observables, it is not permitted to speak of wave packet reduction. While for any given observable one can find a time for which the unwanted interference is as small as you like, for any given time one can find an observable for which it is as big as you do *not* like.

3. Heisenberg Picture

Consider now the Heisenberg picture[3]), in which the states are time-independent and the operators vary. The Heisenberg equations of motion are in general

$$\dot{Q}(t) = [Q(t), -iH]$$

and in particular

$$\dot{x}(t) = 1$$

$$\dot{\vec{\sigma}}_0(t) = -\left(\sum_{n=1}^{\infty} V(x(t)-n)\,\sigma_n^1(t)\right) \quad \hat{k}\times\vec{\sigma}_0(t)$$

$$\dot{\vec{\sigma}}_n(t) = +\left(\sum_{n=1}^{\infty} V(x(t)-n\right)(1-\sigma_0^3(t))\,\hat{\imath}\times\vec{\sigma}_n(t)$$

where $\hat{\imath}$ and $\hat{k}$ are unit vectors in the 1 and 3 directions. Now we could solve these equations forward in time to find subsequent values in terms of initial values, and then to say again what has been said above. But we wish to note rather that the equations can be solved *backwards* in time, to express operators at some initial time in terms of those at any later time. For example, we find

$$\sigma_0^1(0) = \sigma_0^1(t)\cos\theta(t) - \sigma_0^2(t)\sin\theta(t) \tag{21}$$

where

$$\theta(t) = \sum_{n=1}^{\infty} \{F(x(t)-n) - F(x(t)-t-n)\}\,\sigma_n^1(t). \tag{22}$$

Between states which satisfy the Schrödinger equation, matrix elements of σ_0^1 at time zero are equal to the corresponding matrix elements at time t of the combination of observables on the right-hand side of (21). Thus this combination serves the same purpose as that of (17), of giving a constant measure to the persisting coherence – in this case whatever coherence could initially be measured by σ_0^1. It is not, of course, the same construction as (17), and in fact it explicitly invokes $x(t)$, as well as $\sigma_n(t)$, as an observable. But why not?

We note in passing that in the Heisenberg picture there is no complication in considering mixed rather than pure states. Whatever coherence shows up at time 0 in the expectation value of an operator $Q(0)$, will persist and show up at later times in the expectation value of the corresponding combination of $Q(t)$. In this picture the persistence of coherence is directly related to the deterministic character of the Heisenberg equations of motion. This operates backwards as well as forwards in time, and requires a given $Q(0)$ to be some combination of the set $Q(t)$ with any given t.

As written, the summation in (22) is infinite. But for any given wave packet $\chi(x)$, of compact support, it can be terminated without error at some sufficiently large n, growing with time. This is because of (7), which requires F to vanish for large negative arguments. Thus, loosely speaking, the evidence for coherence remains at any finite time in a finite region of the lattice. This will not be generally true in non-relativistic

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[3]) The use of the Heisenberg picture in quantum measurement theory has been advocated, for different reasons, by B. S. De Witt [3].

models. It is associated with the use of interactions and wave packets of compact support, and with the existence in the model of a limiting – indeed universal – velocity, which was taken to be unity.

In *relativistic* theories, however, we again have a limiting velocity, that of light – at least if we have flat unquantized space-time and can avoid the pathologies of Velo and Zwanziger [4]. The local observables in an initial space-time region are then presumably determined by those contained subsequently in a region obtained from the original by expanding its space boundaries with the velocity of light. Presumably the exact formulation of this notion is to be found in the 'primitive causality' of Haag [5]. In so far as it applies we see again that any coherence associated with the initial region must persist, and be detectable subsequently in a bigger but finite region by using the appropriate combination of observables in that region.

4. Conclusion

Clearly there is no room for disagreement about simple mathematics. But there may be disagreement about the physical significance of it. Hepp clearly considers the limit $t \to \infty$ very relevant, while he does 'not, however, accept the ergodic mean as a fundamental solution to the problem of the reduction of wave packets'. In my opinion neither of these approaches provides a *fundamental* solution, but both are quite valuable for indicating how the difference between reducing the wave packet at one time rather than another is extremely hard to see *in practice*. Moreover, both indicate this on the same ground – that the observation of arbitrarily complicated observables, while not excluded in principle, is not possible in practice. It remains true that, whenever it is done, the wave packet reduction is not compatible with the linear Schrödinger equation. And yet at some not-well-specified time, such a reduction is supposed to occur [6]: '...a measurement always causes the system to jump into an eigenstate of the dynamical variable that is measured...'.

The continuing dispute about quantum measurement theory is not between people who disagree on the results of simple mathematical manipulations. Nor is it between people with different ideas about the actual practicality of measuring arbitrarily complicated observables. It is between people who view with different degrees of concern or complacency the following fact: so long as the wave packet reduction is an essential component, and so long as we do not know exactly when and how it takes over from the Schrödinger equation, we do not have an exact and unambiguous formulation of our most fundamental physical theory.

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