An eikonal expansion in quantum electrodynamics

Autor(en): **Quirós, M.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **49 (1976)**

Heft 6

PDF erstellt am: **30.06.2024**

Persistenter Link: https://doi.org/10.5169/seals-114794

An Eikonal Expansion in Quantum Electrodynamics [1])

by **M. Quirós** [2])

University of Geneva

(6. IV. 1976)

Abstract. The classical limit of quantum electrodynamics is studied using functional methods. Two-point Green functions are developed in power series of $\sigma^{\mu\nu}F_{\mu\nu}$. In particular nth order spin corrections are associated to terms coming from $(\sigma F)^n$. The first order spin correction is analyzed in detail. In the limit of zero photon mass it exhibits cuts, in the complex energy plane, whose effects are mainly concentrated on the branching points, with coefficients $(\log \mu)^{-1}$.

1. Introduction

The relativistic eikonal formula, as a generalization of Glauber approximation [1] in potential scattering, was obtained in the past by Lévy and Sucher [2], Chang and Ma [3] and Cheng and Wu [4] using the perturbation approach to field theory, and by Erickson and Fried [5] and Abarbanel and Itzykson [6], among other authors, using functional methods. All these approaches consider the limit $\theta \to 0$, neglecting in this way all possible effects coming from spin–flip amplitudes, as for instance polarization. Spin corrections have been studied by Lévy and Léger [7] in Coulomb scattering and by this author [8] in two-body reactions using perturbative methods. All these works regard the exchange of scalar photons in order to avoid the complications due to the spinor–vector coupling γ_μ. The computation of spin corrections becomes increasingly complicated when Feynman diagrams must be added because the permutation scheme, needed to get the factorization at a given order, becomes also more and more complicated.

In this paper we shall use functional methods to compute spin corrections to the relativistic eikonal formula in quantum electrodynamics.

As it is well known the serie of crossed-ladder graphs is generated by the two-point function of an electron in the presence of an external field. In the case of quantum electrodynamics the Green function satisfies the equation

$$[\not{\partial} + m + ig\not{A}(x)]G(x, y|A) = -i\delta(x - y) \tag{1.1}$$

or, introducing operators P_μ and X_μ with canonical commutation relations,

$$G_A = -[\not{P} - im + g\not{A}(X)]^{-1} \tag{1.2}$$

where $\langle y|G_A|x\rangle \equiv G(x, y|A)$.

[1]) In partial fulfilment of the requirements for the Ph.D. degree at the University of Geneva.

[2]) Supported by the C.I.C.P. Foundation.
Permanent address: Lab. Fisica de Particulas, Instituto de Estructura de la Materia, C.S.I.C., Serrano, 119 Madrid-6 Spain.

From (1.2) the rationalized two-point Green function is given by

$$G_A = -\frac{\not{P} + im + g\not{A}(X)}{P^2 + m^2 + v(X, P) + (g/2)\sigma F(X)} \tag{1.3}$$

where $v(X, P) = gP\cdot A(X) + gA(X)\cdot P + g^2A^2(X)$,

$$\sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu, \gamma_\nu] \quad \text{and} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

In Section 2 we shall expand (1.3) in powers of σF and compute the corresponding contribution to the elastic fermion–fermion amplitude. We shall associate nth order spin corrections with terms of the amplitude coming from $(\sigma F)^n$.

In Section 3 we shall introduce the usual classical limit and in Section 4 we shall restrict ourselves to spin corrections only on one fermion line. The relativistic eikonal formula is obtained as the zeroth order term in this expansion. The first-order correction term is explicitly computed. It contains simple and double spin–flip contributions. The limit $\mu \to 0$ (where μ is the photon mass) can be analytically computed for the double spin–flip contribution. The result is that this amplitude behaves as $(\log \mu)^{-1}$. The mathematics leading to this result are compiled in an Appendix. Thus, the main result of this paper is that spin corrections do not decrease faster than $(\log \mu)^{-1}$ in the infrared limit. This is in contrast with the behaviour in μ of precedent works [7, 8]. Finally the serie of spin–non-flip contributions coming from the total serie in the limit $\theta \to 0$, can be formally summed up.

2. The Elastic Amplitude: An Exact Expression

Let us describe the elastic fermion–fermion reaction

$$e_1(p_1, \lambda) + e_2(p_2, \sigma) \to e_1'(p_1', \lambda') + e_2'(p_2', \sigma') \tag{2.1}$$

by means of the crossed-ladder Feynman graphs. This class of diagrams has been shown to exhibit the property of gauge invariance [9].

The kinematical invariants are defined by

$$s = -(p_1 + p_2)^2 \quad t = -(p_1' - p_1)^2 = -q^2 \quad u = -(p_1' - p_2)^2 \tag{2.2}$$

and the mass-shell conditions by $p^2 + m^2 = (\not{p} - im)u(p) = 0$.

The relation between the S-matrix and the Green-functions is given by the reduction formulas [10]. In this case we can write the T-matrix element as

$$(-i)(2\pi)^4 T_{\lambda'\sigma';\lambda\sigma}(s, t)\delta(P_i - P_f) = e^{\delta}R_1(p_1, p_1')R_2(p_2, p_2')|_{A_1=A_2=0} \tag{2.3}$$

where $P_i = p_1 + p_2$, $P_f = p_1' + p_2'$, δ is the functional operator

$$\delta = \int dz\, d\omega \underset{\mu}{\frac{\delta}{\delta A_1(z)}} D_{\mu\nu}(z - \omega) \underset{\nu}{\frac{\delta}{\delta A_2(\omega)}} \tag{2.4}$$

and $R(p, p')$ is the residue of the two-point Green-function on the mass shell

$$R(p, p') = \lim \bar{u}(p')(\not{p}' - im)\langle p'|G_A|p\rangle(\not{p} - im)u(p). \tag{2.5}$$

The limit in (2.5) indicates the mass-shell conditions, and the free photon propagator in equation (2.4) is chosen in the Lorentz gauge $D_{\mu\nu}(x) = \delta_{\mu\nu}D(x)$.

In order to extract the two residues from (2.5) we shall use the identity [11]

$$e^{\delta} = 1 + \int_0^1 d\lambda e^{\lambda\delta}\delta. \tag{2.6}$$

In equation (2.5) the first term does not contribute to the connected amplitude. Actually, $\bar{u}(p')(\not{p}' - im)\langle p'|p\rangle(\not{p} - im)u(p)$ vanishes on the mass-shell if $q \neq 0$. The second term will give a symmetric expression in G_A by means of the identity

$$\frac{\delta}{\delta A_\mu(z)} G_A = -gG_A\gamma^\mu\delta(X - z)G_A. \tag{2.7}$$

Let us rationalize the Green function and expand the denominator in a power serie of σF as

$$G_A = -\frac{\not{P} + im + g\not{A}(X)}{P^2 + m^2 + v(X,P)} S(X,P) = -\tilde{S}(X,P)\frac{\not{P} + im + g\not{A}(X)}{P^2 + m^2 + v(X,P)} \tag{2.8}$$

where

$$S(X,P) = \sum_{n=0}^{\infty}\left(-\frac{g}{2}\right)^n t^n(X,P) \tag{2.9}$$

and

$$t(X,P) = \sigma F(X)\frac{1}{P^2 + m^2 + v(X,P)} \tag{2.10}$$

$$\tilde{t}(X,P) = \frac{1}{P^2 + m^2 + v(X,P)}\sigma F(X).$$

The only characteristic spin-dependence is given by the serie (2.9), where the electron spin is coupled to the electromagnetic field.

In order to extract the residues from (2.5) one uses an integral representation for the denominator in (2.8), and $\exp\{P^2 + m^2\}$ is factorized out by means of the identities [12]

$$e^{A+B} = e^A T\exp\left(\int_0^1 dse^{-As}Be^{As}\right) = \tilde{T}\exp\left(\int_0^1 dse^{As}Be^{-As}\right)e^A. \tag{2.11}$$

A and B being any non-commuting operators and the symbol $T(\tilde{T})$ meaning chronological (antichronological) ordering with respect to the integration parameter.

Using (2.7), (2.8), (2.11) and the identity

$$e^{\pm i\tau P}f(X)e^{\mp i\tau P} = f(X \mp 2\tau P) \tag{2.12}$$

the residue to the left in (2.5), corresponding to the limit $\bar{u}(p')(\not{p}' - im) \to 0$, is given by

$$-i\langle p'|T\exp\left[i\int_0^\infty d\tau v(X - 2\tau P, P)\right]\bar{u}(p')$$
$$+\frac{g}{2m}\bar{u}(p')\lim_{M\to\infty}\langle p'|\not{A}(X + 2MP)T\exp\left[i\int_0^M d\tau v(X - 2\tau P, P)\right] \tag{2.13}$$

and the second contribution will be negligible after taking the matrix element provided that the function $A_\mu(x)$ vanishes at the infinite. Thus, only the first term in (2.13) will survive.

The residue in (2.5) corresponding to the limit $(\not{p} - im)u(p) \to 0$ can be extracted in a similar way, taking this time the antichronological ordering in (2.11). The result for the amplitude can be written in the following way

$$T_{\lambda'\sigma';\lambda\sigma}(s, t) = \sum_{n_1', n_1, n_2', n_2 = 0}^{\infty} T_{\lambda'\sigma';\lambda\sigma}^{n_1'n_1n_2'n_2}(s, t) \tag{2.14}$$

where the term $T^{n_1'n_1n_2'n_2}$ contains spin effects of order $n = n_1' + n_2' + n_1 + n_2$. Explicitly

$$\begin{aligned}(-i)(2\pi)^4 T_{\lambda'\sigma';\lambda\sigma}^{n_1'n_1n_2'n_2}(s, t)\delta(P_i - P_f) \\ = g^2\left(-\frac{g}{2}\right)^n \int_0^1 d\lambda \int dx_1\, dx_2 D(x_1 - x_2) e^{\lambda\delta} \cdot R^{n_1'n_1}(p_1', p_1; x_1)^\mu_{\lambda'\lambda} \\ \times R_2^{n_2'n_2}(p_2', p_2; x_2)^\mu_{\sigma'\sigma|A_1 = A_2 = 0}\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}R^{n'n}(p', p; x)^\mu = \bar{u}(p')\langle p'|T \exp\left[i\int_0^\infty d\tau' v(X - 2\tau' P, P)\right] t^{n'}(X, P)|x\rangle \\ \times \gamma^\mu \langle x|\tilde{t}^n(X, P)\tilde{T}\exp\left[i\int_0^\infty d\tau v(X + 2\tau P, P)\right]|p\rangle u(p)\end{aligned} \tag{2.16}$$

is the mass-shell residue of the Green function after having introduced the symmetrization (2.6–2.7). Unfortunately this expression is purely formal in the sense that ordered exponentials are not functions defined in a closed way but only through a serie expansion [13]. In order to have a computable amplitude one needs to introduce some approximation, valid in a certain kinematical domain. This is the so-called eikonal or semiclassical approximation which will be used in the following section.

3. The Classical Limit

In the region of very high-energy (very small scattering angle) one can imagine that the reaction will happen nearly in the forward direction. Under this circumstance one can impose a geometrical or semiclassical approximation such as

$$[X_\mu, P_\nu] \sim 0 \tag{3.1}$$

or, equivalently,

$$\langle x|0(P, X)|p\rangle \sim \langle x|p\rangle 0(p, x). \tag{3.2}$$

In conclusion, the amplitude (2.15) is a function of P_μ (operator). The approximation (3.1) is also equivalent to expand this function around the c-number p_μ and keep only the first term. Another possibility would be [6] to expand around $\frac{1}{2}(p + p')_\mu$. In this way the approximated amplitude must be dependent on the particular prescription we have used. Nevertheless, as we shall see later, for very small values of θ all the expressions coincide each other.

The second approximation we shall perform, valid only in the limit of very high energies, is the linearization of the function $v(x, p)$

$$v(x, p) \sim 2gp \cdot A(x). \tag{3.3}$$

Under these approximations the amplitude (2.15) becomes computable and the action of the translation operator in functional space is easily evaluated

$$e^{h\cdot\delta/\delta A}F[A] = F[A + h]. \tag{3.4}$$

Equation (3.4) is the Volterra expansion for functionals, the analogous to Taylor expansion for functions.

Using (3.2–3.4) one can write the amplitudes $T^{n_1'n_1n_2'n_2}$ as

$$\begin{aligned}(-i)(2\pi)^4 T^{n_1'n_1n_2'n_2}_{\lambda'\sigma';\lambda\sigma}(s, t)\delta(P_i - P_f) \\ = g^2\left(\frac{i}{2}g\right)^n \int_0^1 d\lambda \int dx_1\, dx_2 \exp[i(p_1' - p_1)x_1 + i(p_2' - p_2)x_2] \\ \times D(x_1 - x_2)\exp[i\lambda\chi(x_1 - x_2)] \\ \times \int_0^\infty da_1 \cdots da_n F_{\lambda'\sigma';\lambda\sigma}(x_1, x_2, \lambda; a_1, \ldots, a_n; s, t, \mu)\end{aligned} \tag{3.5}$$

where χ is the Lévy–Sucher eikonal function

$$\begin{aligned}\chi(x) = ig^2 \int_0^\infty d\tau_1\, d\tau_2[p_1'p_2'D(x - \tau_1 p_1' + \tau_2 p_2') + p_1'p_2 D(x - \tau_1 p_1' - \tau_2 p_2) \\ + p_1 p_2' D(x + \tau_1 p_1 + \tau_2 p_2') + p_1 p_2 D(x + \tau_1 p_1 - \tau_2 p_2)]\end{aligned} \tag{3.6}$$

and F a very complicated function [14] whose explicit form is not important for the moment. Let us notice, though, that a further approximation is needed in order to get a simpler expression for the amplitude. This is the object of the following section.

4. Spin Effects on One Fermion Line

If we eliminate the σF terms coming from one of the two fermion lines, we get a much simpler, and even explicitly computable, expression for the amplitude, as we shall see. From the mathematical point of view this is equivalent to extract from the serie (2.14) an infinite subserie. Indeed, such a restriction would be very appropriated in order to study the scattering of a spinning particle by a static potential, or the limit where the mass of the particle whose spin has been neglected goes to infinity. Let us keep the σF effects on line 1, that is $n_2 = n_2' = 0$. In order to simplify the notation let us redefine the amplitudes as

$$T^{n_1'n_1} \equiv T^{n_1'n_1 00} \tag{4.1}$$

and the total scattering amplitude in this new approximation is given by

$$T = \sum_{n=0}^{\infty} T^{(n)} \tag{4.2}$$

$$T^{(n)} = \sum_{\substack{n_1', n_1 = 0 \\ (n_1' + n_1 = n)}}^{n} T^{n_1'n_1}. \tag{4.3}$$

$T^{(n)}$ in (4.3) contains all the nth order spin effects that we have considered.

Integration in (3.5) over $x_1 + x_2$ gives the energy-momentum conservation δ-function, and $T^{n_1' n_1}$ can be written as

$$T^{n_1' n_1}_{\lambda'\sigma';\lambda\sigma}(s, t) = ig^2 \int_0^1 d\lambda \int dx\, e^{iqx} D(x)\, e^{i\lambda\chi(x)} (\phi_1)^{-n_1} (\phi_1')^{-n_1'} \bar{u}_{\lambda'}(p_1') \times (\sigma_{\alpha\beta} L_{\alpha\beta})^{n_1'} \gamma^\mu (\sigma_{\alpha\beta} L_{\alpha\beta})^{n_1} u_\lambda(p_1) \bar{u}_{\sigma'}(p_2') \gamma_\mu u_\sigma(p_2) \tag{4.4}$$

with the following definitions

$$\phi_1(x) = \int_0^\infty d\tau [p_1 p_2' D(x + \tau p_2') + p_1 p_2 D(x - \tau p_2)] \tag{4.5}$$

$$\phi_1'(x) = \int_0^\infty d\tau [p_1' p_2' D(x + \tau p_2') + p_1' p_2 D(x - \tau p_2)] \tag{4.6}$$

$$L_{\alpha\beta} = \int_0^\infty d\tau [(\partial_\alpha p'_{2_\beta} - \partial_\beta p'_{2_\alpha}) D(x + \tau p_2') + (\partial_\alpha p_{2_\beta} - \partial_\beta p_{2_\alpha}) D(x - \tau p_2)]. \tag{4.7}$$

4.1. The relativistic eikonal formula

The first term in the serie (4.2) or zeroth order term is given by

$$T^{(0)}_{\lambda'\sigma';\lambda\sigma}(s, t) = ig^2 \int_0^1 d\lambda \int dx e^{iqx} D(x) e^{i\lambda\chi(x)} \bar{u}_{\lambda'}(p_1') \gamma_\mu u_\lambda(p_1) \bar{u}_{\sigma'}(p_2') \gamma_\mu u_\sigma(p_2) \tag{4.8}$$

and this is nothing else than the relativistic generalization of the Glauber eikonal amplitude [1]. This formula has been found in the perturbative approach to field theory by Lévy and Sucher [2] using linearized fermion propagators, Chang and Ma [3] with infinite-momentum techniques and by Cheng and Wu [4] which have computed the high-energy behaviour of Feynman integrals. Using functional methods and Green functions computed with the aid of the old Bloch and Nordsieck [15] approximation, this formula has also been found by Fried [16]. Our method is closer to the one used by Abarbanel and Itzykson [6] even if we have adopted a slightly different approximation (3.2), and different technique (2.5), to extract the residues from the Green function. Nevertheless in the limit $\theta \to 0$ the two methods coincide, as it is expected, and one gets the well-known version of (4.8) as an integral over the impact parameter.

In the limit of very small θ and $\mu \to 0$, equation (4.8) looks like

$$T^{(0)}_{\lambda'\sigma';\lambda\sigma}(s, t) = g^2 \left(\frac{-t}{4\mu^2}\right)^{-i\gamma} \frac{\Gamma(1 + i\gamma)}{\Gamma(1 - i\gamma)} \frac{1}{-t} G_{\lambda'\sigma';\lambda\sigma}[1 + R(\mu)] \tag{4.9}$$

where

$$\gamma = \frac{g^2}{4\pi} \frac{s - 2m^2}{\sqrt{s(s - 4m^2)}} \tag{4.10}$$

and the spinor factor G includes all the θ-dependence.

$$G_{\lambda'\sigma';\lambda\sigma} = \left[\frac{s}{2m^2} \delta_{\lambda'\lambda} \delta_{\sigma'\sigma} + i\theta \frac{s}{4m^2} (\delta_{\lambda'\lambda} \tilde{\delta}_{\sigma'\sigma} + \tilde{\delta}_{\lambda'\lambda} \delta_{\sigma'\sigma}) + \theta^2 \frac{s}{8m^2} \tilde{\delta}_{\lambda'\lambda} \tilde{\delta}_{\sigma'\sigma}\right] \times [1 + O(\theta)] \qquad 4.11)$$

where $\tilde{\delta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the spin–flip matrix and $R(\mu)$ a function bounded as

$$|R(\mu)| \underset{\mu \to 0}{\lesssim} \mu^{1/2-\delta} \qquad (0 < \delta < 1). \tag{4.12}$$

4.2. *The first-order correction term*

The first order spin effects are given by $T^{(1)} = T^{10} + T^{01}$, as defined by (4.3). We get

$$T^{(1)}(s, t) = \frac{g^2}{4} \int_0^1 d\lambda \int dx e^{iqx} D(x) A \tag{4.13}$$

where A is the function

$$A = e^{i\lambda\chi(x)} \bar{u}(p_1') \left[\frac{1}{\phi_1'} \sigma_{\alpha\beta} L_{\alpha\beta} \gamma^\mu + \frac{1}{\phi_1} \gamma^\mu \sigma_{\alpha\beta} L_{\alpha\beta} \right] u(p_1) \bar{u}(p_2') \gamma_\mu u(p_2). \tag{4.14}$$

From the definition of $L_{\alpha\beta}$ (4.7), we deduce

$$A(\theta = 0) = 0. \tag{4.15}$$

In fact, writing

$$\bar{u}_{\sigma'}(p_2') \gamma^\mu u_\sigma(p_2) \underset{\theta \to 0}{\sim} \frac{p_2^\mu}{im} \delta_{\sigma'\sigma} \tag{4.16}$$

and

$$L_{\alpha\beta} \underset{\theta \to 0}{\sim} \psi_\alpha p_{2_\beta} - \psi_\beta p_{2_\alpha} \equiv L_{\alpha\beta}^{(0)} \tag{4.17}$$

where

$$\psi_\alpha = \partial_\alpha \phi \tag{4.18}$$

$$\phi(x) = \int_{-\infty}^{\infty} d\tau D(x - \tau p_2) \tag{4.19}$$

and using the identity

$$\sigma_{\alpha\beta} = i(\delta_{\alpha\beta} - \gamma_\alpha \gamma_\beta) \tag{4.20}$$

we have

$$\sigma_{\alpha\beta} L_{\alpha\beta}^{(0)} = i(\not{p}_2 \not{\psi} - \not{\psi} \not{p}_2) \tag{4.21}$$

from where the relation (4.15) follows.

In this way, the expansion of $A(\theta)$ in powers of θ can be written as

$$A(\theta) = \theta A'(0) + O(\theta^2) \tag{4.22}$$

and the function $A'(0)$ is found to be

$$A'(0) = e^{-i\lambda\chi^{(0)}} (p_1 p_2 \phi(x))^{-1} (S + D) \tag{4.23}$$

where $\chi^{(0)} = -2\gamma K_0(\mu b)$ and S and D are spinor factors containing simple and double spin–flip as

$$S = -\frac{P}{m}\,\bar{u}(p_1)\{\not{p}_2, [\not{e}, \not{e}_2]\}u(p_1)\bar{u}(p_2)u(p_2) \tag{4.24}$$

$$D = i\bar{u}(p_1)\{\gamma^\mu, [\not{b}, \not{p}_2]\}u(p_1)g'_\mu(0) \tag{4.25}$$

with $e_{2_\mu} = \delta_{\mu 2}$, $g_\mu(\theta) = \bar{u}(p'_1)\gamma_\mu u(p_1)$ and the function $u_\sigma(x)$ defined by

$$u_\sigma(x) = \partial_\sigma \int_0^\infty D(x - \tau p_2)\,d\tau. \tag{4.26}$$

It is clear, from (4.24–4.25), that S includes only simple spin–flip contributions, while the double spin–flip effects are included in D. The presence of double spin–flip amplitudes may seem curious to the reader if we remember that the function $A(\theta)$ has been developed only up to the first order in θ, equation (4.22). The explanation is that we have not developed the whole amplitude in θ. It remains the factor $e^{iqx} = e^{i\vec{q}\vec{b}}(1 + O(\theta^2))$ which is not expanded. This means that we keep $e^{i\vec{q}\vec{b}}$, and the integration over $\phi = \text{ang}(\vec{q}, \vec{b})$ will give the corresponding Bessel function J_1, so that $\theta J_1(\theta) \sim \theta^2$ for θ very small. Actually, there will be another double spin–flip term coming from $O(\theta^2)$ in (4.22). We shall consider the contribution given by (4.25).

The functions S and D, given by (4.24–4.25) can be estimated and their contribution to the amplitude $T^{(1)}$ arranged in the following way

$$T^{(1)}_{\lambda'\sigma'\lambda;\sigma} = T^N\delta_{\lambda'\lambda}\delta_{\sigma'\sigma} + T^S\tilde{\delta}_{\lambda'\lambda}\delta_{\sigma'\sigma} + T^D(-1)^{\lambda-1}\tilde{\delta}_{\lambda'\lambda}\tilde{\delta}_{\sigma'\sigma} \tag{4.27}$$

where the spin–non-flip term $T^N \sim \theta$ is largely dominated by $T^{(0)}$. The others are given by

$$T^S = \frac{g^4}{4\gamma m}\,\theta \int_0^1 d\lambda \int_0^\infty dbbJ_0(\sqrt{-t}b)\exp[-2i\lambda\gamma K_0(\mu b)]s \tag{4.28}$$

$$T^D = -i\frac{g^4}{8\gamma}\,\theta^2 \int_0^1 d\lambda \int_0^\infty dbb\,\frac{J_1(\sqrt{-t}b)}{\sqrt{-t}}\exp[-2i\lambda\gamma K_0(\mu b)]d \tag{4.29}$$

where s and d are functions defined by the integrals

$$s = \int_{-\infty}^\infty dx_0\,dx_3 D(x)\phi^{-1}(x)\left[u_3(x) + \left(1 + \frac{4m^2}{s}\right)^{1/2} u_0(x)\right] \tag{4.30}$$

$$d = \int_\infty^\infty dx_0\,dx_3 D(x)\,\frac{\partial}{\partial b}\log\phi(x). \tag{4.31}$$

In equation (4.29) we have made use of the relation $t \sim -\theta^2 s/4$, valid at t fixed $s \to \infty$.

The integral defining $\phi(x)$, (4.19), can be analytically evaluated using the Fourier transform of $D(x)$

$$\phi(x) = (4\pi i m)^{-1}\left[\left(\frac{E}{m}\right)^2\left(x_3 + \frac{P}{E}x_0\right)^2 + b^2\right]^{1/2}$$
$$\times \exp\left\{-\mu\left[\left(\frac{E}{m}\right)^2\left(x_3 + \frac{P}{E}x_0\right)^2 + b^2\right]^{1/2}\right\}. \tag{4.32}$$

On the contrary, the functions $u_\sigma(x)$ appearing in (4.30) are much more difficult to estimate because the non-compensation of principal values coming from the τ-integration, from O to infinity. These functions could be computed, either using the explicit form [5]

$$\int_0^\infty d\tau D(x - \tau p) = \frac{1}{8\pi^2}\int_0^\infty d\xi e^{-i\xi px}\{2\theta(x^2)K_0([x^2(\mu^2 + \xi^2 m^2)]^{1/2}) \\ - \pi i\theta(-x^2)H_0^2([-x^2(\mu^2 + \xi^2 m^2)]^{1/2}) \qquad (4.33)$$

or the Fourier transform of $D(x)$ and Cauchy's theorem. In any way the (x_0, x_3) integrals in (4.30) cannot be analytically computed. One should need, either the use of numerical methods or the restriction to some subdomain of integration, as the light-cone [5]. These problems are postponed to another work and we shall compute now the $\mu \to 0$ limit of (4.29).

The integral defining d (4.31), can be straightforwardly computed

$$d = \frac{i}{2\pi}\mu f(\mu b) \qquad (4.34)$$

with

$$f(\mu b) = \int_0^\infty d\omega K_0[\mu b(1 + \omega^2)^{1/2}]e^{-\mu b\omega} \qquad (4.35)$$

and the amplitude (4.29)

$$T^D = \frac{g^2}{8\pi\gamma}\frac{\theta^2}{\sqrt{-t}}\mu^{-1}\int_0^1 d\lambda\int_0^\infty dbbf(b)J_1\left(\frac{\sqrt{-t}}{\mu}b\right)\exp[-2i\lambda\gamma K_0(b)] \qquad (4.36)$$

The limit $\mu \to 0$ of (4.36) can be evaluated, see Appendix, as

$$T^D = \frac{g^4}{16\pi\gamma}\theta^2\frac{1}{-t}\int_0^1 d\lambda\left(\frac{-t}{4\mu^2}\right)^{-i\lambda\gamma}\frac{\Gamma(1 + i\lambda\gamma)}{\Gamma(1 - i\lambda\gamma)} + R(\mu) \qquad (4.37)$$

and

$$|R(\mu)| \underset{\mu\to 0}{\leqslant} \mu^{\delta/2}(\log\mu)^{-1} \qquad (0 < \delta < \tfrac{1}{2}). \qquad (4.38)$$

Thus, in the same way that the formula (4.9) showed, for the zeroth order amplitude, poles in the γ-plane located at $\gamma = in$ $(n = 1, 2, \ldots)$, in the region $2m^2 < s < 4m^2$, $g^2 < 0$, reproducing the energy levels of bound-states in electrodynamics [9], the first order spin corrections show poles at $\gamma = i(n/\lambda)$, $0 \leqslant \lambda \leqslant 1$, or cuts with branching points at $\gamma_c = in$.

The asymptotic methods for Fourier integrals [17] can be used to write (4.37) as

$$T^D = i\frac{g^4}{16\pi\gamma^2}\frac{\theta^2}{-t\log(-t/4\mu^2)}\left\{\left(-\frac{t}{4\mu^2}\right)^{-i\gamma}\frac{\Gamma(1 + i\gamma)}{\Gamma(1 - i\gamma)} - 1\right\} + O(\log^{-2}\mu). \qquad (4.39)$$

One can compare this expression with the corresponding term coming from the zeroth order (4.9–4.11). The factor $\theta^2/-t$ is always present while (4.39) is damped with respect to (4.9) by a power of s.

In any way we have shown that in the contribution to the amplitude due to the fermion spin there exist a term which does not decrease faster than $(\log\mu)^{-1}$.

The non-considered double spin–flip term, coming from the order θ^2 in $A(\theta)$, is given by an integral like (4.30). Its $\mu \to 0$ limit is not analytically computable and for this reason we shall not write here its integral expression.

4.3. *The forward amplitude*

In the forward direction we have shown (4.13–4.19) that $T^{(1)} = 0$. One can generalize this analysis and show that

$$T^{(2n+1)}(\theta = 0) = 0 \qquad (n = 0, 1, 2, \dots) \tag{4.40}$$

while even terms $T^{(2n)}$ give a non-vanishing contribution to the spin–non-flip amplitude, while their contribution to the spin–flip amplitudes is zero, as needed from angular momentum conservation.

After straightforward computations one finds

$$T^{(2n)}_{\lambda'\sigma';\lambda\sigma} \underset{\theta\to 0}{\sim} -ig^2 \frac{p_1 p_2}{m^2} \int_0^1 d\lambda \int dx e^{iqx} D(x) e^{i\lambda x^0} v^n(x) \delta_{\lambda'\lambda} \delta_{\sigma'\sigma} \tag{4.41}$$

where

$$v(x) = \frac{m^2 \psi^2(x) + (p_2 \psi(x))^2}{4(p_1 p_2)^2 \phi^2(x)}. \tag{4.42}$$

Using the formal serie $\sum a^n = (1 - a)^{-1}$ one can formally add the serie of amplitudes in the forward direction as

$$T_{\lambda'\sigma';\lambda\sigma} \underset{\theta\to 0}{\sim} ig^2 \frac{s - 2m^2}{2m^2} \int_0^1 d\lambda \int dx \exp(iqx) \exp[-2i\lambda\gamma K_0(\mu b)] D(x) \frac{1}{1 - v(x)} \delta_{\lambda'\lambda} \delta_{\sigma'\sigma} \tag{4.43}$$

and from the explicit expression of ϕ (4.32), one can explicitly write $v(x)$ as

$$v(x) = \left[\frac{s}{16m^4}\left(x_3 + \frac{P}{E} x_0\right)^2 + \frac{m^2}{(s - 2m^2)^2}\right] L \tag{4.44}$$

and

$$L = \left[\left(\frac{E}{m}\right)^2 \left(x_3 + \frac{P}{E} x_0\right)^2 + b^2\right]^{-2} + 2\mu \left[\left(\frac{E}{m}\right)^2 \left(x_3 + \frac{P}{E} x_0\right)^2 + b^2\right]^{-3/2}$$
$$+ \mu^2 \left[\left(\frac{E}{m}\right)^2 \left(x_3 + \frac{P}{E} x_0\right)^2 + b^2\right]^{-1}. \tag{4.45}$$

A deeper knowledge of (4.43) would be necessary in order to evaluate spin effects. For instance, in the complex energy plane new singularities corresponding to fine structure splitting could appear. The spin–spin effects, due to terms depending on σF on the two fermion lines would be, in any case, much more difficult to evaluate, because the complexity of the general expression [14].

5. Conclusion

We have studied in this paper the contributions, to the elastic fermion–fermion amplitude, coming from powers of σF, the coupling of fermion spin to the electromagnetic field. Two main applications of the result could be studied in the future.

The first application is concerned with the fine structure energy levels of hydrogen, or positronium, which the eikonal formula is unable to explain. In the same way that the energy levels are not given by any particular term of the spin-independent serie, but by the sum or relativistic eikonal formula, we believe that probably the fine structure splitting is not given by a particular spin correction, but by a serie of spin corrections, such as the one given by equation (4.43). In order to carry out this program the knowledge of Fourier transform of complicated generalized functions is needed.

The second application would be to describe hadronic reactions. One can consider the photons as massive vector mesons (ω, ρ, ...) and the fermions as nucleons. One can try, in this way, to fit $NN \to NN$. This task has been performed by Yao [18] and the result of the fitting disagrees slightly with experimental data. The gap could be fulfilled by the spin terms. Another possibility would be to consider the photons as 'massless hadrons' or 'gluons' and the fermions as nucleons or quarks. It is believed, though, that gauge groups, hidden in strong interactions, are not so simple as $U(1)$ but some non-abelian group, as $SU(n) \times SU(n)$, giving rise to Yang–Mills vector gluons. This is the object of present investigations.

Acknowledgments

I thank Profs. C. Itzykson and H. Ruegg for numerous suggestions and their interest in my work, and Drs. R. Lacaze, B. Morel, B. Petersson, C. Savoy and J. Zinn-Justin for many helpful discussions.

I would like to thank the Theoretical Physics Department of the University of Geneva for its hospitality and financial support.

Appendix

We shall compute in this Appendix $\lim_{\mu\to 0} J$, J being defined by the integral

$$J = \frac{1}{\mu}\int_0^1 d\lambda \int_0^\infty db J_1\left(\frac{\sqrt{-t}}{\mu}b\right)\exp[-2i\lambda\gamma K_0(b)]\int_0^\infty d\omega K_0[(\omega^2+b^2)^{1/2}]e^{-\omega}. \quad (A.1)$$

The ω-integral converges uniformly for $0 \leqslant b \leqslant \infty$ because $K_0(x)$, x real, is a positive, decreasing function, and

$$e^{-\omega}K_0[(\omega^2+b^2)^{1/2}] \leqslant K_0(\omega)e^{-\omega}$$

$$\int_0^\infty d\omega e^{-\omega}K_0(\omega) = 1.$$

One can, under these circumstances, exchange the order of ω and b-integrations and write

$$J = \int_0^1 d\lambda \int_0^\infty d\omega e^{-\omega} j(\omega, \lambda, \mu) \quad (A.2)$$

$$j(\omega, \lambda, \mu) = \frac{1}{\mu}\int_0^\infty db J_1\left(\frac{\sqrt{-t}}{\mu}b\right)\exp[-2i\lambda\gamma K_0(b)]K_0[(\omega^2+b^2)^{1/2}]. \quad (A.3)$$

Let us introduce a cut-off parameter $\epsilon(\mu)$ with the following properties

$$\left.\begin{aligned}&\epsilon(\mu) \to 0\\ &\frac{\epsilon(\mu)}{\mu} \to \infty\end{aligned}\right\}\mu \to 0 \quad (A.4)$$

and one defines j_1 and j_2, and correspondingly J_1 and J_2, as

$$j_1 = \frac{1}{\mu}\int_0^{\epsilon(\mu)} dbJ_1\left(\frac{\sqrt{-t}}{\mu}\,b\right)\exp[-2i\lambda\gamma K_0(b)]K_0[(\omega^2 + b^2)^{1/2}] \tag{A.5}$$

$$j_2 = \frac{1}{\mu}\int_{\epsilon(\mu)}^{\infty} dbJ_1\left(\frac{\sqrt{-t}}{\mu}\,b\right)\exp[-2i\lambda\gamma K_0(b)]K_0[(\omega^2 + b^2)^{1/2}]. \tag{A.6}$$

Thus, in the interval $\epsilon \leqslant b \leqslant \infty$ one can use, in the limit $\mu \to 0$, the asymptotic value of the Bessel function $J_1(z)$

$$J_1(z) \underset{|z|\to\infty}{\sim} \left(\frac{2}{\pi z}\right)^{1/2}\cos z \tag{A.7}$$

and integrating (A.6) by parts, one gets

$$\begin{aligned} j_2 \underset{\mu\to 0}{\sim} \frac{\mu^{1/2}}{(-t)^{3/4}}\Bigg\{&-K_0[(\omega^2 + \epsilon^2)^{1/2}]\sin\left(\frac{\sqrt{-t}}{\mu}\,\epsilon\right)\exp[-2i\lambda\gamma K_0(\epsilon)]\Bigg\} \\ &+ \int_\epsilon^\infty db\,\sin\left(\frac{\sqrt{-t}}{\mu}\,b\right)\exp[-i\lambda\gamma K_0(b)]\Bigg\{\frac{b}{(b^2+\omega^2)^{1/2}}K_1[(b^2+\omega^2)^{1/2}] \\ &\qquad - 2i\lambda\gamma K_0(b)K_0[(b^2+\omega^2)^{1/2}]\Bigg\}. \end{aligned} \tag{A.8}$$

The absolute value of j_2 is bounded by

$$|j_2| \underset{\mu\to 0}{\leqslant} \lambda\gamma(-t)^{-3/4}\mu^{1/2}\log^2\epsilon \tag{A.9}$$

and so

$$|J_2| \leqslant \tfrac{1}{2}\gamma(-t)^{-3/4}\mu^{1/2}\log^2\epsilon \tag{A.10}$$

where use has been made of

$$\int_\epsilon^\infty dbK_1(b) = K_0(\epsilon), \quad \int_\epsilon^\infty dbK_0(b)K_1(b) = \tfrac{1}{2}K_0^2(\epsilon). \tag{A.11}$$

If we impose the particular form $\epsilon(\mu) = \mu^{1-\delta}$ with $0 < \delta < 1$, in order to satisfy the conditions (A.4), equation (A.10), becomes

$$|J_2| \leqslant \frac{\gamma(1-\delta)^2}{2}(-t)^{-3/4}\mu^{1/2}\log^2\mu. \tag{A.12}$$

In order to compute j_1 in the limit $\mu \to 0$ ($\epsilon \to 0$) let us use the serie representation of $K_0(b)$

$$K_0(b) = -\log\frac{b}{2}\sum_{K=0}^{\infty}\frac{b^{2K}}{4^K(K!)^2} + \sum_{K=0}^{\infty}\frac{b^{2K}}{4^K(K!)^2}\psi(K+1) \tag{A.13}$$

in order to write

$$\exp[-2i\lambda\gamma K_0(b)]K_0[(\omega^2+b^2)^{1/2}] = b^{2i\lambda\gamma}K_0(\omega)\Bigg\{1 + \sum_{\substack{n=2,m=1\\(n\geqslant 2m)}}^{\infty} A_{nm}(\omega)b^n\log^m b\Bigg\} \tag{A.14}$$

and these terms behave in absolute value, under the integral (A.5), as $\mu^{n-(n+1)\delta}\log^m\mu$. The condition $n-(n+1)\delta > \frac{1}{2}$ for $n \geqslant 2$ imposes a new restriction to the value of δ

$$0 < \delta < \tfrac{1}{2} \tag{A.15}$$

to be able to write

$$j_1 \underset{\mu\to 0}{\sim} \frac{1}{\mu} K_0(\omega) \int_0^{\epsilon} db b^{2i\lambda\gamma} J_1\left(\frac{\sqrt{-t}}{\mu} b\right). \tag{A.16}$$

The integral (A.16) can be calculated with the aid of [19]

$$\int_0^1 x^\rho J_\nu(ax)\,dx = a^{-\rho-1}\Bigg[(\rho+\nu-1)aJ_\nu(a)S_{\rho-1,\nu-1}(a) - aJ_{\nu-1}(a)S_{\rho,\nu}(a) + 2^\rho \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\rho+\frac{1}{2}\nu)}{(\frac{1}{2}-\frac{1}{2}\rho+\frac{1}{2}\nu)}\Bigg], \qquad (a>0,\ \mathrm{Re}(\rho+\nu)>-1) \tag{A.17}$$

where $S_{\mu\nu}(z)$ is a Lommel function, with asymptotic behaviour [20]

$$S_{\mu,\nu}(z) \underset{|z|\to\infty}{\sim} z^{\mu-1}. \tag{A.18}$$

Let us write j_1 as

$$j_1 = K_0(\omega)[f+g] \tag{A.19}$$

$$f = (-t)^{-1/2}\left(\frac{-t}{4\mu^2}\right)^{-i\lambda\gamma} \frac{\Gamma(1+i\lambda\gamma)}{\Gamma(1-i\lambda\gamma)} \tag{A.20}$$

$$g = \left(\frac{-t}{\mu^2}\right)^{-i\lambda\gamma}\left(\frac{\epsilon}{\mu}\right)\Bigg\{2i\lambda\gamma J_1\left(\sqrt{-t}\frac{\epsilon}{\mu}\right)S_{2i\lambda\gamma-1,0}\left(\sqrt{-t}\frac{\epsilon}{\mu}\right) - J_0\left(\sqrt{-t}\frac{\epsilon}{\mu}\right)S_{2i\lambda\gamma,1}\left(\sqrt{-t}\frac{\epsilon}{\mu}\right)\Bigg\}. \tag{A.21}$$

Taking the limit $\mu\to 0$ one has

$$g \underset{\mu\to 0}{\sim} -\cos\left(\sqrt{-t}\frac{\epsilon}{\mu}\right)\epsilon^{2i\lambda\gamma}(-t)^{-3/4}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\mu}{\epsilon}\right)^{1/2}. \tag{A.22}$$

The term (A.20) gives rise, after ω-integration, to the first term in (4.37) and (A.22), after ω and λ-integration, behaves in absolute value as

$$|R(\mu)| \underset{\mu\to 0}{\lesssim} \left(\frac{2}{\pi}\right)^{1/2}\frac{1}{\gamma}(-t)^{-3/4}\frac{1}{1-\delta}\mu^{\delta/2}(\log\mu)^{-1}. \tag{A.23}$$

Because $\mu^{\delta/2} > \mu^{1/2}$, the highest bound is given by (A.23), which is the result (4.38).

REFERENCES

[1] R. J. GLAUBER, *Lectures in Theoretical Physics*, Vol. 1, edited by W. E. BRITTIN and L. G. DUNHAM (Intersc., New York 1959).

[2] M. LÉVY and J. SUCHER, Phys. Rev. *186*, 1656 (1969); Phys. Rev. *D2*, 1716 (1970).

[3] S. J. CHANG and S. K. MA, Phys. Rev. *180*, 1506 (1969); Phys. Rev. *188*, 2385 (1969); Phys. Rev. Lett. *22*, 1334 (1969).

[4] H. CHENG and T. T. WU, Phys. Rev. *186*, 1611 (1969), and references cited therein.
[5] G. W. ERICKSON and H. M. FRIED, J. Math., Phys. *6*, 414 (1965).
[6] H. D. I. ABARBANEL and C. ITZYKSON, Phys. Rev. Lett. *23*, 53 (1969).
[7] D. LÉGER and M. LÉVY, Nuovo Cimento *25A*, 53 (1975).
[8] B. HUMPERT and M. QUIRÓS, Nucl. Phys. *B59*, 141 (1973); Lett. Nuovo Cimento *7*, 201 (1973).
[9] E. BREZIN, C. ITZYKSON and J. ZINN-JUSTIN, Phys. Rev. *D1*, 2349 (1970).
[10] H. LEHMAN, K. SYMANZIK and W. ZIMMERMAN, Nuovo Cimento *1*, 205 (1955).
[11] C. ORZALESSI, private communication.
[12] M. L. GOLDBERGER and E. N. ADAMS II, J. Chem. Phys. *20*, 240 (1952).
[13] R. P. FEYNMAN, Phys. Rev. *84*, 108 (1951).
[14] M. QUIRÓS, Ph.D. Thesis, University of Geneva, 1975.
[15] F. BLOCH and A. NORDSIECK, Phys. Rev. *52*, 54 (1937).
[16] H. M. FRIED, *Functional Methods and Models in Quantum Field Theory* (M.I.T. Press, Cambridge 1972).
[17] A. ERDELYI, *Asymptotic Expansions* (Dover Pub., Inc. 1956).
[18] Y. P. YAO, Phys. Rev. *D2*, 1342 (1970).
[19] A. ERDELYI et al., *Table of Integral Transforms*, Vol. 2 (McGraw-Hill, Inc. 1954).
[20] A. ERDELYI et al., *Higher Transcendental Functions*, Vol. 2 (McGraw-Hill, Inc. 1954).