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Lense–Thirring Effect and Localizability of Gravitational Energy-Momentum

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Abstract. The Lense–Thirring solution is analysed in connection with the problem of the gravitational energy-momentum localization in general relativity. Using Møller's conditions for the tetrads, it is shown that for great distances from the source the gravitational energy has a well-determined velocity proportional to r^{-1}. The same result is obtained for the analogous problem of electrodynamics, according to the well-known Maxwellian character of general relativity in the case of the weak field. The notion of angular momentum then allows, in principle, the exclusion of some possible gravitational energy-momentum distributions. Thus, in opposition to some authors' opinions, it would not be impossible to localize the gravitational energy-momentum.

1. Introduction

In a recent paper [1] we have developed physical arguments that seem to indicate the possibility of localizing the gravitational energy-momentum (GEM) in general relativity (GR). Mathematically, such a localization can be obtained in the frame of Scherrer's formalism (or of the theory of tetrads if one prefers). Without referring back to the well-known difficulties of this formalism, we intend to expose, in the present paper, a new argument in favour of GEM localizability. For this purpose we analyse, in detail, the famous Lense–Thirring solution of Einstein's equations [2] and its electrodynamic analogue.

2. The Lense–Thirring Solution

We consider an homogeneous sphere, of mass density ρ, at rest in a given inertial co-ordinate system. This body produces a gravitational field (GF), classically described by the Newtonian potential $\psi(r) = -(GM/r)$ ($G \equiv$ Newtonian gravitation constant) and by the Schwarzschild solution in GR. Rotating our sphere with constant angular velocity ω in the classical theory changes nothing in regard to the case $\omega = 0$. But in

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the GR the GF is slightly modified according to the Lense–Thirring solution

$$(g_{\mu\nu}) = \begin{pmatrix} 1-\frac{2a}{r} & -\frac{4a}{5r}\frac{l^2}{r^2}\frac{\omega y}{c} & \frac{4a}{5r}\frac{l^2}{r^2}\frac{\omega c}{c} & 0 \\ -\frac{4a}{5r}\frac{l^2}{r^2}\frac{\omega y}{c} & -1-\frac{2a}{r} & 0 & 0 \\ \frac{4a}{5r}\frac{l^2}{r^2}\frac{\omega x}{c} & 0 & -1-\frac{2a}{r}, & 0 \\ 0 & 0 & 0 & -1-\frac{2a}{r} \end{pmatrix}, \tag{2.1}$$

where $a \equiv GM/c^2$, $M \equiv$ total mass of the sphere, $r \equiv \sqrt{x^2+y^2+z^2}$, and $l \equiv$ radius of the sphere. The $g_{\mu\nu}$ are expressed as functions of orthogonal harmonic co-ordinates (x°, x, y, z), Lorentzian at infinity. They correspond to a weak field, i.e. measured at large distances from the source. Moreover, the terms of order $(v/c)^n$ are neglected for $n \geqslant 2$. For $\omega = 0$ we obtain, of course, the Schwarzschild solution in first approximation.

When $\omega = 0$, the GF is static, and, if one adopts the point of view of GEM localizability, each point is characterized by a definite GE density $-\kappa^{-1}\mathfrak{T}_{0,}{}^{,0}$ $(\kappa \equiv 8\pi G/c^4)$ that tends to the classical potential energy density at the limit of the weak field [1]. When $\omega =$ const. $\neq 0$, the movement of the sphere generates a rotation of the GF (that remains stationary), with a GE flux. Our purpose now is to calculate the velocity of this flux.

To this end we know that we have to impose Møller's supplementary conditions for the tetrads $g^{\lambda,}{}_{,\mu}$ [1, 3]. It is easily obtained[2]):

$$(g^{\lambda,}{}_{,\mu}) = \begin{pmatrix} 1-\frac{a}{r} & -\frac{2a}{5r}\frac{l^2}{r^2}\frac{\omega y}{c} & \frac{2a}{5r}\frac{l^2}{r^2}\frac{\omega x}{c} & 0 \\ \frac{2a}{5r}\frac{l^2}{r^2}\frac{\omega y}{c} & 1+\frac{a}{r} & 0 & 0 \\ -\frac{2a}{5r}\frac{l^2}{r^2}\frac{\omega x}{c} & 0 & 1+\frac{a}{r} & 0 \\ 0 & 0 & 0 & 1+\frac{a}{r} \end{pmatrix} \tag{2.2}$$

For the following calculations it is preferable to use polar co-ordinates. Applying the usual transformation formulae to the four covariant co-ordinate vectors $g^{\lambda,}{}_{,\mu}$, we

[2]) Recall to mind that the tetrads $g^{\gamma,}{}_{,\mu}$ are connected with the Einsteinian metric tensor $g_{\mu\nu}$ by the formulae $g_{\mu\nu} = e_\alpha g^{\alpha,}{}_{,\mu} g^{\alpha,}{}_{,\nu}$.

obtain

$$(g^{\lambda}{}_{\cdot,\mu}) = \begin{pmatrix} 1-\frac{a}{r} & 0 & 0 & \frac{2}{5}\frac{\omega a}{c}\frac{l^2}{r}\sin^2\vartheta \\ \frac{2}{5}\frac{\omega a}{c}\frac{l^2}{r^2}\sin\vartheta\sin\varphi & \left(1+\frac{a}{r}\right)\sin\vartheta\cos\varphi & \left(1+\frac{a}{r}\right)r\cos\vartheta\cos\varphi & -\left(1+\frac{a}{r}\right)r\sin\vartheta\sin\varphi \\ -\frac{2}{5}\frac{\omega a}{c}\frac{l^2}{r^2}\sin\vartheta\cos\varphi & \left(1+\frac{a}{r}\right)\sin\vartheta\cos\varphi & \left(1+\frac{a}{r}\right)r\cos\vartheta\sin\varphi & \left(1+\frac{a}{r}\right)r\sin\vartheta\cos\varphi \\ 0 & \left(1+\frac{a}{r}\right)\cos\vartheta & -\left(1+\frac{a}{r}\right)r\sin\vartheta & 0 \end{pmatrix}. \tag{2.3}$$

We refer the reader to Ref. [4] for detailed expressions of the components $T_{\lambda,}{}^{\cdot\mu}$ of the GEM tensor. Recall simply that

$$f^{\lambda\cdot}{}_{,\mu\nu} \equiv \tfrac{1}{2}(\partial_\mu g^{\lambda\cdot}{}_{,\nu} - \partial_\nu g^{\lambda\cdot}{}_{,\mu})$$

$$f^{\lambda}{}_{\mu\nu} \equiv g_{\mu,}{}^{\cdot\alpha} g_{\nu,}{}^{\cdot\beta} f^{\lambda\cdot}{}_{,\alpha\beta} \quad f^{\lambda} \equiv f^{\alpha\lambda}{}_{\alpha} \text{ etc.,} \tag{2.4}$$

the matrice $(g_{\lambda,}{}^{\cdot\mu})$ being the transposed inverse (transverse) of $(g^{\lambda\cdot}{}_{,\mu})$. The $f^{\lambda}{}_{\mu\nu}$ are listed in the following table:

$f^{0}{}_{01} = -\frac{a}{2r^2}\sin\vartheta\cos\varphi$ $\quad f^{0}{}_{02} = -\frac{a}{2r^2}\sin\vartheta\sin\varphi$ $\quad f^{0}{}_{03} = -\frac{a}{2r^2}\cos\vartheta$

$f^{0}{}_{12} = 2A - 3A\sin^2\vartheta$ $\quad f^{0}{}_{13} = -3A\sin\vartheta\cos\vartheta\sin\varphi$ $\quad f^{0}{}_{23} = 3A\sin\vartheta\cos\theta\cos\varphi$

$f^{1}{}_{01} = 3A\sin^2\vartheta\sin\varphi\cos\varphi$ $\quad f^{1}{}_{02} = -A + 3A\sin^2\vartheta\sin^2\varphi$ $\quad f^{1}{}_{03} = 3A\sin\vartheta\cos\vartheta\sin\varphi$

$f^{1}{}_{12} = \frac{a}{2r^2}\sin\vartheta\sin\varphi$ $\quad f^{1}{}_{13} = \frac{a}{2r^2}\cos\vartheta$ $\quad f^{1}{}_{23} = 0$

$f^{2}{}_{01} = A - 3A\sin^2\vartheta\cos^2\varphi$ $\quad f^{2}{}_{02} = -3A\sin^2\vartheta\sin\varphi\cos\varphi$ $\quad f^{2}{}_{03} = -3A\sin\vartheta\cos\vartheta\cos\varphi$

$f^{2}{}_{12} = -\frac{a}{2r^2}\sin\vartheta\cos\varphi$ $\quad f^{2}{}_{13} = 0$ $\quad f^{2}{}_{23} = \frac{a}{2r^2}\cos\vartheta$

$f^{3}{}_{0i} = 0 \quad f^{3}{}_{12} = 0$ $\quad f^{3}{}_{13} = -\frac{a}{2r^2}\sin\vartheta\cos\varphi$ $\quad f^{3}{}_{23} = -\frac{a}{2r^2}\sin\vartheta\sin\varphi$

$f_0 = 0 \quad f_1 = -\frac{a}{2r^2}\sin\vartheta\cos\varphi$ $\quad f_2 = -\frac{a}{2r^2}\sin\vartheta\sin\varphi$ $\quad f_3 = -\frac{a}{2r^2}\cos\vartheta,$

(2.5)

where

$$A \equiv A(r) \equiv \frac{a}{5r}\frac{\omega}{c}\frac{l^2}{r^2}.$$

The $\underset{i}{H}$ are [4]:

$$\underset{1}{H} = -\frac{3a^2}{2r^4} + 12A^2 \quad \underset{2}{H} = -\frac{3a^2}{4r^4} + 6A^2 \quad \underset{3}{H} = -\frac{a^2}{4r^4}. \tag{2.6}$$

With the assistance of formulae (3.18) of Ref. [4], we are now able to calculate the (physical) GE density $-\kappa^{-1} T_{0,}{}^{,0}$ and the energy flux density $-\kappa^{-1} cr \sin\vartheta\, T_{0,}{}^{,3}$ in the φ-direction.

In the calculation of the GE density, terms of order $A^2 \sim \omega^2$ appear that are the contribution of the rotation. But these terms are generally very small in comparison with the static GE density $-\kappa^{-1}(a^2/r^4)$. For the earth in the field of the sun, for instance, this ratio is of the order $(\omega l/c)^2 (l/r)^2 \sim 10^{-15}$. We therefore neglect this A^2 contribution, the role of which is irrelevant in the following. We then obtain for the GE density,

$$-\frac{1}{\kappa} T_{0,}{}^{,0} = -\frac{1}{\kappa}\frac{a^2}{r^4}; \tag{2.7}$$

and for energy flux density,

$$-\frac{1}{\kappa} cr \sin\vartheta\, T_{0,}{}^{,3} = -\frac{2acA}{\kappa}\frac{\sin\vartheta}{r^2}. \tag{2.8}$$

According to the general formula current density = density · velocity, dividing (2.8) by (2.7) gives the velocity v_{en} of the GE flux in the φ-direction:

$$v_{en} = \frac{2}{5}\frac{\omega l^2 \sin\vartheta}{r}. \tag{2.9}$$

This result is interesting in that it shows, at large distances from the source, that the GE velocity decreases proportionally to r^{-1} and is zero at infinity. We shall refer back to formula (2.9) in the next paragraph, where we consider what we could call the 'electromagnetic Lense–Thirring effect'.

3. The Electromagnetic Analogue of the Lense–Thirring Effect

We consider the same physical situation as in Section 2, but now the rotating sphere carries a constant electric charge density ρ. If this sphere is at rest, it produces an electric field only. But its rotation also generates a magnetic field that corresponds to the Lense–Thirring modification of the Schwarzschild solution in the gravitational case. For $l/r \ll 1$, the classical formulae of electrodynamics allow us to easily calculate the energy and energy flux densities. The velocity of propagation of the energy that is deduced from these expressions is also given by formula (2.9). This result calls attention to the following remark.

Although the electromagnetic energy is quite well localized, it is clear that the equality of the velocities mentioned above do not 'prove' the GEM localizability. In Section 2 we applied Møller's conditions for the tetrads, for we know that this choice at least gives 'good' results in the case of the weak field [1]. But there is no physical argument to justify these supplementary conditions. What we would like to emphasize once more, by means of the above result, is that it does not seem impossible to localize the GEM (in opposition to the Landau–Lifchitz opinion for instance).

4. Angular Momentum and GEM Localizability

In the last paper [5] of a long series devoted to the problem of the GEM localization, Møller expresses the opinion that the major theoretical difficulties of this question are connected with the fact that '... actually nobody has so far been able to give a prescription for measuring the energy of the gravitational field in a small region, in contrast to the total energy for which such prescriptions are easily given.' Notice also that Møller's remark is valid in the Newtonian case as well as in the relativistic case.

It is certainly very difficult, perhaps even impossible, to find an experimental procedure that would lead to the desired result. We think, however, that it is possible to realize substantial improvements in this direction. To this end, with the help of the above results, we study the notion of *angular momentum* according to the following fundamental ideas.

In the Newtonian theory the mass of a source and its potential energy are two distinct entities. For an homogeneous sphere of density ρ, for example, the total mass M is given by the expression $(4\pi/3)l^3\rho$, and the total potential energy by $-(3/5)/GM^2/l$. The situation is different in GR, the gravitational energy (which tends to the classical potential energy in the case of the weak field) is incorporated in the total mass, according to the formula total energy $= Mc^2 =$ material energy + gravitational energy (see, for example, Ref. [6]). It is then imaginable that the gravitational energy contributes to the angular momentum $\vec{B}$ of the sphere when the latter is rotating. It is not true in the classical case, where only the material energy is considered in the evaluation of $\vec{B}$.

If one admits this point of view, it is then clear that the angular momentum depends on the GE distribution (in the following, 'gravitational' and 'potential' have the same meaning). Of course, knowing only the value of the angular momentum does not imply an unambiguous determination of this distribution. Nevertheless, it allows us to exclude some localizations that could, perhaps, be postulated. This shows that the GE distribution should at least satisfy certain constraints inconsistent with the non-tensorial character of the energy-momentum complex of Einstein's theory. To define these ideas, we consider the following situations.

a) An homogeneous sphere of density ρ rotates with constant angular velocity ω.
b) We first admit that ρ also contains the potential energy, supposed uniformly distributed throughout the sphere (there is then no energy out of the sphere). Calculation of the corresponding angular momentum $\vec{B}_1$.
c) We then postulate a GE distribution according to the classical formula $u_{\text{pot}} = -(\vec{G}^2/8\pi G)$. Calculation of the corresponding angular momentum $\vec{B}_2$.
d) Comparison of $\vec{B}_1$ with $\vec{B}_2$. If $\vec{B}_1 \neq \vec{B}_2$ it will be shown by this example that some GE distributions are to be excluded. Indeed, a definite system cannot have, simultaneously, two different angular momenta!

Before we complete this programme, we recall to mind that Møller's remark on the non-measurability of GEM density is not only true in GR but also in Newtonian theory. For simplicity, we could of course try to solve our problem in this formalism, but the integrals representing the angular momentum do not then converge. On the other hand, a pure GR treatment of this question is very complicated, mainly because of the non-tensorial character of angular momentum density. It should then be emphasized that *we shall work in the frame of the Einstein–Maxwell theory mentioned above. This formalism just gives the velocity formula (2.9), and the r^{-1} dependence of v_{en} is responsible for the fact that the angular momentum integral converges.*

a) To avoid unnecessary complications, without connection with the nature of our problem, we suppose that the angular velocity ω is small enough, so that the relativistic mass variation is negligible. For the sun, for instance, the terms in $(\omega l/c)^2$ are of order 10^{-11} and can be ignored. We set aside, in the same way, that part of energy due to rotation, at least out of the sphere.

b) The z-axis, being the rotation axis, we find for the single non-zero component B_{1z} of the angular momentum

$$B_{1z} = \int_{\text{sphere}} \rho v_{en} r \sin\vartheta \, dV = \tfrac{2}{5} M\omega l^2, \tag{4.1}$$

where $v_{en} = \omega r \sin\vartheta$ is the velocity of the matter contained in the volume element dV.

c) In the above formula M is the total mass connected with the constant density ρ by the formula

$$M = \frac{4\pi}{3} l^3 \rho, \tag{4.2}$$

ρ includes also the potential energy. From now on we designate by $\sigma = \text{const.}$ the purely 'material' density, with the total 'material' mass $M_\sigma = (4\pi/3) l^3 \sigma$. If the mass of the potential energy is added to the latter we obtain

$$M = M_\sigma - \frac{3}{5} \frac{G M_\sigma^2}{l c^2}. \tag{4.3}$$

We then calculate B_{2z}, replacing in (4.1) the integral on the sphere by an integral on the whole space. The latter converges because $v_{en} \sim r^{-1}$ for great r. The contribution ΔB_{2z} to the angular momentum B_{2z}, corresponding to the exterior of a sphere of radius R, with $l/R \ll 1$, is given by the relation

$$\Delta B_{2z} = \int_{r>R} \frac{u_{pot}}{c^2} v_{en} r \sin\vartheta \, dV = -\frac{1}{20\pi} \frac{G M_\sigma^2 \omega l^2}{c^2} \int_{r>R} \frac{\sin^3\vartheta}{r^2} \, dr \, d\vartheta \, d\varphi$$

$$= -\frac{2}{15} \frac{G M_\sigma^2 \omega l^2}{c^2 R}, \tag{4.4}$$

with $u_{pot} = -(G M_\sigma^2 / 8\pi r^4)$. Evidently the calculation of B_{2z} necessitates the exact value of v_{en} in the whole space. Unfortunately the exact determination of the energy velocity is rather difficult. However we tried, with the help of several reasonable hypotheses, some approximate evaluations and our calculations seem to show clearly that B_{1z} and B_{2z} have not the same value.

d) If it is admitted that $\vec{B}_1 \neq \vec{B}_2$, in the actual state of our knowledge only an experiment could show which of the two considered GE distributions has to be excluded. We think, of course, that the $\vec{B}_2$ distribution has the best chance of being the right one.

5. Conclusion

We find that we are still very far from direct measurement of the GE density. Nevertheless, the notion of angular momentum seems to be a valuable way of excluding possible GEM distributions. Even if our above considerations are limited to the 'classical' case (Newtonian and Einstein–Maxwellian formalisms), they indicate that the corresponding GR problem is far from being solved in the sense of the non-localizability of GEM.

References

[1] J. Chevalier, Sur la localisation de l'énergie-impulsion gravitationnelle dans la théorie relativiste de la gravitation (thèse d'habilitation, Genève, 1974).
[2] J. Lense and H. Thirring, Phys. Z. *19*, 156 (1918).
[3] C. Møller, Mat. Fys. Skr. Dan. Vid. Selsk. *1*, No. 10 (1961).
[4] W. Scherrer, Helv. Phys. Acta *38*, 215 (1965).
[5] C. Møller, Mat. Fys. Medd. Dan. Vid. Selsk. *35*, No. 3 (1966).
[6] J. Chevalier and E. C. G. Stueckelberg de B., Helv. Phys. Acta *45*, 587 (1972).