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Nonlinear $0(n + 1)$-Symmetric Field Theories, Symmetry Breaking and Finite Energy Solutions

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Abstract. We introduce a class of $0(n + 1)$-symmetric, nonlinear field theories, containing the standard chiral models as special cases. Our class is parametrized by a function α and we give explicit transformations between any two theories corresponding to different choices of the function α. We then exhibit classical 'plane wave' and static solutions of the equations of motions of these theories. However we have to introduce symmetry breaking potentials to find solutions leading to a finite total energy.

Introduction

As recently stressed [1], the study of classical solutions of nonlinear field equations is essential for a method of quantizing interacting fields based on semiclassical approximations. We also recall that other approaches to quantization rely on the knowledge of all classical solutions of the nonlinear field equations [2]. Furthermore, in the relativistic theory of extended particles, the classical solutions of its governing nonlinear partial differential equations describe simple excitations or modes, which can be quantized [3].

The purpose of this paper is to discuss a class of $0(n + 1)$-symmetric nonlinear field theories from a group theoretical and differential geometric point of view. We are also interested in breaking the symmetry to get models with finite total energy.

I. A Class of $0(n + 1)$-Symmetric Nonlinear Field Theories

Consider a set $\{\Psi^1,\ldots,\Psi^{n+1}\}$ of $(n + 1)$ real scalar fields on the Minkowski space, satisfying the equation

$$\Psi^{1^2} + \cdots + \Psi^{n^2} + \Psi^{n+1^2} = \frac{1}{k}, \qquad k > 0. \tag{1}$$

These fields can be looked upon as coordinates of the sphere S_n embedded in R^{n+1}.

We now parametrize the upper hemisphere of S_n by new coordinates ϕ^i (fields)

$$\Psi^i = \alpha(\phi)\phi^i, \qquad i = 1,\ldots, n, \tag{2}$$

$$\Psi^{n+1} = \left(\frac{1}{k} - \alpha^2\phi^2\right)^{1/2} > 0, \tag{3}$$

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where

$$\phi = (\phi^{1^2} + \cdots + \phi^{n^2})^{1/2}. \tag{4}$$

From the metric on S_n

$$ds^2 = d\Psi^1\, d\Psi^1 + \cdots + d\Psi^n\, d\Psi^n + d\Psi^{n+1}\, d\Psi^{n+1} \tag{5}$$

we get the induced metric on the upper hemisphere of S_n:

$$ds^2 = g_{ik}\, d\phi^i\, d\phi^k = \alpha^2(\phi)\{d\phi^1\, d\phi^1 + \cdots + d\phi^n\, d\phi^n\} + \beta(\phi)\{\phi^1\, d\phi^1 + \cdots + \phi^n\, d\phi^n\}^2, \tag{6}$$

where

$$\beta(\phi) = \frac{1}{1 - k\alpha^2\phi^2}\left[\frac{2\alpha\alpha'}{\phi} + (\alpha')^2 + k\alpha^4\right], \tag{7}$$

and $\alpha' = d\alpha/d\phi$.

Equation (7) can also be written as

$$(1 - k\alpha^2\phi^2)(\alpha^2 + \beta\phi^2) = (\alpha'\phi + \alpha)^2. \tag{8}$$

Each function $\alpha(\phi)$ labels an $0(n + 1)$-symmetric theory. Two parametrizations $\alpha_{(1)}$ and $\alpha_{(2)}$ imply the following nonlinear transformation between the corresponding fields $\phi^i_{(1)}$ and $\phi^i_{(2)}$

$$\phi^i_{(1)} = \frac{\phi_{(1)}}{\phi_{(2)}}\,\phi^i_{(2)}, \tag{9}$$

where

$$\phi_{(k)} = \left(\sum_{i=1}^{n} (\phi^i_{(k)})^2\right)^{1/2}, \qquad k = 1, 2, \tag{10}$$

and

$$\phi_{(1)}\alpha(\phi_{(1)}) = \phi_{(2)}\alpha(\phi_{(2)}). \tag{11}$$

The group $0(n + 1)$ acts transitively on S_n and the nonlinear transformations (9) follow from equation (2). The subgroup $0(n)$ of $0(n + 1)$ acts linearly on the fields $\{\phi^1, \ldots, \phi^n\}$, thus the class given by the parametrization (2) is best suited to treat theories with $0(n)$-symmetric potentials.

Equations (1) and (2) can be generalized to other manifolds embedded in R^{n+1} and different symmetry breaking; these problems will be the subject of a subsequent report.

The action density corresponding to the metric (6) with notation (4) is

$$\mathscr{L}_0 = \tfrac{1}{2}[\alpha^2(\phi)\partial_\mu\phi^i\partial^\mu\phi_i + \beta(\phi)(\phi^i\partial_\mu\phi_i)(\phi^j\partial^\mu\phi_j)]. \tag{12}$$

The equations of motion are

$$\partial_\mu\partial^\mu\phi^i = -\frac{2\alpha'}{\alpha\phi}(\phi^j\partial^\mu\phi_j)\partial_\mu\phi^i + \frac{\phi^i}{\alpha^2 + \beta\phi^2} \times \left[(\partial_\mu\phi^k\partial^\mu\phi_k)\left(\frac{\alpha\alpha'}{\phi} - \beta\right) + (\phi^k\partial_\mu\phi_k)(\phi^j\partial^\mu\phi_j)\,\frac{4\alpha'\beta - \alpha\beta'}{2\phi\alpha}\right], \tag{13}$$

with $\alpha^2 + \beta\phi^2 \neq 0$. Note that $\alpha^2 + \beta\phi^2 = \text{const}$ corresponds to normal coordinates. The case $\alpha^2 + \beta\phi^2 = 0$ is a singular model for which we obtain $\alpha = c/\phi$, $\beta = c^2/\phi^4$ and $c \neq 1/\sqrt{\varkappa}$. The energy-momentum tensor for (12) is

$$T^{\mu\nu} = \alpha^2\partial^\mu\phi^i\partial^\nu\phi_i + \beta(\phi^i\partial^\mu\phi_i)(\phi^k\partial^\nu\phi_k) - \eta^{\mu\nu}\mathscr{L}_0. \tag{14}$$

Some special cases of (6), used in the literature [4] are shown below

α	β	
1	$\dfrac{k}{1-k\phi^2}$	(15)
$\dfrac{2}{1+k\phi^2}$	0	(16)
$\dfrac{1}{\sqrt{1+k\phi^2}}$	$-\dfrac{k}{(1+k\phi^2)^2}$	(17)
$\dfrac{\sin(\sqrt{k}\,\phi + c)}{\sqrt{k}\,\phi}$	$\dfrac{1}{\phi^2}\left[1 - \dfrac{\sin^2(\sqrt{k}\,\phi + c)}{k\phi^2}\right]$	(18)

Geometrically, different choices of the function α correspond to different parametrizations of the upper hemisphere of S_n. The corresponding Lagrangians look quite differently, but the transformation between two parametrizations connects the solutions of the associated field equations. Note that the case (18) is the model in normal coordinates, i.e. $\alpha^2 + \beta\phi^2 = 1$.

II. Classical Solutions

II.1. 'Plane wave' solutions

We first look at solutions of the form

$$\phi^i = \phi^i(s), \quad s = q^\mu x_\mu, \quad q^\mu q_\mu = q^2 \neq 0.$$

For this type of solutions the equation of motion (13) reduces to the equations for a geodesic on S_n,

$$\ddot{\phi}^j = -\frac{2\alpha'}{\alpha\phi}(\phi^i\dot{\phi}_i)\dot{\phi}^j + \frac{4\beta\alpha' - \alpha\beta'}{2\alpha\phi}\frac{1}{\alpha^2+\beta\phi^2}(\phi^i\dot{\phi}_i)^2\phi^j$$
$$+ \frac{\alpha\alpha' - \beta\phi}{\phi(\alpha^2+\beta\phi^2)}(\dot{\phi}^i\dot{\phi}_i)\phi^j. \tag{19}$$

Here the dot denotes the derivative with respect to s. The energy density (14) for the solutions of (19) becomes

$$T^{00} = \tfrac{1}{2}q_E^2 g_{ik}\dot{\phi}^i\dot{\phi}^k, \qquad q_E^2 = q_0^2 + \vec{q}^{\,2} \tag{20}$$

and is a constant since $\{\phi^i\}$ constitutes a geodesic. Hence the 'plane wave' solutions lead to a constant kinetic energy density.

The group $0(n+1)$ maps geodesics into geodesics, because it is the isometry group of S_n, and hence it is sufficient to solve equation (19) for $\phi^i = \delta^{i1}\phi^1$, and then use the nonlinear action of $0(n + 1)$ to recover all other geodesics. Thus we get the equation

$$\ddot{\phi}^1 = -\frac{1}{2}\dot{\phi}^{1^2}\frac{d}{d\phi^1}\ln(\alpha^2 + \beta\phi^{1^2}). \tag{21}$$

Generally, the 'plane wave' solutions are stable in the geodesic sense [5]. In normal coordinates the geodesics are straight lines. For $\alpha = (1 + k\phi^{1^2})^{-1/2}$, $\beta = -k(1 + k\phi'^2)^{-2}$ we find from (21)

$$\phi^1(s) = \frac{1}{\sqrt{k}}\tan[\sqrt{k}\,(c_1 s + c_2)].$$

II.2. Static solutions

We will now look for static spherically symmetric solutions through the equation

$$\phi^i = A^i f(r), \quad A^i A_i = 1, \quad r = |\vec{x}|. \tag{22}$$

From (13) the equation of motion then reads

$$f'' = -\frac{2f'}{r} - \frac{1}{2}(f')^2\frac{\partial}{\partial f}\ln(\alpha^2 + \beta f^2), \tag{23}$$

where $f' = df/dr$.

For normal coordinates $\alpha^2 + \beta f^2 = 1$, one gets

$$f'' = -2\frac{f'}{r}. \tag{24}$$

The energy density for the fields (22) is

$$T^{00} = (\alpha^2 + \beta f^2)(f')^2 \tag{25}$$

and hence one sees that the solutions of (24) again do not lead to a finite total energy. To achieve this we have to break the symmetry.

III. Symmetry Breaking

Even though solutions of field equations with locally finite energy have recently been advocated [6] we want to introduce a symmetry breaking potential so that we still retain linear $0(n)$-symmetry but find a finite total energy.

With the lagrangian $\mathscr{L}_0$ from (12) let us look at, with eq. (4),

$$\mathscr{L} = \mathscr{L}_0 + V(\phi). \tag{26}$$

The equations of motion become

$$\partial_\mu\partial^\mu\phi^i = -\frac{2\alpha'}{\alpha\phi}(\phi^j\partial^\mu\phi_j)\partial_\mu\phi^i + \frac{\phi^i}{\alpha^2 + \beta\phi^2} \times \left[(\partial_\mu\phi^k\partial^\mu\phi_k)\left(\frac{\alpha\alpha'}{\phi} - \beta\right) + (\phi^k\partial_\mu\phi_k)(\phi^j\partial^\mu\phi_j)\frac{4\alpha'\beta - \alpha\beta'}{2\alpha\phi}\right] + \frac{V'\phi^i}{\phi(\alpha^2 + \beta\phi^2)}. \tag{27}$$

The 'plane wave' solutions then have to satisfy the following equation:

$$\ddot{\phi}^j = \frac{-2\alpha'}{\alpha\phi}(\phi^i\dot{\phi}_i)\phi^j + \frac{4\beta\alpha' - \alpha\beta'}{2\alpha\phi}\frac{1}{\alpha^2 + \beta\phi^2}(\phi^i\dot{\phi}_i)^2\phi^j + \frac{\alpha\alpha' - \beta\phi}{\phi(\alpha^2 + \beta\phi^2)}(\dot{\phi}^i\dot{\phi}_i)\phi^j + \frac{\phi^j V'}{q^2\phi(\alpha^2 + \beta\phi^2)}. \tag{28}$$

The equation for static spherically symmetric solutions are

$$f'' = -\frac{2f'}{r} - \frac{1}{2}(f')^2\frac{\partial}{\partial f}\ln(\alpha^2 + \beta f^2) - \frac{V'}{\alpha^2 + \beta f^2}. \tag{29}$$

For the special case of 0(4)-symmetric field theories with a symmetry breaking potential one can look for static solutions of the Skyrme type [7]

$$\phi^i = A^i_k\frac{x^k}{r}f(r), \quad A^i_kA^l_i = \delta^l_k, \quad r = |\vec{x}|. \tag{30}$$

Our equations of motion then read

$$f'' = \frac{2f}{r^2} - \frac{2f'}{r} - \frac{1}{2}(f')^2\frac{\partial}{\partial f}\ln(\alpha^2 + \beta f^2) + \frac{2f^2(\alpha\alpha' - \beta f)}{(\alpha^2 + \beta f^2)r^2} - \frac{V'}{\alpha^2 + \beta f^2}. \tag{31}$$

The expressions for the energy density for the 'plane wave', static spherically symmetric, and Skyrme type solutions are, respectively

(i) $T^{00} = -q^2g^{ik}(\phi)\dot{\phi}_i\dot{\phi}_k - V(\phi),$ (32)

(ii) $T^{00} = \frac{1}{2}(\alpha^2 + \beta f^2)f'^2 - V(f),$ (33)

(iii) $T^{00} = \frac{1}{2}(\alpha^2 + \beta f^2)f'^2 + \frac{\alpha^2 f^2}{r^2} - V(f).$ (34)

We now want to discuss two examples, both in normal coordinates, i.e. $\alpha^2 + \beta\phi^2 = 1$. This means

$$\alpha(\phi) = \frac{\sin\sqrt{k}\,\phi}{\sqrt{k}\,\phi}, \quad \beta(\phi) = \frac{1}{\phi^2}\left[1 - \frac{\sin^2(\sqrt{k}\,\phi)}{k\phi^2}\right]. \tag{35}$$

(*a*) *Static spherically symmetric case with ϕ^6 symmetry breaking* [8]
The equation of motion (29) becomes

$$f'' = -2\frac{f'}{r} - V' \tag{36}$$

where

$$V(f) = \lambda f^6 \tag{37}$$

yielding the one parameter family of solutions

$$f(r) = \frac{b}{(a^2 + r^2)^{1/2}}, \quad b = \left(\frac{a^2}{2\lambda}\right)^{1/4}. \tag{38}$$

For $a^2/\lambda = \text{const}$, and both λ and a going to zero, we observe a free point source behaviour for f, i.e. $f(r) \sim r^{-1}$.

The energy density has the form (33)

$$T^{00} = E(r) = \tfrac{1}{2}f'^2 - V(f) \tag{39}$$

and the total energy is finite and is given by

$$M = 4\pi \int_0^\infty r^2\, dr E(r) = \frac{\pi^2}{4\sqrt{2\lambda}}. \tag{40}$$

(*b*) *Skyrme type case with symmetry breaking*

We want to look for potentials that decompose equation (31) as follows

$$f'' = -V', \quad rf' = f + (\alpha\alpha' - \beta f)f^2 \tag{41}$$

with integration constants B, C, V_0 these equations lead to

$$f(r) = (1/\sqrt{k}) \tan^{-1}(Br) + C, \quad V(r) = -\tfrac{1}{2}f'^2 + V_0. \tag{42}$$

The energy density (34) becomes

$$T^{00} = E(r) = f'^2 + \frac{\alpha^2 f^2}{r^2},$$

or

$$E(r) = \frac{B^2}{k}[(1 + B^2r^2)^{-2} + (1 + B^2r^2)^{-1}] - V_0. \tag{43}$$

Hence the total energy $M = 4\pi \int_0^\infty r^2\, dr E(r)$ does not exist [9].

Finally we want to translate example (35–37) into the model $\alpha(\phi) = 2/(1 + k\phi^2)$, $\beta(\phi) = 0$. The transformation from normal coordinates to this model follows from (9), (11) and results in

$$\chi^i = \frac{1 - \cos(\sqrt{k}\,\phi)}{\sqrt{k}\,\phi \sin(\sqrt{k}\,\phi)}\phi^i, \qquad 0 \leqslant \sqrt{k}\,\phi < \pi \tag{44}$$

with

$$\chi^i = A^i g(r), \quad \phi^i = A^i f(r), \quad A^i A_i = 1 \tag{45}$$

we get

$$g(r) = \frac{1 - \cos(\sqrt{k}f)}{\sqrt{k}\sin(\sqrt{k}f)}. \tag{46}$$

The equation of motion in the new model is

$$g'' = -\frac{2g'}{r} + \frac{2kg}{1 + kg^2}g'^2 - \frac{1}{4}\frac{dV}{dg}(1 + kg^2)^2 \tag{47}$$

with

$$V(g) = \lambda k^{-3}\left(\sin^{-1}\left(\frac{2\sqrt{k}\,g}{1 + kg^2}\right)\right)^6. \tag{48}$$

The solution of these equations is

$$g(r) = \frac{1 - \cos\{B[k/(\alpha^2 + r^2)]^{1/2}\}}{\sqrt{k}\sin\{B[k/(\alpha^2 + r^2)]^{1/2}\}}. \tag{49}$$

Note that the point transformation between fields (i.e. different parametrizations) should be regular. For the case (44) the condition $\sqrt{k}f < \pi$, is

$$\left(\frac{\alpha^2}{2\lambda}\right)^{1/4} \frac{\sqrt{k}}{(\alpha^2 + r^2)^{1/2}} < \pi. \tag{50}$$

Hence a should be so chosen that $4\sqrt{k^2/2\lambda a^2} < \pi$. This is precisely the condition for the Beltrami metric g_{ik} to be regular, for in this case

$$g_{ik} = \frac{\sin(\sqrt{k}f)}{\sqrt{k}f}\,\delta_{ik} + \frac{kf^2 - \sin^2(\sqrt{k}f)}{k^2f^4}\,\phi_i\phi_k. \tag{51}$$

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