Dynamics of the Dicke laser model in the classical region. I

Autor(en): **Scharf, G.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **48 (1975)**

Heft 3

PDF erstellt am: **12.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-114677

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

Helvetica Physica Acta
Vol. 48, 1975.

Birkhäuser Verlag Basel

Dynamics of the Dicke Laser Model in the Classical Region – I

by **G. Scharf**

Institute für Theoretische Physik der Universität Zürich, Schönberggasse 9, 8001 Zürich, Switzerland

(19. II. 75)

Abstract. The dynamics of the Dicke model describing N spins ($s = \frac{1}{2}$) interacting with a single mode of the radiation field is investigated in the limit $N \to \infty$. It is shown that this limit corresponds to the classical limit $h \to 0$ of quantum mechanics. The time evolution of the initial state with all N spins in the upper level is then studied, using the semi-classical asymptote of quantum mechanics. The result is applied to calculate the second-order normal ordered photon correlation function, which is of particular interest because it shows a slow decay of coherence for large times.

1. Introduction

The quantum-optical model of Dicke [1], describing the collective interaction of radiation and matter, has been found calculable to a large extent. This is due to the fact that there exists a linear correspondence between the Dicke Hamiltonian and the Hamiltonian of a quantum mechanical problem with one degree of freedom [2]. The latter cannot be solved within the class of classical special functions; however, the related classical mechanical problem can be integrated. It is then possible to set up the semi-classical asymptote of the quantum mechanical problem. The small parameter h (Planck's constant in ordinary quantum mechanics), which is going to 0 in the classical limit, is proportional to N^{-1} in the present case (see next section), where N is the number of radiating atoms. Hence, in contrast to ordinary quantum mechanics, this parameter h is here at our disposal, and the classical limit $h \to 0$ corresponds to a large number of radiating atoms. This has been observed in a different context by Hepp and Lieb [3].

The semi-classical solution becomes exact in the following two limiting cases: i) for fixed h as the time t goes to 0, ii) for $h \to 0$ and fixed t. In our previous calculations [2, 4, 5] we followed case i). The advantage of this approach is that it applies to the quantum region; that means, higher orders in N^{-1} can be calculated. Its disadvantage is the restriction to small t. For simple observables such as the photon number, this is no serious shortcoming, because it is easy to extend the results to large t [2]. However, for more complicated observables, such as the correlation functions, this is a difficult problem. Indeed, we did not succeed by method i) in the case of the normal ordered photon correlation function [5]

$$g(t, t_1) = (\psi_0, a^+(t_1)\, a^+(t)\, a(t)\, a(t_1)\, \psi_0). \tag{1.1}$$

In this paper a different method is developed for the limiting case ii) above. This method gives results valid for arbitrary t, but only the leading order in N^{-1}.

The classical limit ii) has been extensively studied in ordinary quantum mechanics, especially by Maslov [6]. Unfortunately, these methods cannot be taken over immediately to the present situation for various reasons, as will be seen below. The convergence of the semi-classical solution has to be studied in a different manner. This interesting problem will be considered in detail in Part II of the paper. Here, the method is formally introduced (Section 2), and is applied to calculate the correlation function (1.1) where the previous method failed (Section 3). The results are discussed in the last section. It is found that $g(t, t_1)$ factorizes approximately into a product of the photon numbers

$$g(t, t_1) = n(t)\,n(t_1) + g'(t, t_1). \tag{1.2}$$

The additional term $g'(t, t_1)$ is uniformly small in t, but not in t_1. It increases slowly with t_1: after N oscillations of the system, it has reached the same order of magnitude as the main term. That means, that the normal ordered correlation function (1.1) shows a slow decay of coherence.

2. The Semi-Classical Solution

It is our aim to study the dynamics defined by the Hamiltonian

$$\mathscr{H} = \sum_{i=1}^{N} s_i^+ s_i^- + (1-\delta)\,a^+ a - g\sum_{i=1}^{N}(a^+ s_i^- + s_i^+ a). \tag{2.1}$$

Here $s_i^\pm$ are the usual spin flip operators ($s = \frac{1}{2}$) for the atom i and a^+, a are the creation and annihilation operators of the photon mode. It follows from first principles that the coupling constant g depends on the quantization volume V of the radiation field according to

$$g = O(V^{-\frac{1}{2}}), \tag{2.2}$$

or, assuming a fixed density N/V of radiating atoms,

$$g = O(N^{-\frac{1}{2}}). \tag{2.3}$$

The detuning parameter δ, which measures the deviation from resonance, is of the order

$$\delta = O(1). \tag{2.4}$$

The Schrödinger equation for the Hamiltonian (2.1) can be reduced to a $N+1$-dimensional matrix equation [2]

$$i\frac{d\psi(t)}{dt} = \mathscr{H}\psi(t), \quad \hbar = 1 \tag{2.5}$$

where

$$\psi(t) = \{e_n(t)\} \in \mathbb{C}^{N+1}, \quad n = 0, 1, \ldots N \tag{2.6}$$

and

$$\mathscr{H}_{n,n+1} = \mathscr{H}_{n+1,n} = -g[(R-n)(N-n)(n+1)]^{\frac{1}{2}} \qquad (2.7)$$

$$\mathscr{H}_{n,n} = R - \delta(R-n-1),$$

and $\mathscr{H}_{n,n'} = 0$ for $n' \neq n, n+1$. The integer

$$R = N + 2L \qquad (2.8)$$

is a conserved quantity, and $2L = 0, 1, \ldots$ is the number of excess photons, i.e. the number of photons being present if all N atoms are in the upper state.

The matrix equation (2.5) is transformed into the Schrödinger equation of a quantum mechanical problem having one degree of freedom in $L^2(0, \infty)$ by means of the linear transformation [2, 4]

$$y(z,t) = \sum_{n=0}^{N} \binom{N}{n}^{\frac{1}{2}} ((R-n)!)^{-\frac{1}{2}} e_n(t) z^{N+L-n} \exp - \tfrac{1}{4}z^2 - \frac{\delta}{2g} z, \qquad (2.9)$$

namely

$$\frac{i}{g}\frac{\partial y}{\partial t} = -\tfrac{1}{2}\left(z\frac{\partial^2 y}{\partial z^2} + \frac{\partial^2}{\partial z^2} zy\right) + \left(\tfrac{1}{4}z^3 + \frac{1}{2}\frac{\delta}{g}z^2 - Ez + \frac{L^2}{z}\right)y \qquad (2.10)$$

where

$$E = N + L + 1 - \frac{\delta^2}{4g^2} = O(N). \qquad (2.11)$$

Let us introduce the intensive quantities

$$\eta = g^2 E, \quad \lambda = g^2 L, \quad x = gz. \qquad (2.12)$$

Then equation (2.10) takes the form

$$ih\frac{\partial y}{\partial t} = -\frac{h^2}{2}\left(x\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2}{\partial x^2} xy\right) + V(x)y \overset{\text{def}}{=} Hy \qquad (2.13)$$

where

$$h = g^2 = O(N^{-1}) \qquad (2.14)$$

and

$$V(x) = \tfrac{1}{4}x^3 + \frac{\delta}{2}x^2 - \eta x + \frac{\lambda^2}{x}. \qquad (2.15)$$

The dependence on N in equation (2.13) is now concentrated in the parameter h (2.14), which appears in exactly the same way as Planck's constant in ordinary quantum mechanics. Consequently, for large N, the semi-classical solution of equation (2.13) is significant.

For sufficiently small t, the semi-classical solution is given by

$$y(x,t)=(e^{-(i/h)Ht}y_0)(x)=\int_0^\infty dx_0\,K(x,x_0,t)\,y_0(x_0) \tag{2.16}$$

where

$$K(x,x_0,t)=\phi\exp\frac{i}{h}S(x,x_0,t) \tag{2.17}$$

$$\phi^2=-\frac{1}{2\pi ih}\frac{\partial^2 S(x,x_0,t)}{\partial x\,\partial x_0} \tag{2.18}$$

and S is the classical action function. The latter has been calculated before [2, 4]

$$S(x,x_0,t)=\int_{x_0}^{x}dx'\left(\eta-\tfrac{1}{4}x'^2-\frac{\delta}{2}x'-\frac{\varepsilon}{x'}-\frac{\lambda^2}{x'^2}\right)^{\frac{1}{2}}+\varepsilon t. \tag{2.19}$$

The (negative) energy $\varepsilon(x,x_0,t)$ is implicitly given by one of the following relations

$$t=\int_{x_0}^{x}dx'\,R^{-\frac{1}{2}}(x') \tag{2.20}$$

or

$$x_0=x+\frac{p'(t)\,R(x)^{\frac{1}{2}}+\frac{1}{2}R'(x)\,[p-\frac{1}{24}R''(x)]+\frac{1}{24}RR'''(x)}{2(p-\frac{1}{24}R''(x))^2+\frac{1}{2}R} \tag{2.21}$$

where

$$R(x)=-x^4-2\delta x^3+4\eta x^2-4\varepsilon x-4\lambda^2. \tag{2.22}$$

The primes denote the derivatives with respect to the argument and

$$p=p(t;g_2,g_3) \tag{2.23}$$

is Weierstrass' p-function with the invariants

$$\begin{aligned}g_2&=\tfrac{4}{3}\eta^2-2\delta\varepsilon+4\lambda^2\\ g_3&=-\tfrac{8}{27}\eta^3+\tfrac{2}{3}\eta\delta\varepsilon+\tfrac{8}{3}\eta\lambda^2+\delta^2\lambda^2+\varepsilon^2.\end{aligned} \tag{2.24}$$

Together with x_0, the square-root $R(x_0)^{\frac{1}{2}}$ can be expressed rationally by x and $R(x)^{\frac{1}{2}}$ [7]

$$\begin{aligned}\sqrt{R(x_0)}=\{&p'[(p-\tfrac{1}{24}R'')^2R'-\tfrac{1}{4}RR'+\tfrac{1}{6}RR'''(p-\tfrac{1}{24}R'')]\\ &+\sqrt{R(x)}\,[-(12p^2-g_2)((p-\tfrac{1}{24}R'')^2+\tfrac{1}{4}R)\\ &+4p'^2(p-\tfrac{1}{24}R'')]\}/[2(p-\tfrac{1}{24}R'')^2+\tfrac{1}{2}R]^2.\end{aligned} \tag{2.25}$$

Let us assume a quantum mechanical initial state y_0 of the form

$$y_0(x)=\rho(x)\exp\frac{\varphi(x)}{h}, \tag{2.26}$$

then according to equation (2.16)

$$y(x,t)=\int_0^\infty dx_0\,\phi(x,x_0,t)\,\rho(x_0)\exp\frac{1}{h}[iS(x,x_0,t)+\varphi(x_0)]. \tag{2.27}$$

We evaluate this integral for $h \to 0$ by the saddle-point method. The saddle-points $x_0' = x_0'(x,t)$ are the solutions of

$$\frac{\partial}{\partial x_0'}\left[iS(x,x_0',t)+\varphi(x_0')\right]=0. \tag{2.28}$$

In general, the saddle-points x_0' are complex (in the resonant case $\delta = 0$ they are purely imaginary, see equation (3.5) below), so that no proper classical mechanical path exists. This is one, but not the main reason, why Maslov's method [6] cannot be directly applied. According to the saddle-point method, the path of integration in equation (2.27) is deformed in such a way that it runs through the saddle-points x_0' in the direction of steepest descent ϑ. This direction ϑ is given by

$$\vartheta + 2\arg\frac{\partial^2}{\partial x_0'^2}(iS+\varphi)=\pi. \tag{2.29}$$

One saddle-point x_0' gives the following contribution to the integral (2.27)

$$\left(-\frac{1}{2\pi i h}\frac{\partial^2 S(x,x_0',t)}{\partial x\,\partial x_0'}\frac{2\pi h}{\left|\dfrac{\partial^2}{\partial x_0'^2}(iS+\varphi)\right|}\right)^{\frac{1}{2}} e^{i\vartheta}\, e^{(i/h)\,S(x,x_0',t)}\, y_0(x_0')$$

$$=\left(\frac{\partial x_0'}{\partial x}\right)^{\frac{1}{2}} e^{(i/h)\,S(x,x_0',t)}\, y_0(x_0'), \tag{2.30}$$

where the last equality follows from equation (2.28) by implicit differentiation.

In general, i.e. for larger t, we have $n(x,t)$ saddle-points $x_k = x_k(x,t)$, $k = 1, \dots n$. Then the semi-classical solution reads

$$y(x,t)=\sum_{k=1}^{n(x,t)}\left(\frac{\partial x_k}{\partial x}\right)^{\frac{1}{2}}\exp\frac{i}{h}S(x,x_k,t)\,y_0(x_k). \tag{2.31}$$

This, apart from the phase factor, is a shift of the initial wave-packet. The square-root of the Jacobian derivative ensures the conservation of the L^2-norm.

3. Applications

The most interesting initial state ψ_0, regarding the laser, is the fully excited state of all N atoms in the upper level given by

$$\psi_0 = \{0, 0, \dots 1\} \tag{3.1}$$

in $\mathbb{C}^{N+1}$ (2.6). The related mechanical initial state (2.9)

$$y_0 = (2L)^{-\frac{1}{2}} Z^L \exp\left(-\tfrac{1}{4}z^2 - \frac{\delta}{2g}z\right)$$

$$= \text{const}\exp\frac{1}{h}\left(\lambda\log x - \tfrac{1}{4}x^2 - \frac{\delta}{2}x\right) \tag{3.2}$$

is of the form (2.26). The equation (2.28) determining the saddle-points x_k now reads as follows

$$i\left(\eta - \tfrac{1}{4}x_k^2 - \frac{\delta}{2}x_k - \frac{\varepsilon_k}{x_k} - \frac{\lambda^2}{x_k^2}\right)^{\frac{1}{2}} + \frac{\lambda}{x_k} - \tfrac{1}{2}x_k - \frac{\delta}{2} = 0$$

or

$$\varepsilon_k = \left(\eta + \frac{\delta^2}{4} - \lambda\right)x_k - \lambda\delta. \tag{3.3}$$

Inserting this for ε in equation (2.21), we get a transcendental equation for the saddle-points. For later discussion, let us write down this equation for the resonant case $\delta = 0$ and $x = 0$: with

$$\varepsilon_k = h(N+1)\,x_k \tag{3.4}$$

we get

$$x_k = -\frac{igLP'}{P^2 - L^2 + (N+1)P}, \tag{3.5}$$

where

$$P(t) = \frac{1}{h}p(t; g_2, g_3) - \frac{E}{3} = p\left(gt; \frac{g^2}{h^2}, \frac{g_3}{h^3}\right) - \frac{E}{3}. \tag{3.6}$$

As a first application of the semi-classical solution and for later use, we will calculate the mean photon number $n(t)$

$$n(t) = (\psi_0, a^+(t)\,a(t)\,\psi_0), \tag{3.7}$$

which has already been obtained by the previous method [2, 4]. It was shown there that $n(t)$ can be expressed in terms of the quantum mechanical time evolution (2.16) as follows:

$$n(t) = \lim_{x \to 0} x^{-L} e^{(i/h)Ht}\left[\frac{1}{h}\left(\tfrac{1}{2}x^2 + \frac{\delta}{2}x + \lambda\right) + x\frac{\partial}{\partial x}\right] e^{-(i/h)Ht}\, y_0(x). \tag{3.8}$$

It is important to realize that in this exact expression the photon number is given by the value of a quantum mechanical wave-packet at a *single point* $x = 0$. This is the main reason why the usual treatment of the classical limit [6] cannot be taken over, because in the latter only L^2-convergence is obtained. This is sufficient for ordinary quantum mechanics, but here it is not. Indeed, for special values of t there are exceptional points (foci), where the semi-classical solution (2.31) diverges locally. A more refined treatment is then necessary. We shall return to this interesting point in Part II of the paper.

Using equation (2.31) in equation (3.7), in leading order $O(h^{-1}) = O(N)$, we need only differentiate the exponent of (2.31) and in the exponent we must only differentiate with respect to the explicit variable x, the x-dependence of x_k gives no contribution because of equation (2.28). Since

$$x\frac{\partial}{\partial x}S(x, x_k, t) = \tfrac{1}{2}(-x^4 - 2\delta x^3 + 4\eta x^2 - 4\varepsilon_k x - 4\lambda^2)^{\frac{1}{2}} \overset{\text{def}}{=} \tfrac{1}{2}R_k(x)^{\frac{1}{2}}, \tag{3.9}$$

we get

$$n(t) = \frac{1}{h}\lim_{x\to 0} x^{-L} e^{(i/h)Ht} \sum_k \left(\tfrac{1}{2}x^2 + \frac{\delta}{2}x + \lambda + \frac{i}{2}R_k(x)^{\frac{1}{2}}\right)$$

$$\times \left(\frac{\partial x_k}{\partial x}\right)^{\frac{1}{2}} e^{(i/h)S(x,\,x_k,\,t)}\, y_0(x_k). \tag{3.10}$$

Under the inverse transformation $\exp(i/h)Ht$, all saddle-points x_k are shifted back to $x = 0$, ε_k (3.3) goes into ε_0

$$\varepsilon_0 = -\lambda\delta, \tag{3.11}$$

and x under the sum in equation (3.10) becomes the initial point $\tilde{x}$ corresponding to the saddle-point $x = 0$, which is uniquely given by equation (2.21) with $x = 0$, $\varepsilon = \varepsilon_0$ (3.11)

$$\tilde{x} = \frac{i\lambda p' - \lambda\delta\left(p - \dfrac{\eta}{3}\right) + \delta\lambda^2}{\left(p - \dfrac{\eta}{3}\right)^2 - \lambda^2}. \tag{3.12}$$

Then

$$n(t) = \frac{1}{h}\left(\tfrac{1}{2}\tilde{x}^2 + \frac{\delta}{2}\tilde{x} + \lambda + \frac{i}{2}\sqrt{R(\tilde{x})}\right). \tag{3.13}$$

The square-root in equation (3.13) is obtained from equation (2.25) with $x = 0$, $\varepsilon = \varepsilon_0$. Substituting this and equation (3.12) into equation (3.13), we finally get

$$n(t) = 2\frac{\lambda}{h}\frac{p + \frac{2}{3}\eta + \dfrac{\delta^2}{4}}{p - \frac{1}{3}\eta + \lambda} = 2L\frac{P + E + \dfrac{\delta^2}{4h}}{P + L} = 2L + 2L\frac{N}{P + L}, \tag{3.14}$$

where $P(t)$ is defined in equation (3.6). The invariants of the p-function are given by equation (2.24) with $\varepsilon = \varepsilon_0$ (3.11.)

Both these invariants and equation (3.14) are the results in leading order in N^{-1}. By our previous method, one more order in N^{-1} could be calculated

$$n(t) = 2L + (2L+1)\frac{N}{P + L + L} \tag{3.15}$$

with the invariants

$$\frac{g_2}{h^2} = \tfrac{4}{3}E^2 + 2\frac{\delta^2}{h}(L+1) + 4L^2$$

$$\frac{g_3}{h^3} = -\tfrac{8}{27}E^3 - \frac{2}{3}\frac{\delta^2}{h}E(L+1) + \tfrac{8}{3}EL^2 + 2\frac{\delta^2}{h}L(L+1) + 2LN. \tag{3.16}$$

We now turn to the normal ordered photon correlation function (1.1), where the present method is more powerful. In mechanical terms, we have to calculate [5]

$$g(t, t_1) = \lim_{x \to 0} x^{-L} e^{(i/h)Ht_1} x^{\frac{1}{2}} e^{(i/h)H_- t_2} \left[\frac{1}{h}\left(\tfrac{1}{2}x^2 + \frac{\delta}{2}x + \lambda_-\right) + x\frac{\partial}{\partial x}\right]$$

$$e^{-(i/h)H_- t_2}\left[\frac{1}{h}\left(\tfrac{1}{2}x^{3/2} + \frac{\delta}{2}x^{\frac{1}{2}} + \lambda x^{-\frac{1}{2}}\right) + x^{\frac{1}{2}}\frac{\partial}{\partial x}\right] e^{-(i/h)Ht_1} y_0(x) \tag{3.17}$$

with

$$t_2 = t - t_1, \quad \lambda_- = \frac{1}{h}(L - \tfrac{1}{2}), \quad \eta_- = \frac{1}{h}(E - \tfrac{1}{2}) \tag{3.18}$$

and H_- is given by the right side of equation (2.13), with λ, η in the potential $V(x)$ (2.15) substituted by λ_-, η_-. Inserting the semi-classical solution as above, we get

$$g(t, t_1) = \ldots e^{-(i/h)H_- t_2} \frac{1}{h}\sum_k \left(\tfrac{1}{2}x^{3/2} + \frac{\delta}{2}x^{\frac{1}{2}} + \lambda x^{-\frac{1}{2}} + \frac{i}{2}x^{-\frac{1}{2}} R_k(x)^{\frac{1}{2}}\right)$$

$$\times \left(\frac{\partial x_k}{\partial x}\right)^{\frac{1}{2}} e^{(i/h)S(x, x_k, t_1)} y_0(x_k) \tag{3.19}$$

where x_k is given by equation (2.21) with

$$\varepsilon_k = \left(\eta + \frac{\delta^2}{4} - \lambda\right) x_k - \lambda\delta. \tag{3.20}$$

The exponential of H_- shifts x to saddle-points x'_m and the x_k to x_{km},

$$g(t, t_1) = \cdots \frac{1}{h^2}\sum_m \left(\tfrac{1}{2}x^2 + \frac{\delta}{2}x + \lambda_- + \frac{i}{2}R_m^-(x)^{\frac{1}{2}}\right)\left(\frac{\partial x'_m}{\partial x}\right)^{\frac{1}{2}}$$

$$\times e^{(i/h)S_-(x, x'_m, t_2)} \sum_k \left(\tfrac{1}{2}x'^{\,3/2}_m + \frac{\delta}{2}x'^{\frac{1}{2}}_m + \lambda x'^{-\frac{1}{2}}_m + \frac{i}{2}x'^{-\frac{1}{2}}_m R_k(x'_m)^{\frac{1}{2}}\right)$$

$$\times \left(\frac{\partial x_{km}}{\partial x'_m}\right)^{\frac{1}{2}} e^{(i/h)S(x'_m, x_{km}, t_1)} y_0(x_{km}). \tag{3.21}$$

The saddle-points x'_m are determined by

$$\frac{\partial}{\partial x'_m}[S_-(x, x'_m, t_2) + S(x'_m, x_{km}, t_1)] = 0 \tag{3.22}$$

and the x_{km} by equation (2.21) with $x = x_k$ and

$$\varepsilon_{km} = \left(\eta + \frac{\delta^2}{4} - \lambda\right) x_{km} - \lambda\delta. \tag{3.23}$$

From equation (3.22) it follows

$$\varepsilon_m' - \varepsilon_{km} = O(h). \tag{3.24}$$

The inverse exponential of H_- shifts all x_m' back to x, which can be combined with the factor $x^{\frac{1}{2}}$ in equation (3.17). The points x_{km} move back to x_k, and x in equation (3.21) becomes $\tilde{x}_k(x, t_2)$ determined by

$$\tilde{\varepsilon}_k = \varepsilon_m'|_{x_m'=x} = \varepsilon_{km}|_{x_{km}=x_k} + O(h) = \varepsilon_k + O(h). \tag{3.25}$$

Finally, the inverse exponential of H_1 shifts all x_k to $x = 0$, x changes into $\tilde{x}$ defined by

$$\tilde{\varepsilon} = \varepsilon_0 = -\lambda\delta \tag{3.26}$$

and the $\tilde{x}_k$ become $\tilde{\tilde{x}}(\tilde{x}, t_2)$ which is uniquely defined by

$$\tilde{\tilde{\varepsilon}} = \tilde{\varepsilon}_k|_{\tilde{x}_k=\tilde{\tilde{x}}} = \varepsilon_k|_{x_k=0} + O(h) = -\lambda\delta + O(h). \tag{3.27}$$

Summing up

$$\begin{aligned} g(t, t_1) = \frac{1}{h^2}&\left(\tfrac{1}{2}\tilde{\tilde{x}}^2 + \frac{\delta}{2}\tilde{\tilde{x}} + \lambda_- + \frac{i}{2}\sqrt{R^-(\tilde{\tilde{x}})}\right) \\ &\times\left(\tfrac{1}{2}\tilde{x}^2 + \frac{\delta}{2}\tilde{x} + \lambda + \frac{i}{2}\sqrt{R(\tilde{x})}\right), \end{aligned} \tag{3.28}$$

where $\tilde{x}$ and $R(\tilde{x})$ are given by the classical solution (2.21), (2.25) with the initial point $x = 0$, time t_1 and energy $\tilde{\varepsilon}$ (3.26), while $\tilde{\tilde{x}}$ and $R^-(\tilde{\tilde{x}})$ correspond to the initial point $\tilde{x}$, time t_2 and a *different energy* $\tilde{\tilde{\varepsilon}}$ (3.27). If the energy difference of the order $O(h)$ is neglected, we have

$$\tilde{\tilde{x}}(\tilde{x}, t_2) = \tilde{\tilde{x}}(0, t_2) \tag{3.29}$$

and the correlation function factorizes into a product of the photon numbers

$$g(t, t_1) = n(t)\,n(t_1). \tag{3.30}$$

However, there is one place where the difference in the energies cannot be neglected, not even in leading order, namely in the invariants of the p-function

$$p_2 = p(g_2^-, g_3^-, t - t_1) \tag{3.31}$$

appearing in the equation for $\tilde{\tilde{x}}$. At all other places the error remains small uniformly in time, but since the p-functions p_1, p_2 with slightly different invariants have different periods, they come out of phase and the error increases with time (see next section). Therefore we must treat the first factor in equation (3.28) more carefully, retaining the difference of the invariants.

Since the semi-classical solution gives the invariants only in the leading order, we have to refer to the quantum mechanical calculation at this point and we take the invariants g_2^-, g_3^- from equation (3.16) with

$$E_- = E - \tfrac{1}{2}, \quad L_- = L - \tfrac{1}{2} \tag{3.32}$$

instead of E, L. Then the straightforward procedure to calculate $\tilde{\tilde{x}}$ would be to substitute in equation (2.21), written down for $\tilde{\tilde{x}}$, the expression (3.12) for the initial point $\tilde{x}$

(which stands in place of x in equation (2.21)). This, however, leads to tedious algebraic manipulations which can be reduced by the following trick: Without the difference in the invariants, the first factor in equation (3.28) would give

$$n_-(t) = 2L - 1 + 2LN\frac{1}{P(t) + L + \frac{1}{2}}. \tag{3.33}$$

Here we express the p-function by means of the addition formula

$$P(t) = P(t_1 + t_2) = \frac{1}{4}\left(\frac{P_1' - P_2'}{P_1 - P_2}\right)^2 - P_1 - P_2 - E \tag{3.34}$$

in terms of $p(t_1)$ and $p(t_2)$. The desired result is then obtained by transforming the resulting elliptic function to the normal form (with the derivatives P_1', P_2' in the nominator) and taking *different* invariants g_2, g_3 and $g_{\bar{2}}, g_{\bar{3}}$ in P_1 and P_2, respectively. In this transformation, the usual formula for the derivatives

$$(P_{1,2}')^2 = \frac{1}{h^3}(p_{1,2}^3 - g_2 p_{1,2} - g_3) \tag{3.35}$$

must be used, where in g_2 and g_3 (2.24) only the leading order

$$\varepsilon = -\delta\lambda = -\frac{\delta}{h}L \tag{3.36}$$

has to be taken with

$$(P_{1,2}')^2 = 4(P_{1,2}^2 - L^2)(P_{1,2} + E) - 2\frac{\delta^2}{h}LP_{1,2} - 2\frac{\delta^2}{h}L^2. \tag{3.37}$$

In this way we finally get

$$\frac{1}{P(t) + L} = \frac{A}{B} \tag{3.38}$$

where

$$\begin{aligned} A = {} & \tfrac{1}{2}P_1'P_2' + P_1P_2(P_1 + P_2) + 2\left(N - \frac{\delta^2}{4h}\right)P_1P_2 - L\left(L + \frac{\delta^2}{2h}\right)(P_1 + P_2) \\ & + L(P_1^2 + P_2^2) - 2L^2\left(N + L + \frac{\delta^2}{4h}\right) \\ B = {} & P_1^2P_2^2 + 2LP_1P_2(P_1 + P_2) + L^2(P_1^2 + P_2^2) + 4L(N + L)P_1P_2 \\ & + 4L^2(N + L)(P_1 + P_2) - 2L^3(P_1 + P_2) - 4L^3(N + L) + 4L^2(N + L)^2 + L^4 \end{aligned} \tag{3.39}$$

which leads to the following result for the correlation function

$$g(t, t_1) = n(t_1)\left(2L + 2LN\frac{A}{B}\right). \tag{3.40}$$

This rather complicated formula will be simplified and discussed in the next section.

4. Discussion

The result (3.15) for the mean photon number $n(t)$ has already been discussed in Refs. [2, 4]. We add to those findings some remarks concerning the general (non-resonant) case. For the properties of the p-function, the fundamental cubic equation

$$4e^3 - g_2 e - g_3 = 0 \tag{4.1}$$

is of central importance. In the leading order $O(N)$, it has the following simple solutions $e_1 \geqslant e_2 \geqslant e_3$

$$\begin{aligned} e_{1,3} &= -\frac{E}{6} + \frac{L}{2} \pm \tfrac{1}{2}\sqrt{(E+L)^2 + 2L\delta^2/h} \\ e_2 &= \frac{E}{3} - L. \end{aligned} \tag{4.2}$$

The parameter k^2 of the elliptic functions then becomes

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \tfrac{1}{2} + \frac{E - 3L}{2\sqrt{(E+L)^2 + 2L\delta^2/h}}. \tag{4.3}$$

It follows from the definition (2.11) that

$$0 \leqslant k^2 \leqslant 1. \tag{4.4}$$

The value $k^2 = \tfrac{1}{2}$ corresponds to the so-called super-radiant case $E = 3L$, i.e. $R = 2N + 1 - \delta^2/4h$.

The p-function $P(t)$ (3.6) assumes its minimum value $P_{\min}$

$$P_{\min} = e_1 - \frac{E}{3} \tag{4.5}$$

at the half-period

$$\omega = \frac{K(k)}{\sqrt{e_1 - e_3}}, \tag{4.6}$$

$K(k)$ is the complete elliptic integral of the first kind. The minimum $P_{\min}$ corresponds to a maximum of the mean photon number (3.15)

$$n_{\max} = 2L + \frac{N}{1 + (\sqrt{(E+L)^2 + 2L\delta^2/h} - E - L)/4L}. \tag{4.7}$$

This formula shows that the total energy N stored in the N excited atoms is converted into radiation only in the resonant case $\delta = 0$. For $\delta = O(1) \neq 0$, some finite fraction of the N quanta remains in the atomic system.

Let us now consider the normal ordered correlation function (3.40). As has already been noticed in the last section, the second factor in equation (3.40) simplifies to $n_-(t)$ (3.33) if the invariants of P_1, P_1' are set equal to $g_{\bar{2}}$, $g_{\bar{3}}$. Therefore we write

$$g(t, t_1) = n(t_1)\left[2L + 2LN\frac{A}{B}\bigg|_- + 2LN\left(\frac{A}{B} - \frac{A}{B}\bigg|_-\right)\right], \tag{4.8}$$

where the subscript '–' means that the invariants of *all* p-functions are $g_{\bar{2}}, g_{\bar{3}}$. Since the difference in the invariants is one order in N smaller than the invariants themselves, it is sufficient to calculate the first-order corrections

$$\frac{A}{B}-\frac{A}{B}\bigg|_{-}=\frac{\partial}{\partial P_1}\frac{A}{B}\bigg|_{-}(P_1-P_{\bar{1}}) \tag{4.9}$$

where

$$P_1-P_{\bar{1}}=\frac{\partial P_{\bar{1}}}{\partial g_2}(g_2-g_{\bar{2}})+\frac{\partial P_{\bar{1}}}{\partial g_3}(g_3-g_{\bar{3}}). \tag{4.10}$$

In calculating the derivative with respect to P_1 in equation (4.9), P_1' has to be expressed by equation (3.35) by means of P_1. We then have

$$\frac{\partial}{\partial P_1}\frac{A}{B}\bigg|_{-}=\frac{1}{P_{\bar{1}}'}\frac{\partial}{\partial t_1}\frac{A}{B}\bigg|_{-}=\frac{1}{P_{\bar{1}}'}\frac{\partial}{\partial t_1}\frac{1}{P_{-}(t_1+t_2)+L}$$

$$=\frac{1}{P_{\bar{1}}^{1}}\frac{\partial}{\partial t}\frac{1}{P_{-}+L}=-\frac{P_{-}'}{P_{1-}'(P_{-}+L)^2}. \tag{4.11}$$

The right side of equation (4.10) is computed using the formula [8]

$$\frac{\partial p_1}{\partial g_2}=\frac{1}{\Delta}\left\{p_1'\left(-\tfrac{9}{2}g_3\zeta_1+\frac{g_2^2}{4}t_1\right)-9g_3p_1^2+\frac{g_2^2}{2}p_1+\tfrac{3}{2}g_2g_3\right\}$$

$$\frac{\partial p_1}{\partial g_3}=\frac{1}{\Delta}\{p_1'(3g_2\zeta_1-\tfrac{9}{2}g_3t_1)+6g_2p_1^2-9g_3p_1-g_2^2\} \tag{4.12}$$

where

$$\Delta=g_2^3-27g_3^2 \tag{4.13}$$

and $\zeta_1=\zeta(t_1)$ is Weierstrass' ζ-function. Since the correction we are calculating becomes important only for large t_1, we must concentrate on the first terms in equation (4.12) containing $\zeta(t_1)$ and t_1. All other terms are uniformly bounded in t_1 apart from the poles at multiples of ω (see below). For simplicity let us consider the resonant case $\delta=0$. Then, using

$$g_2-g_{\bar{2}}=\tfrac{4}{3}E+4L \tag{4.14}$$

$$g_3-g_{\bar{3}}=-\tfrac{4}{9}E^2+\tfrac{8}{3}LE+\tfrac{4}{3}L^2, \tag{4.14}$$

we get

$$P_1-P_{\bar{1}}=\frac{P_{\bar{1}}'}{4L(E+L)}\left[\zeta_{\bar{1}}+\left(\frac{E}{3}+L\right)t_1\right]. \tag{4.15}$$

This expression has the defect that it introduces poles of the fourth order at multiples of ω, which are actually compensated by the neglected terms in equation (4.12). The bracket in equation (4.15) increases linearly with t_1 like

$$\left(\frac{\eta}{\omega}+\frac{E}{3}+L\right)t_1=\frac{E(k)}{K(k)}(E+L)\,t_1 \tag{4.16}$$

where $\eta = \zeta(\omega)$ and $E(k)$ is the complete elliptic integral of the second kind. We make an error uniformly small in t_1 by using in equation (4.9)

$$P_1 - P_1^- = \frac{P_1^{-\prime}}{4L}\frac{E(k)}{K(k)}t_1, \tag{4.17}$$

instead of equation (4.15). This finally leads to the following more transparent formula for the correlation function

$$g(t, t_1) = n(t_1)\left[n_-(t) - \frac{1}{2}\frac{E(k)}{K(k)}N\frac{P_-'(t)}{(P_- + L)^2}t_1\right]. \tag{4.18}$$

We have checked both results, (3.40) and (4.18), numerically in the super-radiant case with $N = 50$, $R = 101$. We find that the difference between the numerical and the analytical results are uniformly small and nicely periodic in *both* time variables t, t_1 for both formulae (3.40) and (4.18) (see Figs. 1 and 2). Since the deviation is obviously one order in N smaller than g itself, both results (3.40) and (4.18) are confirmed. However, to our surprise, the latter formula (4.18) seems to be better, because the periodic structure in t is more simple (Fig. 2) and the remainder is slightly smaller. By combination with the previous result for small times [5], it should be possible to refine the formulae, including the next order in N^{-1}.

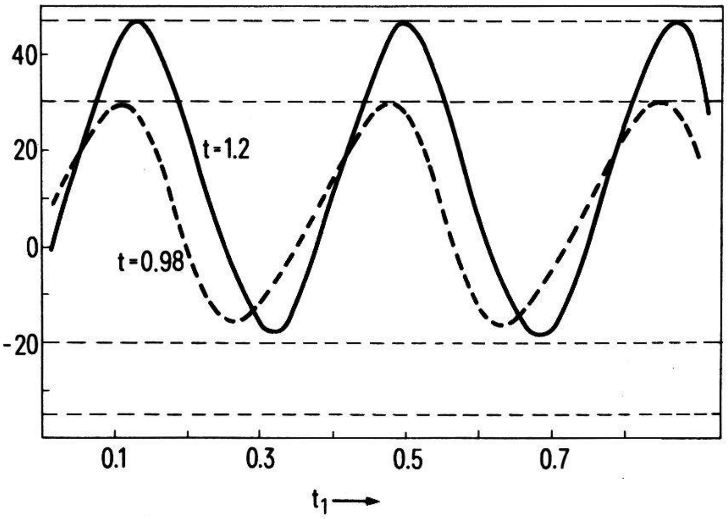

Figure 1

Differences between the numerical and analytical values of the correlation function $g(t, t_1)$ as a function of t_1 for fixed t and $N = 50$, $R = 101$. The solid curve corresponds to the formula (3.40), the dashed curve to the formula (4.18). The horizontal solid and dashed lines are uniform upper and lower bounds of the functions.

One important consequence of equation (4.18) is the decay in the coherence of the radiation for increasing t_1. As a measure of the coherence let us consider the quantity

$$\frac{g(t, t_1) - n(t_1)n_-(t)}{n(t_1)n_-(t)} = -\frac{N}{4L}\frac{E(k)}{K(k)}\frac{P_-'(t)}{(P_- + L)(P_- + E)}t_1. \tag{4.19}$$

If this is of the order $O(1)$, i.e. $t_1 = O(N^{\frac{1}{2}})$, one can say that the coherence is completely destroyed. Since the period 2ω (4.6) is of the order $O(N^{-\frac{1}{2}})$, this occurs after $O(N)$ oscillations of the system. That means that the coherence decays very slowly. Indeed, it must be investigated whether the semi-classical solution can be maintained up to these

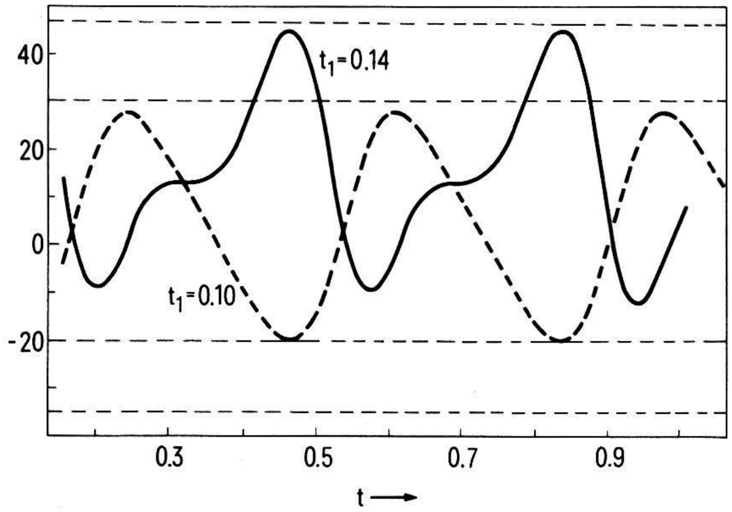

Figure 2
Differences between the numerical and analytical values of the correlation function $g(t, t_1)$ as a function of t for fixed t_1. The solid curve corresponds to equation (3.40), the dashed curve to equation (4.18).

large times $t = O(h^{-\frac{1}{2}})$. This is again a problem which goes beyond the usual discussion of the classical limit and will be considered in Part II of the paper.

It is interesting to compare equation (4.19) with the corresponding result for the time-ordered correlation function $g_1(t, t_1)$,

$$g_1(t, t_1) = (\psi_0, a^+(t_1)\, a(t_1)\, a^+(t)\, a(t)\, \psi_0), \tag{4.20}$$

which looks quite similar, namely [5]

$$\frac{g_1(t, t_1) - n(t)n(t_1)}{n(t)\,n(t_1)} = \frac{1}{2}\frac{N}{4L}\frac{P'}{(P+L)(P+E)}\frac{P_1'}{(P_1+L)(P_1+E)}. \tag{4.21}$$

This quantity, however, is of the order $O(N^{-1})$ for arbitrary t, t_1. Consequently, the time-ordered correlation function shows a high degree of factorization for all times, in contrast to the normal ordered one.

References

[1] R. Dicke, Phys. Rev. *93*, 99 (1954).
[2] G. Scharf, Ann. Phys. *83*, 71 (1974).
[3] K. Hepp and E. H. Lieb, Helv. Phys. Acta *46*, 573 (1973).
[4] R. Weiss, Helv. Phys. Acta *46*, 546 (1973).
[5] G. Scharf and R. Weiss, Helv. Phys. Acta *47*, 505 (1974).
[6] V. P. Maslov, *Théorie des perturbations et méthodes asymptotic* (Dunod Gauthier-Villars, Paris 1972).
[7] K. Weierstrass, *Mathematische Werke*, Vol. V, p. 4 (1915).
[8] M. Abramowitz and A. Stegun (Eds.), *Handbook of Mathematical Functions* (Dover Publications, New York 1965), p. 641.