Absolute continuity for a 1-dimensional model of the Stark-Hamiltonian

Absolute Continuity for a 1-Dimensional Model of the Stark-Hamiltonian

by **P. A. Rejto**[1]) **and Kalyan Sinha**[2])

Département de Physique Théorique, 32, Bd. d'Yvoy, CH-1211 Genève-4, Suisse

(11. XI. 1975)

Abstract. Absolute continuity of a generalized Stark-like Hamiltonian in presence of short-range potential is proved.

1. Introduction

In this paper, we study an idealized model of the Stark effect. Firstly, we consider a one-dimensional version and secondly, we replace the singular coulomb potential by a shortrange, viz. L_1-potential. On the other hand, we replace pure Stark potential, i.e. the potential due to an uniform field by more general Stark-like potential, which is described in detail in Section 2.

The following questions arise naturally. Is the spectrum of the Hamiltonian absolutely continuous? Does the probability of finding a particle in finite space region at time t decay to zero as time increases to infinity? [1] Also, does the particle spend finite time in a finite space region? Here we answer in the affirmative the first question. The second one is dealt with elsewhere [2] and the third will be answered in a future communication.

Physically, it is well-known [3] that the Stark potential adjoins a barrier of finite height and large width to the short-range potential. Therefore, a quantum mechanical particle can tunnel through the barrier. Another way of describing the same phenomenon is to say that the system will undergo ionization when the Stark field is switched on. Hence the motion of the particle is expected to be infinite or equivalently the spectrum of the Hamiltonian is expected to be continuous.

In order to prove absolute continuity, we use a method described in [4] and applied in [5] to another situation. In Section 3, we state the properties which imply the absolute continuity of a self-adjoint operator. In particular, we introduce the abstract notion of an approximating family of operators that approximate a given operator. In Section 4, we construct a candidate for the approximating family for Stark-like potentials, using the JWKB-approximation method [3]. For this, we subdivide the real line into three parts. To every given interval $\mathscr{I}$ in energy we associate an interval I^m in space, called turning interval, in the interior of which lie all the

---

[1]) On sabbatical leave from the School of Mathematics, University of Minnesota, Minneapolis, Minn. 55455, USA.

[2]) Supported by Fonds National Suisse.

turning points. We start with a JWKB-approximate solution, in the region to the left of I^m and continue it through the turning interval to the right of I^m. This along with another one starting from the right of I^m gives us a pair of approximate solutions of Schrödinger equation.

In Sections 5 and 6, we verify that indeed the construction in Section 4 provides us with an approximating family. In Sections 7 and 8, we verify additional Conditions $A_1(\mathscr{I})$ and $A_2(\mathscr{I})$ to arrive at the conclusion in Section 9.

2. Formulation of the Result

We give the formulation of a condition, similar to the one due to Walter [6]. This condition differs from those of Walters, Titchmarsh [7] and Neumark [8] on two counts. Firstly, the basic interval is $(-\infty, \infty)$ instead of $(0, \infty)$ and secondly we have added a shortrange part. Also, since $p_2(+\infty) \neq p_2(-\infty)$, the method of splitting p_2 into two parts, one shortrange and the other with no turning points, as given in [9] does not work.

Condition B. The potential p is a twice continuously differentiable real function defined on $(-\infty, \infty)$ such that

$$\lim_{\xi \to \pm\infty} p(\xi) = p(\pm\infty) = \mp\infty \tag{2.1}$$

and

$$\int_{\pm 1}^{\pm\infty} \frac{d\xi}{|p(\xi)|^{1/2}} = \infty. \tag{2.2}$$

Furthermore, there is a positive number $\tilde{\xi}$ large enough such that the potential

$$r(p)(\xi) = \frac{5}{16}\left(\frac{p'(\xi)}{p(\xi)}\right)^2 - \frac{1}{4}\frac{p''(\xi)}{p(\xi)} \tag{2.3}$$

satisfies the estimate

$$\int_{\pm\tilde{\xi}}^{\pm\infty} |r(p)(\xi)| \frac{d\xi}{|p(\xi)|^{1/2}} < \infty. \tag{2.4}$$

A potential satisfying this condition is called Stark-like potential. We call a potential short-range if

$$p \in L_1(\mathbf{R}) \cap L_{2,\mathrm{loc}}(\mathbf{R}). \tag{2.5}$$

We also state a set of simplifying conditions on the potential under which some of the conclusions follow relatively easily.

Condition S. A potential p is Stark-like if it satisfies (2.1), (2.2) and if for some positive $\tilde{\xi}$

$$\int_{\pm\tilde{\xi}}^{\pm\infty} \left(\frac{r(p)(\xi)}{|p(\xi)|^{1/2}}\right)^n d\xi < \infty, \qquad n = 1,2. \tag{2.6}$$

A potential p is short-range if

$$p \in L_1(\mathbf{R}) \cap L_2(\mathbf{R}). \tag{2.7}$$

Next we assume that the potential p admits a decomposition

$$p = p_1 + p_2 \tag{2.8}$$

where p_2 is Stark-like and p_1 is short-range, satisfying either Condition B or Condition S.

Then we set

$$D(L(p)) = \left\{ \begin{matrix} f \in L_2(\mathbf{R})/f \text{ and } f' \text{ are locally absolutely} \\ \text{continuous in } \mathbf{R} \text{ and } -f'' + pf \in L_2(\mathbf{R}) \end{matrix} \right\} \tag{2.9}$$

and define the operator $L(p)$ mapping $D(L(p))$ into $L_2(\mathbf{R})$ by

$$L(p)f(\xi) = -f''(\xi) + p(\xi)f(\xi). \tag{2.10}$$

Theorem 2.1. Suppose that the potential function p satisfies assumption (2.8) with Condition B and define operator $L(p)$ by relations (2.9) and (2.10). Suppose further that this operator is essentially self-adjoint on $D(L(p)) \cap \dot{C}(\mathbf{R})$. Then this operator is absolutely continuous, that is to say,

$$L(p) = (L(p))_{\text{a.c.}}. \tag{2.11}$$

Theorem 2.2. Suppose that the potential function satisfies assumption (2.8) with Condition S and define $L(p)$ as before. Then this operator is absolutely continuous.

We prove these two theorems from an abstract theorem, stated in Section 3, proved elsewhere [9]. Under the set of simplifying assumptions, self-adjointness of $L(p)$ is proved in Section 8. For the general case, if short-range part is zero, then our essential self-adjointness assumption is implied by assumption (2.2) [10].

It is easy to see that one-dimensional Stark potential, given as $p_2(\xi) = -eE\xi$ satisfies Condition S, which in fact implies also Condition B.

3. An Abstract Criterion for Absolute Continuity

Let A be a given self-adjoint operator acting on a given abstract Hilbert space $\mathscr{H}$. We start with a lemma which gives a simple sufficient condition for a part of A to be absolutely continuous. To formulate it we need some notations. To a given interval of reals, $\mathscr{I}$, and angle α, we assign two open regions of the complex plane by setting

$$\mathscr{R}_{\pm}(\mathscr{I}) = \{\mu \mid \operatorname{Re}\mu \in \mathscr{I}^0, 0 < \pm \arg\mu < \alpha\} \tag{3.1}$$

where $\mathscr{I}^0$ denotes the interior of the interval $\mathscr{I}$. As usual, we denote by $\mathscr{B}(\mathscr{H})$ the space of everywhere defined bounded operators on $\mathscr{H}$. For a possibly unbounded operator T and for μ in $\rho(T)$, the resolvent set of T, we set

$$R(\mu, T) = (\mu I - T)^{-1} \in \mathscr{B}(\mathscr{H}). \tag{3.2}$$

Lemma 3.1. Suppose that to A and to the given compact interval $\mathscr{I}$ there is a dense subset S such that for each pair of vectors (f, g) in $S \times S$

$$\sup_{\mu \in \mathscr{R}_{\pm}(\mathscr{I})} |(R(\mu, A)f, g) - (R(\bar{\mu}, A)f, g)| < \infty. \tag{3.3}$$

Then $A(\mathscr{I})$, the part of A over $\mathscr{I}$, is absolutely continuous.

It was observed elsewhere [11] that this lemma is an elementary consequence of the resolvent loop-integral formula.

For a class of Schrödinger operators it is possible to factorize the resolvent in a manner which allows one to establish the rather general assumptions of Lemma 3.1. To describe such factorizations we make a digression on forms. Accordingly let $\mathscr{G}$ be an abstract Banach space and F a functional on $\mathscr{G} \times \mathscr{G}$ which is linear in the first argument and conjugate linear in the second argument, in short a sesquilinear form. In analogy to the notion of the norm of an operator we define the norm of the form $[F]$ by

$$\|[F]\| = \sup_{f \neq 0, g \neq 0} \frac{|[F](f, g)|}{\|f\|_{\mathscr{G}} \|g\|_{\mathscr{G}}} \tag{3.4}$$

and denote by $F(\mathscr{G})$ the space of forms for which this norm is finite. Next let A be a bounded operator on $\mathscr{G}$. We define the product $[F]A$ to be the form determined by

$$[F]A(f, g) = [F](Af, g). \tag{3.5}$$

Then clearly

$$\|[F]A\| \leqslant \|[F]\| \, \|A\|. \tag{3.6}$$

So far the Banach space $\mathscr{G}$ was independent of our Hilbert space $\mathscr{H}$. Now we impose our first requirement, namely that both $\mathscr{G}$ and $\mathscr{H}$ can be embedded in a metric space $\mathscr{M}$ in such a manner

$$\mathscr{G} \cap \mathscr{H} \text{ is dense in } \mathscr{H} \text{ and in } \mathscr{G}. \tag{3.7}$$

Clearly an operator T in $\mathscr{H}$ defines a form on $D(T) \cap \mathscr{G} \times D(T) \cap \mathscr{G}$; namely the form

$$[T]_{\mathscr{G}}(f, g) = [T]_{\mathscr{H}}(f, g) = (Tf, g). \tag{3.8}$$

If

$$D(T) \cap \mathscr{G} \text{ is dense in } \mathscr{G}, \tag{3.9}$$

and the closure of this form is in $F(\mathscr{G})$ we denote it by the same symbol $T_{\mathscr{G}}$. In this case we say that the operator T determines a form in $F(\mathscr{G})$. Note that in view of assumption (3.7) assumption (3.9) holds for each T in $\mathscr{B}(\mathscr{H})$. If in addition to assumption (3.9)

$$T(D(T) \cap \mathscr{G}) \subset \mathscr{G}, \tag{3.10}$$

and the closure of this operator is in $\mathscr{B}(\mathscr{G})$ we denote it by $T_{\mathscr{G}}$. In this case we say that the operator T determines an operator in $\mathscr{B}(\mathscr{G})$.

These definitions allow us to state our key definition.

Definition 3.1. ($\mathscr{I}$). The family of operators $A_0(\mu)$ is an approximating family to the given operator A over the given interval $\mathscr{I}$ if there are open regions $\mathscr{R}_{\pm}(\mathscr{I})$ of the form (3.1) such that for each μ in $\mathscr{R}_{\pm}(\mathscr{I})$,

$$\mu \in \rho(A_0(\mu)), \quad \text{i.e.} \quad R(\mu, A_0(\mu)) \in \mathscr{B}(\mathscr{H}). \tag{3.11}$$

Furthermore there is a space $\mathscr{G}$ satisfying assumption (3.7) such that with reference to it the two conditions that follow hold:

Condition $G_1(\mathscr{I})$. For each μ in $\mathscr{R}_{\pm}(\mathscr{I})$ the approximate resolvent operator, $R(\mu, A_0(\mu))$ determines a sesquilinear form in $F(\mathscr{G})$ for which

$$\sup_{\mu \in \mathscr{R}_{\pm}(\mathscr{I})} \|[R(\mu, A_0(\mu))]_{\mathscr{G}}\| < \infty. \tag{3.12}$$

Condition $G_2(\mathscr{I})$. For each μ in $\mathscr{R}_\pm(\mathscr{I})$ the operator determines an operator in $\mathscr{B}(\mathscr{G})$, i.e.

$$T(\mu)_{\mathscr{G}} \equiv ((A - A_0(\mu))R(\mu, A_0(\mu)))_{\mathscr{G}} \in \mathscr{B}(\mathscr{G}). \tag{3.13}$$

These operators depend norm-continuously on μ and admit continuous extensions on to the closures $\bar{\mathscr{R}}_\pm(\mathscr{I})$.

An example of Pavlov and Petras [12] concerning Hölder gentle perturbations implies that the existence of a family of approximating operators alone is not a sufficient condition for absolute continuity. Therefore, in analogy with such perturbations [11], [13], we introduce two additional conditions.

Condition $A_1(\mathscr{I})$. For each ω in $\mathscr{I}$ each of the two limit operators $(I - T_\pm(\omega))_G$ admit inverses in $\mathscr{B}(\mathscr{G})$.

Condition $A_2(\mathscr{I})$. For each μ in the open region $\mathscr{R}_\pm(\mathscr{I})$ the original resolvent $R\ (\mu; A)$ determines a sesquilinear form in $F(\mathscr{G})$. This sesquilinear form is such that

$$[R(\mu, A)]_{\mathscr{G}} = [R(\mu, A_0(\mu))]_{\mathscr{G}}(I - T(\mu))_{\mathscr{G}}^{-1}. \tag{3.14}$$

It is not difficult to show that these two additional conditions together with the existence of a family of approximating operators are sufficient for absolute continuity, In fact, the following theorem was proved in the report [9].

Theorem 3.1. Let A be a given self-adjoint operator and let $\mathscr{I}$ be a given compact interval. Suppose that A admits a family of approximating operators over $\mathscr{I}$, in the sense of Definition 3.1. Suppose further that Conditions $A_{1,2}(\mathscr{I})$ hold. Then $A(\mathscr{I})$ is absolutely continuous.

4. Construction of JWKB Approximate Potentials

Application of JWKB approximation method in quantum mechanics goes back to the early days [14]. In most of its applications it is used to construct an approximate eigenfunction (proper or improper). These approximate eigenfunctions, in turn, have been used to compute transmission and reflection coefficients, currents [3] and various other quantities of interest [15]. JWKB approximation method is basically a semi-classical approximation in which the leading term in the asymptotic expansion of the solution of the Schrödinger equation, in the limit of Planck's constant h tending to zero, is kept. Details of this can be found in Ref. [16].

It is well-known that JWKB approximation is valid in regions away from the turning point for every given value of energy ω, where a turning point is defined as the solution of the equation $p_2(\xi) = \omega$. Since we shall be interested in an interval of energy $\mathscr{I}$ instead of a given value of energy ω, we shall need an interval I^m, called turning interval, such that

$$\text{distance } (\mathscr{I}, p_2(\mathbf{R} - I^m)) \neq 0. \tag{4.1}$$

That such an interval can be constructed follows easily from the continuity of p_2 and property (2.1). We also designate the boundary points of I^m as a and b respectively and they are chosen such that

$$p_2(\xi) - \omega > 0 \qquad \text{for all } \xi \leqslant a, \omega \in \mathscr{I} \tag{4.2}$$

and

$$p_2(\xi) - \omega < 0 \qquad \text{for all } \xi \geqslant b,\ \omega \in \mathscr{I}. \tag{4.3}$$

Thus I^m is the compact interval $[a, b]$ and JWKB approximation is valid only in $\mathbf{R} - I^m$. In the interval I^m, we have a lot of freedom to choose the approximate solution. As the simplest possibility, we choose here the linear approximation in the turning interval I^m. In the following we denote by I^l and I^r the semi-infinite regions $(-\infty, a)$ and (b, ∞) respectively so that

$$\mathbf{R} = I^l \cup I^m \cup I^r.$$

With the aid of the above construction, we define the JWKB approximate potential $q(\mu)$ by

$$q(\mu)(\xi) = \begin{cases} p_2(\xi) + r(p_2 - \mu)(\xi), & \xi \notin I^m \\ \mu, & \xi \in I^m \end{cases} \tag{4.4}$$

where $\mu \in \mathscr{R}_\pm(\mathscr{I})$. This is motivated by the JWKB approximation method to obtain approximate solutions of the equation

$$f''(\xi) + (\mu - p_2(\xi))f(\xi) = 0. \tag{4.5}$$

Let $\sqrt{z}$ denote the branch of the square-root function defined by the property

$$\operatorname{Re} \sqrt{z} > 0, \qquad \text{for } z \notin (-\infty, 0]. \tag{4.6}$$

With the aid of this function we define

$$w^\pm(\mu)(\xi) = \pm\sqrt{p_2(\xi) - \mu} - \frac{1}{4}\frac{p_2'(\xi)}{p_2(\xi) - \mu}, \qquad \xi \notin I^m \tag{4.7}$$

and write

$$k^l(\mu)(\xi) = \begin{cases} \exp\left(\int_a^\xi w^+(\mu)(\sigma)\, d\sigma\right), & \xi \in I^l \\ \alpha^l(\mu) + \beta^l(\mu)\xi, & \xi \in I^m \\ \gamma^+(\mu)\exp\left(\int_b^\xi w^+(\mu)(\sigma)\, d\sigma\right) + \gamma^-(\mu)\exp\left(\int_b^\xi w^-(\mu)(\sigma)\, d\sigma\right), & \xi \in I^r \end{cases} \tag{4.8}$$

$$k^r(\mu)(\xi) = \begin{cases} \delta^+(\mu)\exp\left(\int_a^\xi w^+(\mu)(\sigma)\, d\sigma\right) + \delta^-(\mu)\exp\left(\int_a^\xi w^-(\mu)(\sigma)\, d\sigma\right), & \xi \in I^l \\ \alpha^r(\mu) + \beta^r(\mu)\xi, & \xi \in I^m \\ \exp\left(\int_b^\xi w^-(\mu)(\sigma)\, d\sigma\right), & \xi \in I^r. \end{cases} \tag{4.9}$$

The constants appearing in (4.8) and (4.9) are chosen such as to make $k^l(\mu)$, $k^l(\mu)'$

and $k^r(\mu)$, $k^r(\mu)'$ continuous everywhere. This leads to

$$\left.\begin{aligned}\alpha^l(\mu) &= 1 - aw^+(\mu)(a), \quad \beta^l(\mu) = w^+(\mu)(a)\\ \gamma^+(\mu) &= \frac{w^+(\mu)(a) - w^-(\mu)(b) - (b-a)w^+(\mu)(a)w^-(\mu)(b)}{w^+(\mu)(b) - w^-(\mu)(b)}\\ \gamma^-(\mu) &= \frac{w^+(\mu)(a) - w^+(\mu)(b) - (b-a)w^+(\mu)(a)w^+(\mu)(b)}{w^-(\mu)(b) - w^+(\mu)(b)}\end{aligned}\right\} \qquad (4.10)_l$$

$$\left.\begin{aligned}\alpha^r(\mu) &= 1 - bw^-(\mu)(b), \quad \beta^r(\mu) = w^-(\mu)(b)\\ \delta^+(\mu) &= \frac{w^-(\mu)(b) - w^-(\mu)(a) + (b-a)w^-(\mu)(a)w^-(\mu)(b)}{w^+(\mu)(a) - w^-(\mu)(a)}\\ \delta^-(\mu) &= \frac{w^+(\mu)(a) - w^-(\mu)(b) - (b-a)w^+(\mu)(a)w^-(\mu)(b)}{w^+(\mu)(a) - w^-(\mu)(a)}.\end{aligned}\right\} \qquad (4.10)_r$$

An elementary algebra together with the above choice of constants imply that both functions $k^l(\mu)$, $k^r(\mu)$ satisfy the differential equation

$$k(\mu)''(\xi) + (\mu - q(\mu)(\xi))k(\mu)(\xi) = 0. \qquad (4.11)$$

We shall make essential use of this fact in subsequent sections. There we shall show that the family of approximating operators defined by $A_0(\mu) = L(q(\mu))$ satisfies the assumptions of the abstract Theorem 3.1.

5. A Lemma on Approximating Potentials

Let $p(\mu)$ be a given family of potentials and let $\mathscr{I}$ be a given interval. Recall that definition (2.10) assigns to each of them the operator $L(p(\mu))$. In this section we formulate conditions which ensure that this family of operators approximates the operator $L(p)$ over the interval $\mathscr{I}$.

Condition I($\mathscr{I}$). The family of potentials $p(\mu)$ is such that the operator $L(p(\mu))$ satisfies assumption (3.11). Furthermore for each point ω of $\mathscr{I}$ each of the two limit functions exists,

$$\lim_{\epsilon\to+0} p(\omega \pm i\epsilon)(\xi) = p_\pm(\omega)(\xi), \qquad (5.1)$$

and this convergence is uniform in ω in $\mathscr{I}$ and ξ in any compact subset of $\mathbf{R}$.

Condition II($\mathscr{I}$). There is a strictly positive function s and there are regions $\mathscr{R}_\pm(\mathscr{I})$ of the form (3.1) such that for each μ in these regions resolvent kernel of the operator $L(p(\mu))$ satisfies the estimate

$$|R(\mu, L(p(\mu)))(\xi,\eta)| \leqslant 0(1)s(\xi)s(\eta), \qquad (5.2)$$

where the constant 0(1) is uniformly bounded for μ in $\mathscr{R}_\pm(\mathscr{I})$. With reference to this function, these potentials satisfy the estimate

$$\int_{-\infty}^{\infty} \sup_{\mu\in\mathscr{R}_\pm(\mathscr{I})} |p(\xi) - p(\mu)(\xi)|\cdot s^2(\xi)\, d\xi < \infty. \qquad (5.3)$$

Furthermore for each point ω of $\mathscr{I}$ each of the two limit kernels exists,

$$\lim_{\epsilon \to +0} R(\omega \pm i\epsilon, L(p(\omega \pm i\epsilon)))(\xi, \eta) = R_{\pm}(\omega, L(p(\omega)))(\xi, \eta) \tag{5.4}$$

and this convergence is uniform in ω in $\mathscr{I}$ and (ξ, η) in any compact subset of $\mathbf{R} \times \mathbf{R}$.

In the following lemma we use these conditions to formulate conditions ensuring that $L(p(\mu))$ approximates $L(p)$. Recall that in Definition 3.1 this approximation property was stated with reference to a given space $\mathscr{G}$. In the following we define such a space by defining a norm on the space of measurable functions.

Lemma 5.1. Suppose that to the given potential p there is a family of potentials $p(\mu)$ satisfying Conditions $I(\mathscr{I})$ and $II(\mathscr{I})$. For each real function x such that

$$\int_{-\infty}^{\infty} s^2(\xi) \exp\left(-x(\xi) \int_0^{|\xi|} s^2(\sigma)\, d\sigma\right) d\xi < \infty, \tag{5.5}$$

define

$$n(\xi) = \sup_{\mu \in \mathscr{R}_{\pm}(\mathscr{I})} |p(\xi) - p(\mu)(\xi)| + \exp\left(-x(\xi) \int_0^{|\xi|} s^2(\sigma)\, d\sigma\right) \tag{5.6}$$

and

$$\|f\|_{\mathscr{G}} = \left\| \left(\frac{1}{n}\right)^{1/2} f \right\|_{\mathscr{H}}. \tag{5.7}$$

Suppose further, that with reference to such a norm the operator

$$(L(p) - L(p(\mu)))R(\mu, L(p(\mu))) \tag{5.8}$$

satisfies assumptions (3.9) and (3.10). Then over the interval $\mathscr{I}$, with reference to this norm, the family of operators $L(p(\mu))$ approximates $L(p)$.

Proof. It suffices to verify Conditions $G_1(\mathscr{I})$ and $G_2(\mathscr{I})$ under the hypotheses listed above.

Condition $G_1(\mathscr{I})$. (3.11) tells us that $R(\mu, L(p(\mu))) \in \mathscr{B}(\mathscr{H})$. From the definition (5.6) of $\mathscr{G}$ it follows that

$$\mathscr{H} \cap \mathscr{G} = \{f | f \in \mathscr{H} \text{ and } M(n^{1/2})f \in \mathscr{H}\}.$$

Therefore,

$$(R(\mu, L(p(\mu)))f, g) = (M(n^{1/2})R(\mu, L(p(\mu)))M(n^{1/2})M(1/n)^{1/2}f,\ M(1/n)^{1/2}g) \quad \text{on } \mathscr{H} \cap \mathscr{G} \times \mathscr{H} \cap \mathscr{G}.$$

If it is also true that $M(n^{1/2})R(\mu, L(p(\mu)))M(n^{1/2}) \in \mathscr{B}(\mathscr{H})$, then applying Schwarz inequality, we obtain

$$|(R(\mu, L(p(\mu)))f, g)| \leqslant \|M(n^{1/2})R(\mu, L(p(\mu)))M(n^{1/2})\|_{\mathscr{H}} \|f\|_{\mathscr{G}} \|g\|_{\mathscr{G}},$$

which in its turn implies that

$$\sup_{\mu \in \mathscr{R}_{\pm}(\mathscr{I})} \|[R(\mu, L(p(\mu)))]_{\mathscr{G}}\| \leqslant \sup_{\mu \in \mathscr{R}_{\pm}(\mathscr{I})} \|M(n^{1/2})R(\mu, L(p(\mu)))M(n^{1/2})\|_{\mathscr{H}}.$$

Hence $G_1(\mathscr{I})$ is implied by

$$\sup_{\mu\in\mathscr{R}_{\pm}(\mathscr{I})} \|M(n^{1/2})R(\mu, L(p(\mu)))M(n^{1/2})\|_{\mathscr{H}} < \infty. \tag{5.9}$$

To prove this estimate, set

$$X(\mu) = M(n^{1/2})R(\mu, L(p(\mu)))M(n^{1/2}). \tag{5.10}$$

Then the closure of this operator, which we denote by the same symbol, is an integral operator and its kernel is given by

$$X(\mu)(\xi, \eta) = n^{1/2}(\xi)R(\mu, L(p(\mu)))(\xi, \eta)n^{1/2}(\eta). \tag{5.11}$$

Inserting (5.2) in (5.11) we obtain

$$|X(\mu)(\xi, \eta)| \leqslant 0(1)n^{1/2}(\xi)s(\xi)\cdot n^{1/2}(\eta)s(\eta). \tag{5.12}$$

It is worthwhile to note that though the operator $X(u)$ is not, in general, even normal, its kernel however is majorized by the kernel of a symmetric operator (in fact, rank one as we shall see below). Now,

$$\iint_{R\times R} \left|X(\mu)(\xi, \eta)\right|^2 d\xi\, d\eta \leqslant 0(1)\left(\int_{-\infty}^{\infty} n(\xi)s^2(\xi)\, d\xi\right)^2.$$

By virtue of definition (5.6) and relations (5.3), (5.5); the integral on the right is easily seen to be finite. Since the constant 0(1) is uniformly bounded in μ for μ in $\mathscr{R}_{\pm}(\mathscr{I})$ and since the Hilbert–Schmidt norm of an operator majorizes its operator norm, we have established (5.9) or equivalently verified $G_1(\mathscr{I})$.

From the definition (5.7) of $\mathscr{G}$ it is clear that $M(1/n)^{1/2}$ defines an isometry mapping $\mathscr{G}$ onto $\mathscr{H}$ and similarly $M(n^{1/2})$ defines an isometry mapping $\mathscr{H}$ onto $\mathscr{G}$. Denoting these isometries by the same symbol,

$$M\left(\frac{1}{n}\right)^{1/2}\mathscr{G} = \mathscr{H}, \quad M(n^{1/2})\mathscr{H} = \mathscr{G}, \tag{5.13}$$

and $M(1/n)^{1/2}$ and $M(n^{1/2})$ are unitary transformations, being inverses to each other.

We set

$$T(\mu) = (L(p) - L(p(\mu)))R(\mu, L(p(\mu))). \tag{5.14}$$

Then $T(\mu)$ satisfies, by assumption, domain conditions (3.9). We claim that if the closure of the operator $M(1/n)^{1/2}T(\mu)M(n^{1/2})$ is in $B(H)$, then $T(\mu)$ satisfies (3.10) and defines an operator $T(\mu)_{\mathscr{G}}$ in $\mathscr{G}$ and is in $\mathscr{B}(\mathscr{G})$. This is seen as follows:

For all $f \in D(T(\mu)) \cap \mathscr{G}$, which is dense in $\mathscr{G}$

$$\|T(\mu)f\|_{\mathscr{G}} = \left\|M\left(\frac{1}{n}\right)^{1/2}T(\mu)M(n^{1/2})M\left(\frac{1}{n}\right)^{1/2}f\right\|_{\mathscr{H}} \leqslant \left\|M\left(\frac{1}{n}\right)^{1/2}T(\mu)M(n^{1/2})\right\|_{\mathscr{H}}\|f\|_{\mathscr{G}}.$$

Therefore, $T(\mu)$ maps $D(T(\mu)) \cap \mathscr{G}$ into $\mathscr{G}$ verifying (3.10) and also establishing

$$\|T(\mu)\|_{\mathscr{G}} \leqslant \left\|M\left(\frac{1}{n}\right)^{1/2}T(\mu)M(n^{1/2})\right\|_{\mathscr{H}}.$$

Hence in order to verify the first half of $G_2(\mathscr{I})$ it suffices to establish

$$\sup_{\mu\in\mathscr{R}_{\pm}(\mathscr{I})}\left\|M\left(\frac{1}{n}\right)^{1/2}T(\mu)M(n^{1/2})\right\|_{\mathscr{H}} < \infty. \tag{5.15}$$

Denoting by

$$Y(\mu) = M\left(\frac{1}{n}\right)^{1/2} T(\mu) M(n^{1/2}), \tag{5.16}$$

we see that $Y(\mu)$ is an integral operator given by its kernel

$$Y(\mu)(\xi, \eta) = \left(\frac{n(\eta)}{n(\xi)}\right)^{1/2} (p(\xi) - p(\mu)(\xi)) R(\mu, p(\mu))(\xi, \eta). \tag{5.17}$$

Definition (5.6) clearly implies that

$$\frac{1}{n(\xi)} |(p(\xi) - p(\mu)(\xi))| \leqslant n(\xi)^{1/2},$$

hence

$$| Y(\mu)(\xi, \eta)| \leqslant |X(\mu)(\xi, \eta)| \tag{5.18}$$

and therefore the previous estimate on $X(\mu)$ enables us to reach the required conclusion.

As for the second half of $G_2(\mathscr{I})$, we need to observe first that the kernel $Y(\mu)(\xi, \eta)$ is a continuous function of μ in $\mathscr{R}_{\pm}(\mathscr{I})$ and that

$$| Y(\mu)(\xi, \eta)| \leqslant 0(1) n^{1/2}(\xi) s(\xi) \cdot n^{1/2}(\eta) s(\eta) \tag{5.19}$$

which is $L_2(\mathbf{R} \times \mathbf{R})$ as observed before. By Lebesque dominated convergence, $Y(\mu)$ depend continuously in Hilbert–Schmidt and therefore, also in operator-norm on μ for μ in $\mathscr{R}_{\pm}(\mathscr{I})$ and the same conclusion follows for $T(\mu)_{\mathscr{G}}$.

Properties (5.1), (5.4) and definition (5.16) implies

$$\lim_{\mu_1 \to \omega} \lim_{\mu_2 \to \omega} | Y(\mu_1)(\xi, \eta) - Y(\mu_2)(\xi, \eta)| = 0 \tag{5.20}$$

and this limit is uniform in ω in $\mathscr{I}$ and (ξ, η) in any compact subset of $\mathbf{R} \times \mathbf{R}$.

This along with the fact that $Y(\mu)(\xi, \eta)$ is uniformly majorized by a function in $L_2(\mathbf{R} \times \mathbf{R})$ as in (5.19) helps us conclude that

$$\lim_{\mu_1 \to \omega} \lim_{\mu_2 \to \omega} \| Y(\mu_1) - Y(\mu_2)\| = 0, \tag{5.21}$$

uniformly in ω in $\mathscr{I}$. This implies that $T(\mu)_{\mathscr{G}}$ admits norm-continuous extensions onto the closures of $\mathscr{R}_{\pm}(\mathscr{I})$.

6. The Family of Operators $L(q(\mu))$

In this section we want to verify all the hypotheses of Lemma 5.1 so that the conclusion of Lemma 5.1 enables us to conclude that $L(q(\mu))$ constitutes an approximating family of the given operator $L(p)$.

In this direction, we want to show that the basic part of Weyl construction can be carried out to obtain the resolvent kernel of $L(q(\mu))$, though the operator $L(q(\mu))$ is not even symmetric. In fact we have already seen in Section 4 that functions $k^l(\mu)$ and $k^r(\mu)$ satisfy the equation (4.11). Also by construction,

$$k^l(\mu) \in L_2(-\infty, -1) \quad \text{and} \quad k^r(\mu) \in L_2(1, \infty). \tag{6.1}$$

That their Wronskian is non-zero is shown in

Lemma 6.1. To every given compact interval $\mathscr{I}$, there exists a region $\mathscr{R}_\pm(\mathscr{I})$ such that for every complex μ in the closures of $\mathscr{R}_\pm(\mathscr{I})$, the Wronskian

$$W(k^l(\mu), k^r(\mu)) \neq 0. \tag{6.2}$$

Proof. An inspection of solutions $k^l(\mu)$ and $k^r(\mu)$ convinces us that it is easiest to compute the Wronskian in the turning interval I^m. Doing that, the Wronskian turns out to be

$$W(k^l(\mu), k^r(\mu)) = w^+(\mu)(a) - w^-(\mu)(b) - (b - a)w^+(\mu)(a)w^-(\mu)(b). \tag{6.3}$$

We have seen in (4.2) and (4.3) that

$$p_2(a) - \omega > 0 \quad \text{and} \quad p_2(b) - \omega < 0.$$

Therefore, recalling definition (4.7)

$$\operatorname{Im} w^+(\omega)(a) = 0 \quad \text{and} \quad \operatorname{Im} w^-(\omega)(b) \neq 0,$$

from which it follows that

$$\operatorname{Im} \frac{1}{w^+(\omega)(a)} \neq \operatorname{Im} \frac{1}{w^-(\omega)(b)}. \tag{6.4}$$

Then

$$\operatorname{Im}\left\{\frac{W(k^l(\omega), k^r(\omega))}{w^+(\omega)(a)w^-(\omega)(b)}\right\} = \operatorname{Im}\left(\frac{1}{w^-(\omega)(b)} - \frac{1}{w^+(\omega)(a)}\right) \neq 0$$

by virtue of (6.4) and hence the Wronskian is non-zero for all $\omega \in \mathscr{I}$.

On the other hand the Wronskian is a continuous function of μ, where $\omega = \operatorname{Re} \mu \in \mathscr{I}$ and therefore by continuity, we can find the angle α in definition (3.1) small enough so that W remain non-vanishing for all $\mu \in \mathscr{R}_\pm(\mathscr{I})$, where $\mathscr{R}_\pm(\mathscr{I})$ is constructed with the angle α so determined. In the sequel, by $\mathscr{R}_\pm(\mathscr{I})$ we shall mean the region $\mathscr{R}_\pm(\mathscr{I})$ determined in Lemma 6.1, so that (6.2) is valid for all $\mu \in \mathscr{R}_\pm(\mathscr{I})$.

Having established (6.2), following the Weyl construction we define a kernel $K(\mu)(\xi, n)$ by

$$K(\mu)(\xi, \eta) = \frac{1}{W(k^l(\mu), k^r(\mu))}\begin{cases} k^l(\mu)(\xi)k^r(\mu)(\eta), & \xi \leqslant \eta \\ k^l(\mu)(\eta)k^r(\mu)(\xi), & \xi \geqslant \eta \end{cases} \tag{6.5}$$

where $\mu \in \mathscr{R}_\pm(\mathscr{I})$.

First we claim that the corresponding operator, that is the operator defined by

$$K(\mu)f(\xi) = \int K(\mu)(\xi, \eta)f(\eta)\, d\eta, \qquad f \in L_2(\mathbf{R})$$

is bounded. According to a result of Schur–Holmgren–Carleman [17], this is implied by the following

Lemma 6.2. Setting

$$t(\mu)(\eta) = \begin{cases} 1 & \eta \in I^m \\ |p_2(\eta) - \mu|^{-1/4} & \eta \notin I^m \end{cases}$$

one has

$$\left(\sup_{\xi} \frac{1}{t(\mu)(\xi)} \int |K(\mu)(\xi, \eta)| t(\mu)(\eta)\, d\eta\right) \cdot \left(\sup_{\eta} \frac{1}{t(\mu)(\eta)} \int |K(\mu)(\xi, \eta)| t(\mu)(\xi)\, d\xi\right) < \infty. \tag{6.6}$$

Proof. We introduce a non-negative function $v(\mu)$ by

$$v(\mu)(\xi) = \begin{cases} \operatorname{Re} \int_a^\xi \sqrt{p_2(\sigma) - \mu}\, d\sigma, & \xi \in I^l \\ \xi - a, & \xi \in I^m \\ \operatorname{Re} \int_b^\xi \sqrt{p_2(\sigma) - \mu}\, d\sigma + b - a, & \xi \in I^r. \end{cases} \tag{6.7}$$

From the definition of $k^l(\mu)$ and $t(\mu)$, it follows that

$$|k^l(\mu)(\xi)| \leqslant C(\mu) t(\mu)(\xi) \exp(v(\mu)(\xi)), \tag{6.8}$$

where

$$C(\mu) = \max\{|p_2(a) - \mu|^{1/4}, |\alpha^l| + |\beta^l| b, (|\gamma^+| + |\gamma^-|)|p_2(b) - \mu|^{1/4}\}. \tag{6.9}$$

Similarly,

$$|k^r(\mu)(\eta)| \leqslant D(\mu) t(\mu)(\eta) \exp(-v(\mu)(\eta)) \tag{6.10}$$

where

$$D(\mu) = \max\{|p_2(a) - \mu|^{1/4}(|\delta^+| + |\delta^-|), (|\alpha^r| + |\beta^r| b) e^{b-a}, |p_2(b) - \mu|^{1/4} e^{b-a}\}. \tag{6.11}$$

Inserting estimates (6.8), (6.10) in definition (6.5) and remembering that $W(k^l(\mu), k^r(\mu)) \neq 0$ and that $v(\mu)$ is a non-decreasing function we obtain that to each non-real complex number μ in $\mathscr{R}_\pm(\mathscr{I})$ there is a constant $0(1)$ such that for every (ξ, η) in $\mathbf{R} \times \mathbf{R}$,

$$|K(\mu)(\xi, \eta)| = 0(1) t(\mu)(\xi) t(\mu)(\eta) \exp(-|v(\mu)(\xi) - v(\mu)(\eta)|). \tag{6.12}$$

Therefore,

$$\frac{1}{t(\mu)(\xi)} |K(\mu)(\xi, \eta)| t(\mu)(\eta) = 0(1) t(\mu)^2(\eta) \exp(-|v(\mu)(\xi) - v(\mu)(\eta)|).$$

By virtue of a Lemma formulated elsewhere [18], one verifies that $t^2(\mu)(\eta) = 0(1) v(\mu)'(\eta)$ and one obtains

$$\begin{aligned} &\frac{1}{t(\mu)(\xi)} \int |K(\mu)(\xi, \eta)| t(\mu)(\eta)\, d\eta \\ &\quad = 0(1) \int \exp(-|v(\mu)(\xi) - v(\mu)(\eta)|) v(\mu)'(\eta)\, d\eta \\ &\quad = 0(1)[2 - \exp(-|v(\mu)(\infty) - v(\mu)(\xi)|) - \exp(-|v(\mu)(\xi) - v(\mu)(-\infty)|)] \end{aligned}$$

and this leads to

$$\sup_{\xi} \left(\frac{1}{t(\mu)(\xi)} \int |K(\mu)(\xi, \eta)| t(\mu)(\eta)\, d\eta\right) < \infty.$$

Interchanging the variables ξ and η in the above chain of estimates, we arrive at

$$\sup_{\eta}\left(\frac{1}{t(\mu)(\eta)}\int |K(\mu)(\xi,\eta)|t(\mu)(\xi)\,d\xi\right) < \infty$$

and together these two estimates imply (6.6).

Now we state and prove the main theorem of this section which verifies the first part of definition (3.1).

Theorem 6.1. Let μ be a non-real complex number in $\mathscr{R}_{\pm}(\mathscr{I})$ and let $q(\mu)$ be the potential defined by equation (4.4) and $L(q(\mu))$ be the operator given by equation (2.10). Then μ is in the resolvent set of $L(q(\mu))$, i.e.

$$\mu \in \rho(L(q(\mu))). \tag{6.13}$$

Proof. We have already seen in Lemma 6.2 that the integral operator $K(\mu)$ is bounded and defined everywhere. Therefore, to arrive at the conclusion of the theorem, it suffices to show that the integral operator $K(\mu)$ is the inverse of the operator $(\mu I - L(q(\mu)))$, i.e.

$$(\mu I - L(q(\mu)))K(\mu) = I \qquad \text{on } L_2(\mathbf{R}) \tag{6.14}$$

and

$$K(\mu)(\mu I - L(q(\mu))) = I \qquad \text{on } D(L(q(\mu))). \tag{6.15}$$

From definition (6.5) it follows that for all $f \in L_2(\mathbf{R})$,

$$K(\mu)f(\xi) = \frac{1}{W}\left(k^r(\mu)(\xi)\int_{-\infty}^{\xi} k^l(\mu)(\eta)f(\eta)\,d\eta + k^l(\mu)(\xi)\int_{\xi}^{\infty} k^r(\mu)(\eta)f(\eta)\,d\eta\right). \tag{6.16}$$

Since $k^l(\mu)$ and $k^r(\mu)$ are continuous and square-integrable at $-\infty$ and $+\infty$ respectively, it is clear that functions $k^l(\mu)\cdot f$ and $k^r(\mu)\cdot f$ belong to $L_1(-\infty,\xi)$ and $L_1(\xi,\infty)$ respectively. Therefore the two integrals in (6.16) define locally absolutely continuous functions of ξ [10]. Also $k^l(\mu)$ and $k^r(\mu)$ are continuously differentiable as is evident from $(4.10)_{l,r}$ and hence locally absolutely continuous, establishing the result that $K(\mu)f(\xi)$ is locally absolutely continuous. Differentiating (6.16) with respect to ξ, we obtain

$$K(\mu)f'(\xi) = \frac{1}{W}\left[k^r(\mu)'(\xi)\int_{-\infty}^{\xi} k^l(\mu)(\eta)f(\eta)\,d\eta + k^l(\mu)'(\xi)\int_{\xi}^{\infty} k^r(\mu)(\eta)f(\eta)\,d\eta\right]. \tag{6.17}$$

Similarly, observing that $k^r(\mu)$ and $k^l(\mu)$ are piecewise twice continuously differentiable by virtue of the differential equation (4.11) that they satisfy and the fact that $q(\mu)$ is piecewise continuous, we conclude that $K(\mu)f'(\xi)$ is also locally absolutely continuous.

Now we are in a position to compute

$$\begin{aligned}
&-K(\mu)f''(\xi) + q(\mu)(\xi)K(\mu)f(\xi)\\
&= \frac{1}{W}\left[\{-k^r(\mu)''(\xi) + q(\mu)(\xi)k^r(\mu)(\xi)\}\int_{-\infty}^{\xi} k^l(\mu)(\eta)f(\eta)\,d\eta\right.\\
&\qquad \left. + \{-k^l(\mu)''(\xi) + q(\mu)(\xi)k^l(\mu)(\xi)\}\int_{\xi}^{\infty} k^r(\mu)(\eta)f(\eta)\,d\eta - Wf(\xi\right]\\
&= \mu K(\mu)f(\xi) - f(\xi) \in L_2(\mathbf{R})
\end{aligned}$$

and conclude that $K(\mu)f \in D(L(q(\mu)))$ for all $f \in L_2(\mathbf{R})$ and then (6.14) follows from the above equation.

In order to obtain the second relation (6.15) it suffices to show that $\mu I - L(q(\mu))$ is one-to-one. Let $f \in D(L(q(\mu)))$ and $(\mu I - L(q(\mu)))f = 0$. Then from the definition of $D(L(q(\mu)))$, it follows that $f(\xi)$ satisfies the differential equation (4.11).

Since it is a second order differential equation and we already know that $k^l(\mu)$ and $k^r(\mu)$ are two linearly independent solutions, there exists two constants Γ^r and Γ^l such that

$$f = \Gamma^l k^l(\mu) + \Gamma^r k^r(\mu).$$

We have seen in (6.1) that $k^l(\mu) \in L_2(-\infty, -1)$ and $k^r(\mu) \in L_2(1, \infty)$. It has been shown elsewhere [18] that by virtue of condition (2.2) $k^l(\mu)$ and $k^r(\mu)$ are not square-integrable at $+\infty$ and $-\infty$ respectively. Therefore, though $f \in L_2(R)$, both $k^l(\mu)$ and $k^r(\mu)$ do not belong to $L_2(R)$, leading us to conclude that $\Gamma^l = \Gamma^r = 0$, i.e.

$$f = 0,$$

or equivalently we have shown that $(\mu I - L(q(\mu)))$ is one-to-one. This completes the proof of the theorem 6.1.

Relation (5.1) for $q(\mu)$ follows from definition (4.4) of $q(\mu)(\xi)$ and the fact that it is a piecewise continuous function of ξ and μ. This observation and the conclusion of Theorem 6.1 verifies Condition I($\mathscr{I}$).

In (6.12) observing the fact that $v(\mu)$ is a non-decreasing function, we obtain

$$|K(\mu)(\xi, \eta)| \leqslant \left| \frac{C(\mu)D(\mu)}{W(k^l(\mu), k^r(\mu))} \right| t(\mu)(\xi)t(\mu)(\eta), \tag{6.18}$$

where constants $C(\mu)$ and $D(\mu)$ are given in (6.10), (6.12).

Recalling from Section 4 that the construction of the turning interval I^m ensures that all the constants α, β, γ, δ etc. appearing in the expression (6.9) and (6.11) are continuous functions of μ and are uniformly bounded for μ real in $\mathscr{I}^0$. Therefore it follows that $C(\mu)$ and $D(\mu)$ are uniformly bounded for μ in $\mathscr{R}_\pm(\mathscr{I})$.

Now setting,

$$s(\xi) = \sup_{\mu \in \mathscr{R}_\pm(\mathscr{I})} t(\mu)(\xi) \tag{6.19}$$

we get the desired relation (5.2).

From the proof of Lemma 6.1 and definition 6.5 it is clear that the Wronskian $W(k^l(\mu), k^r(\mu))$ is a continuous function of μ while $k^l(\mu)$ and $k^r(\mu)$ are continuous functions of μ and their arguments. Insertion of this observation in (6.5) yields (5.4) and uniform convergence.

As a consequence of conditions (2.1) and the construction of the turning interval I^m, we observe that there exists a constant χ such that

$$s^2(\xi) \leqslant \chi \inf_{\mu \in \mathscr{R}_\pm(\mathscr{I})} |1/p_2(\xi) - \mu|^{1/4}, \qquad \xi \notin I^m. \tag{6.20}$$

Since the supremum of a product majorizes the infimum of the first factor times the supremum of the second, we have that

$$\inf_{\mu \in \mathscr{R}_\pm(\mathscr{I})} \left| \frac{1}{p_2(\xi) - \mu} \right|^{1/2} \cdot \sup_{\mu \in \mathscr{R}_\pm(\mathscr{I})} |(p - q(\mu))(\xi)| \leqslant \sup_{\mu \in \mathscr{R}_\pm(\mathscr{I})} \frac{|(p - q(\mu)(\xi))|}{|p_2(\xi) - \mu|^{1/2}}, \tag{6.21}$$

whenever $\xi \notin I^m$.

Since p_1 is a shortrange potential, we observe that

$$\int_{-\infty}^{a} |p_1(\xi)|s^2(\xi)\,d\xi < \infty, \quad \int_{b}^{\infty} |p_1(\xi)|s^2(\xi)\,d\xi < \infty. \tag{6.22}$$

Similarly, definition (4.4) and property (2.4) along with the fact that in neighbourhoods of $\pm\infty$, $|p_2(\xi)/(p_2(\xi) - \mu)|$ is close to 1 show that

$$\int_{-\infty}^{a} \sup_{\mu\in\mathscr{R}_{\pm}(\mathscr{I})} |(p_2 - q(\mu))(\xi)| \frac{d\xi}{|p_2(\xi) - \mu)^{1/2}} < \infty \tag{6.23}$$

and

$$\int_{b}^{\infty} \sup_{\mu\in\mathscr{R}_{\pm}(\mathscr{I})} |(p_2 - q(\mu))(\xi)| \frac{d\xi}{|p_2(\xi) - \mu|^{1/2}} < \infty.$$

On the other hand in the turning interval $[a, b]$, $s(\xi) = 1$ and since p_2 and $q(\mu)$ are continuous functions,

$$\int_{b}^{a} \sup_{\mu\in\mathscr{R}_{\pm}(\mathscr{I})} |p_2(\xi) - q(\mu)(\xi)|s^2(\xi)\,d\xi < \infty. \tag{6.24}$$

This combining with (6.22) and (6.23) yields (5.3), completing the verification of Condition II($\mathscr{I}$).

Now we are left with the task of verifying assumptions (3.9) and (3.10). First we note that under the simplifying assumption Condition S, $T(\mu)$ is a bounded (in fact Hilbert–Schmidt) operator in $\mathscr{H}$ and therefore (3.9) follows.

To prove (3.9) under Condition B, we choose a function x so that in addition to relation (5.5)

$$\lim_{|\xi|\to\infty} x(\xi) = 0. \tag{6.25}$$

That such an x can be found is seen as follows. Since

$$\int_{-\infty}^{\infty} s^2(\xi) \exp\left(-\int_{0}^{|\xi|} s^2(\sigma)\,d\sigma\right) d\xi = \Phi < \infty,$$

we can find positive numbers ξ_n $(n = 1, 2, 3, \ldots)$ so that

$$\int_{\xi_n}^{\infty} d\xi s^2(\xi) \exp\left(-\frac{1}{2^n}\int_{0}^{\xi} s^2(\sigma)\,d\sigma\right) = \frac{1}{2^n}\Phi. \tag{6.26}$$

Clearly ξ_n tends to $+\infty$ as n increases. Now, define

$$x(\xi) = \frac{1}{2^n}, \tag{6.27}$$

for $\xi_{n-1} \leqslant \xi < \xi_n$, with $\xi_0 = 0$ and $x(-\xi) = x(\xi)$.

Then it is easy to see that $x(\xi)$ thus constructed satisfy both (5.5) and (6.25).

Next for every non-real $\mu \in \mathscr{R}_{\pm}(\mathscr{I})$ it suffices to exhibit a set $S(\mu)$ such that

$$S(\mu) \text{ is a dense subset of } \mathscr{G} \tag{6.28}$$

and

$$S(\mu) \subset D(T(\mu)). \tag{6.29}$$

To this end, define

$$N(\mu) = \left\{f \,\middle|\, f \in \dot{C}(\mathbf{R}), \int_{-\infty}^{\infty} k^{l,r}(\mu)(\eta)n^{1/2}(\eta)f(\eta)\,d\eta = 0\right\} \tag{6.30}$$

where $\dot{C}(\mathbf{R})$ denotes the space of continuous functions with bounded support. Then set

$$S(\mu) = M(n^{1/2})N(\mu). \tag{6.31}$$

Noting from (5.6) that $n^{1/2} \in L_{2,\mathrm{loc}}(\mathbf{R})$ and from (5.13) that $M(n^{1/2})\mathscr{H} = \mathscr{G}$, we conclude that

$$S(\mu) \subset \mathscr{H} \cap \mathscr{G}. \tag{6.32}$$

By virtue of (5.13) the denseness of $N(\mu)$ in $\mathscr{H}$ implies the denseness of $S(\mu)$ in $\mathscr{G}$. For this, define

$$N^l(\mu) = \left\{f \in \dot{C}(\mathbf{R}) \,\middle|\, \int_{-\infty}^{\infty} k^l(\mu)(\eta)n^{1/2}(\eta)f(\eta)\,d\eta = 0\right\}. \tag{6.33}$$

From the properties of the function x it is not difficult to see that $k^l(\mu)n^{1/2} \notin \mathscr{H}$. Therefore the linear functional in (6.33) is unbounded on the domain $\dot{C}(\mathbf{R})$ and from [22], we conclude that its null-space $N^l(\mu)$ is dense in $\mathscr{H}$. Next, on this set $N^l(\mu)$, define the functional ϕ by

$$\phi(f) = \int k^r(\mu)(\eta)n^{1/2}(\eta)f(\eta)\,d\eta, \qquad f \in N^l(\mu). \tag{6.34}$$

Then clearly $N(\mu)$ is the nullspace of this functional ϕ. Another application of the same argument as above yields that $N(\mu)$ is dense in $N^l(\mu)$. Since the topology involved is always the same, that is the topology of $L_2(\mathbf{R})$, $N(\mu)$ is also dense in $\mathscr{H}$. To prove (6.29), we recall formula (6.5). This shows that

$$R(\mu, L(q(\mu)))S(\mu) \subset \dot{C}(\mathbf{R}).$$

It is also clear from Condition B and definition (4.4) that

$$p - q(\mu) \in L_{2,\mathrm{loc}}(\mathbf{R}).$$

These two together imply (6.29).

7. Proof of Condition $A_1(\mathscr{I})$

It is clear from the last section that $Y(\mu)$ is compact in $\mathscr{H}$ and so is $T(\mu)_{\mathscr{G}}$ in $\mathscr{G}$. It is also true then that the two limit operators $T_\pm(\omega)_{\mathscr{G}}$ are also compact. Therefore, $(I - T_\pm(\omega))_{\mathscr{G}}$ is invertible if it is one-to-one [19]. This one-to-one property follows from

Lemma 7.1. Let ω in $\mathscr{I}$ be an exceptional point and h in $\mathscr{G}$ be the corresponding exceptional vector, or equivalently

$$(I - T_+(\omega))_{\mathscr{G}}h = 0 \quad \text{or} \quad (I - T_-(\omega))_{\mathscr{G}}h = 0. \tag*{(7.1)$_\pm$}$$

Then

$$h = 0. \tag{7.2}$$

In order to prove Lemma 7.1 we use the following abstract result, the proof of which is given elsewhere [9]. Before giving the statement, we give the

Condition $G_3(\mathscr{I})$. The family of approximating operators can be extended to the closures $\mathscr{R}_\pm(\mathscr{I})$ so that

$$(A - A_0(\mu))^* = (A - A_0(\bar{\mu})), \tag{7.3}$$

in particular

$$(A - A_0(\mathrm{Re}))^* = (A - A_0(\mathrm{Re}\,\mu)).$$

The family of operators

$$F(\mu) = (A - A_0(\mathrm{Re}\,\mu))R(\mu, A_0(\mu)) \tag{7.4}$$

satisfies assumptions (3.9) and (3.10). Furthermore, for ω in $\mathscr{I}$,

$$\lim_{\mu \to \omega} \|F(\mu) - T(\mu)\|_{\mathscr{G}} = 0. \tag{7.5}$$

Proposition 7.1. Let the approximating family $A_0(\mu)$ satisfy Condition $G_3(\mathscr{I})$ and let the form $[R(\mu, A_0(\mu))]_{\mathscr{G}}$ admit continuous extension onto closures $\mathscr{R}_\pm(\mathscr{I})$. Suppose that the point ω in $\mathscr{I}$ and the vector h in $\mathscr{G}$ are such that

$$(I - T_+(\omega))_{\mathscr{G}} h = 0 \quad \text{or} \quad (I - T_-(\omega))_{\mathscr{G}} h = 0.$$

Then the jump of the form $[R(\mu, A_0(\mu\ )]_{\mathscr{G}}$ at this exceptional point and at this exceptional vector is zero. That is to say,

$$[R_+(\omega, A_0(\omega))]_{\mathscr{G}}(h, h) - [R_-(\omega, A_0(\omega))]_{\mathscr{G}}(h, h) = 0. \tag{7.6}$$

Verification of $G_3(\mathscr{I})$. It is easy to see from the definition (4.4) that $q(\bar{\mu}) = \overline{q(\mu)}$ and also this function can be continuously extended to the closures $\mathscr{R}_\pm(\mathscr{I})$. In particular $q(\omega)$ is real for ω in $\mathscr{I}$. Next we have to verify that

$$\lim_{\mu \to \omega} \|F(\mu)_{\mathscr{G}} - T(\mu)_{\mathscr{G}}\| = 0 \tag{7.7}$$

where

$$F(\mu) = (L(p) - L(q(\mathrm{Re}\,\mu)))R(\mu, L(q(\mu))).$$

Similarly as in Section 5, we note that

$$F(\mu)_{\mathscr{G}} - T(\mu)_{\mathscr{G}} \text{ is unitarily equivalent to } M\left(\frac{1}{n}\right)^{1/2}(F(\mu) - T(\mu))M(n^{1/2}). \tag{7.8}$$

Setting

$$Z(\mu) = M\left(\frac{1}{n}\right)^{1/2}(F(\mu) - T(\mu))M(n^{1/2}), \tag{7.9}$$

we note that this is an integral operator with kernel

$$Z(\mu)(\xi, \eta) = \left(\frac{n(\eta)}{n(\xi)}\right)^{1/2}(q(\mu)(\xi) - q(\mathrm{Re}\,\mu)(\xi))R(\mu, L(q(\mu)))(\xi, \eta). \tag{7.10}$$

As seen in Section 6, $R(\mu, L(q(\mu)))(\xi, \eta)$ converges as $\mu \to \omega$ uniformly in ω in $\mathscr{I}$

and (ξ, η) in compact of $\mathbf{R} \times \mathbf{R}$ and so does $q(\mu)(\xi)$, uniformly in ω in $\mathscr{I}$ and ξ in compact of $\mathbf{R}$. Therefore

$$\lim_{\mu \to \omega} |Z(\mu)(\xi, \eta)| = 0,$$

uniformly in ω in $\mathscr{I}$ and ξ, η in any compact subset of $\mathbf{R} \times \mathbf{R}$. Also, since

$$Z(\mu)(\xi, \eta) = Y(\operatorname{Re} \mu)(\xi, \eta) - Y(\mu)(\xi, \eta)$$

it follows from (5.18) that

$$|Z(\mu)(\xi, \eta)| \leqslant 2|X(\mu)(\xi, \eta)|$$

and therefore calculations similar to those in Section 5 yield

$$\lim_{\mu \to \omega} \|Z(\mu)\|_{\text{H.S.}} = 0,$$

which imply (7.5).

Proof of Lemma 7.1. Firstly, we note that $h \in \mathscr{G}$ implies

$$\int_{-\infty}^{\infty} \frac{1}{n(\eta)} |h(\eta)|^2 \, d\eta < \infty \tag{7.11}$$

and therefore

$$\int_{-\infty}^{\infty} s(\eta)|h(\eta)| \, d\eta = \int_{-\infty}^{\infty} s(\eta) n^{1/2}(\eta) \cdot n^{-1/2}(\eta)|h(\eta)| \, d\eta$$
$$\leqslant \left(\int_{-\infty}^{\infty} s^2(\eta) n(\eta) \, d\eta\right)^{1/2} \left(\int_{-\infty}^{\infty} \frac{1}{n(\eta)} |h(\eta)|^2 \, d\eta\right)^{1/2} < \infty$$

where we have used Schwarz inequality and (5.4).

Therefore, for each fixed $\xi \in \mathbf{R}$, the family of functions $R(\mu, L(q(\mu)))(\xi, \eta)h(\eta)$ admits an integrable majorant, namely $s(\eta)|h(\eta)|$. Hence the following limits do exist,

$$g_{\pm}(\omega)(\xi) = \lim_{\epsilon \to +0} R(\omega \pm i\epsilon, L(q(\omega \pm i\epsilon)))h(\xi).$$

At the same time it follows from (6.5) that

$$g_{\pm}(\omega)(\xi) = \frac{k^r_{\pm}(\omega)(\xi)}{W(k^l(\omega), k^r_{\pm}(\omega))} \int_{-\infty}^{\xi} k^l(\omega)(\eta) h(\eta) \, d\eta$$
$$+ \frac{k^l(\omega)(\xi)}{W(k^l(\omega), k^r_{\pm}(\omega))} \int_{\xi}^{\infty} k^r_{\pm}(\omega)(\eta) h(\eta) \, d\eta. \tag{7.12}$$

Here we have also used that

$$k^l_+(\omega) = k^l_-(\omega) = k^l(\omega)$$

and that this function is real. This follows from the fact that according to definition (4.8) at the point a for each μ the functions $k^l(\mu)$ and $k^l(\mu)'$ are real and from the fact that both $k^l_{\pm}(\omega)$ satisfy the same differential equation with real coefficients; viz. (4.10), ω replacing μ.

It is worthwhile to note that the above relations are valid whether ω is an exceptional point or not.

The hypothesis $(7.1)_\pm$ allows us to apply proposition 7.1 to the exceptional point ω and the exceptional vector h. This yields

$$[R_+(\omega, L(q(\omega)))]_{\mathscr{G}}(h, h) - [R_-(\omega, L(q(\omega)))]_{\mathscr{G}}(h, h) = 0$$

or equivalently

$$\int_{-\infty}^{\infty} \overline{h(\xi)}(g_+(\omega)(\xi) - g_-(\omega)(\xi))\, d\xi = 0. \tag{7.13}$$

According to (7.12)

$$g_+(\omega)(\xi) - g_-(\omega)(\xi) = \left[\frac{k^r(\omega)}{W(k^l(\omega), k^r(\omega))}\right]_-^+(\xi) \int_{-\infty}^{\xi} k^l(\omega)(\eta)h(\eta)\, d\eta$$
$$+ k^l(\omega)(\xi) \int_{\xi}^{\infty} \left[\frac{k^r(\omega)}{W(k^l(\omega), k^r(\omega))}\right]_-^+(\eta)h(\eta)\, d\eta. \tag{7.14}$$

Next we compute

$$\frac{\left[\dfrac{k^r(\omega)}{W(k^l(\omega), k^r(\omega))}\right]_-^{+\prime}(a)}{\left[\dfrac{k^r(\omega)}{W(k^l(\omega), k^r(\omega))}\right]_-^{+}(a)} = \lim_{\epsilon\to+0} \frac{\left(\dfrac{W(\bar{\mu})k^r(\mu)'(a) - W(\mu)k^r(\bar{\mu})'(a)}{|W(\mu)|^2}\right)}{\left(\dfrac{W(\bar{\mu})k^r(\mu)(a) - W(\mu)k^r(\bar{\mu})(a)}{|W(\mu)|^2}\right)}$$

where we have used $\mu = \omega + i\epsilon$ and $W(\mu)$ for the Wronskian $W(k^l(\mu), k^r(\mu))$. Evaluating $W(\mu)$ at the point $\xi = a$ and substituting, we get that

$$\frac{[k^r(\omega)/W(\omega)]_-^{\pm\prime}(a)}{[k^r(\omega)/W(\omega)]_-^{\pm}(a)} = \frac{k^l(\omega)'(a)(W(k^r_+(\omega), k^r_-(\omega))/|W(\omega)|^2)}{k^l(\omega)(a)(W(k^r_+(\omega), k^r_-(\omega))/|W(\omega)|^2)} = \frac{k^l(\omega)'(a)}{k^l(\omega)(a)} \tag{7.15}$$

where we have used the property that $k^l(\omega)$ and $k^l(\omega)'$ are real functions. Therefore the function $[k^r(\omega)/W(\omega)]_-^{\pm}(\xi)$ satisfies the same condition at $\xi = a$, as does the function $k^l(\omega)(\xi)$ and of course, satisfies the same differential equation. Thus there exists a complex constant $\theta(\omega)$ such that

$$\left[\frac{k^r(\omega)}{W(\omega)}\right]_-^+(\xi) = \theta(\omega)k^l(\omega)(\xi). \tag{7.16}$$

In fact it is clear from (7.15) that

$$\theta(\omega) = \frac{W(k^r_+(\omega), k^r_-(\omega))}{|W(\omega)|^2}, \tag{7.17}$$

which is non-zero by virtue of the fact $k^r_+(\omega)(\xi)$ and $k^r_-(\omega)(\xi)$ has different asymptotic properties at $\xi \to +\infty$.

Inserting (7.16) in (7.14) and computing the L.H.S. of (7.13), we conclude that

$$\int_{-\infty}^{\infty} k^l(\omega)(\eta)h(\eta)\, d\eta = 0. \tag{7.18}$$

Going back to relation (7.14), we obtain

$$g_+(\omega)(\xi) = \int_{\xi}^{\infty} \frac{k^l(\omega)(\xi)k^r_+(\omega)(\eta) - k^l(\omega)(\eta)k^r_+(\omega)(\xi)}{W(k^l(\omega), k^r_+(\omega))} h(\eta)\, d\eta. \tag{7.19}$$

From definition (6.7), estimates (6.8) and (6.10), it is seen that

$$\lim_{\mu \to \omega \pm i0} v(\mu)(\xi) = b - a,$$

for all ξ in a neighbourhood of $+\infty$, i.e. for $\xi > \xi_0$, ξ_0 being a large positive number. Therefore for $\eta \geqslant \xi > \xi_0$

$$\left| \frac{k^l(\omega)(\xi) k^r_+(\omega)(\eta)}{W(k^l(\omega), k^r_+(\omega))} \right| \leqslant 0(1) t(\omega)(\xi) t(\omega)(\eta). \tag{7.20}$$

Inserting this estimate in (7.19) and utilizing (7.11), we obtain

$$\sup_{\xi > \xi_0} \left| \frac{g_+(\omega)(\xi)}{t(\omega)(\xi)} \right| < \infty. \tag{7.21}$$

Under hypothesis $(7.1)_+$ and definition of g_+, it follows that

$$h = T_+(\omega)_{\mathscr{G}} h = (p - g(\omega)) g_+(\omega). \tag{7.22}$$

This along with (7.19) yields

$$g_+(\omega)(\xi) = \int_\xi^\infty \frac{k^l(\omega)(\xi) k^r_+(\omega)(\eta) - k^l(\omega)(\eta) k^r_+(\omega)(\xi)}{W(k^l(\omega), k^r_+(\omega))} (p - q(\omega))(\eta) g_+(\omega)(\eta)\, d\eta$$

and using (7.20), we arrive at

$$\left| \frac{g_+(\omega)(\xi)}{t(\omega)(\xi)} \right| \leqslant 0(1) \int_\xi^\infty |(p - g(\omega))(\eta)| t(\omega)^2(\eta) \left| \frac{g_+(\omega)(\eta)}{t(\omega)(\eta)} \right| d\eta$$

where $\xi > \xi_0$.

The function $|(p - q(\omega))(\eta)| t(\omega)^2(\eta)$ is known to be integrable from relations (6.22), (6.23), (6.24) and therefore, remembering (7.21), we conclude that $|g_+(\omega)(\xi)/t(\omega)(\xi)| = 0$ for large enough ξ.

Since $t(\omega)(\xi)$ is finite everywhere, we see that

$$g_+(\omega)(\xi) = 0 \tag{7.23}$$

for large enough ξ.

From (7.19) it follows that $g_+(\omega)$ admits a locally absolutely continuous first derivative. Hence equation (4.11) together with (7.19) tells us that $g_+(\omega)$ satisfies:

$$g_+(\omega)''(\xi) + (\omega - q(\omega)(\xi)) g_+(\omega)(\xi) = h(\xi).$$

Insertion of (7.22) in the above equation yields

$$g_+(\omega)''(\xi) + (\omega - p(\xi)) g_+(\omega)(\xi) = 0 \tag{7.24}$$

(7.23) and (7.24) together lead us to conclude that $g_+(\omega)(\xi) = 0$, which by virtue of (7.21) implies $h = 0$. This completes the proof of lemma 7.1 and of Condition $A_1(\mathscr{I})$.

8. The Proof of Condition $A_2(\mathscr{I})$

In this section we verify the Condition $A_2(\mathscr{I})$ under Conditions S and B respectively.

Lemma 8.1. Under the simplifying Condition S,

$$[R(\mu, L(p))]_{\mathscr{G}} = [R(\mu, L(q(\mu)))]_{\mathscr{G}} (I - T(\mu))_{\mathscr{G}}^{-1}.$$

Proof. We have seen in Section 6 that (2.6) and (2.7) imply that for all non-real complex μ in $\mathscr{R}_\pm(\mathscr{I})$,

$$T(\mu) \equiv (L(p) - L(q(\mu)))R(\mu, L(q(\mu))) \tag{8.1}$$

is compact in $\mathscr{B}(\mathscr{H})$.

Therefore $(I - T(\mu))$ is invertible if it is one-to-one. We also known from [20] that $D(L(p)) = D(L(q(\mu))$ for a fixed μ in $\mathscr{R}_\pm(\mathscr{I})$ so that we can write

$$(\mu I - L(p)) = (I - T(\mu))(\mu I - L(q(\mu))) \qquad \text{on } D(L(p)). \tag{8.2}$$

Since $L(p)$ is a symmetric operator, $(\mu I - I(p))$ is one-to-one and we know from Theorem 6.1 that $(\mu I - L(q(\mu)))$ is onto. This fact combined with (8.2) yields

$$(I - T(\mu))^{-1} \in \mathscr{B}(\mathscr{H}). \tag{8.3}$$

In the proof of Lemma 5.1 we have seen that $T(\mu)_\mathscr{G}$ is unitarily equivalent to the closure of $M(1/n)^{1/2}T(\mu)M(n^{1/2})$ in $\mathscr{H}$. We have also seen that this operator is compact. Therefore $(I - T(\mu))_\mathscr{G}$ is invertible if it is one-to-one in $\mathscr{G}$. For this it suffices to show that $(I - M(1/n)^{1/2}T(\mu)M(n^{1/2}))$ is one-to-one in $\mathscr{H}$. To that end, we first claim that the closure of $T(\mu)M(n^{1/2})$ is a bounded operator. This follows by writing its kernel,

$$(T(\mu)M(n^{1/2})(\xi, \eta) = (p - q(\mu))(\xi)R(\mu, L(q(\mu)))(\xi, \eta)n^{1/2}(\eta) \tag{8.4}$$

and using relation

$$|(T(\mu)M(n^{1/2}))(\xi, \eta)| \leqslant 0(1)|(p - q(\mu))(\xi)|s(\xi)\cdot n^{1/2}(\eta)s(\eta).$$

Then relations (2.6), (2.7) and (5.3) imply that this is a Hilbert–Schmidt operator.

Next we note that if the closure of the product of a closable operator and a bounded operator is bounded then it is equal to the product of the closures. Therefore by virtue of the observation that both operators $M(1/n)^{1/2}T(\mu)M(n^{1/2})$ and $T(\mu)M(n^{1/2})$ are bounded, we conclude that

$$M\left(\frac{1}{n}\right)^{1/2} T(\mu)M(n^{1/2}) = M\left(\frac{1}{n}\right)^{1/2} \cdot (T(\mu)M(n^{1/2})). \tag{8.5}$$

Then,

$$\left(I - M\left(\frac{1}{n}\right)^{1/2} T(\mu)M(n^{1/2})\right)f = 0 \tag{8.6}$$

imply, by virtue of (8.5), that

$$f \in D(M(n^{1/2})) \tag{8.7}$$

and

$$M(n^{1/2})f = (T(\mu)M(n^{1/2}))f. \tag{8.8}$$

Applying again the previous considerations about the closure of the product of operators, we arrive at

$$(T(\mu)M(n^{1/2})) = T(\mu)\cdot M(n^{1/2}).$$

Inserting this in (8.8) and using (8.3), we conclude that $f = 0$. And this completes the proof of Lemma 8.1.

Corollary. Under the simplifying assumptions (2.6) and (2.7), the operator $L(p)$ is self-adjoint.

Proof. Since by (8.3), $(I - T(\mu))$ is onto and by Theorem 6.1 $(\mu I - L(q(\mu)))$ is onto, the relation (8.2) implies that the operator $(\mu I - L(p))$ is onto. Since $L(p)$ is a symmetric operator, this implies that $L(p)$ is self-adjoint.

For the general situation we need the following extension of Kato's resolvent formula, the proof of which is given elsewhere [21].

Proposition 8.1. Let the given operator A_1 be essentially self-adjoint on the given set $S \subset H$ and let the operators A_0 and V be such that

$$A_1 = A_0 + V \qquad \text{on } S. \tag{8.9}$$

Suppose that V and its adjoint V^* can be factored as

$$V = (V^{1/2})(V^{1/2}), \qquad V^* = (V^{1/2})^*(V^*)^{1/2} = (V^*)^{1/2}(V^{1/2})^* \qquad \text{on } S. \tag{8.10}$$

Suppose further that

$$\mu \in \rho(A_0) \cap \rho(A_1) \tag{8.11}$$

and that

$$V^{1/2}R(\mu, A_0) \in \mathscr{B}(\mathscr{H}) \tag{8.12}$$

$$R(\mu, A_0)V^{1/2} \in \mathscr{B}(\mathscr{H}) \tag{8.13}$$

and that the following operator is compact,

$$V^{1/2}R(\mu, A_0)V^{1/2} \in \mathscr{S}_\infty(\mathscr{H}). \tag{8.14}$$

Then

$$(I - V^{1/2}R(\mu, A_0)V^{1/2})^{-1} \in \mathscr{B}(\mathscr{H}) \tag{8.15}$$

and

$$R(\mu, A_1) = R(\mu, A_0) + R(\mu, A_0)V^{1/2}(I - V^{1/2}R(\mu, A_0)V^{1/2})^{-1}V^{1/2}R(\mu, A_0). \tag{8.16}$$

We apply this abstract result to the case where $A_1 = L(p)$, $A_0 = L(p(\mu))$ is the approximating family and $V = M(p - p(\mu))$ and to the set $S = D(L(p)) \cap \dot{C}(\mathbf{R})$, then it is one of the assumptions of Theorem 2.1 that this operator is essentially self-adjoint on S. The next three hypotheses are verified if we set

$$V^{1/2} = M(p - p(\mu))^{1/2}, \quad (V^*)^{1/2} = M(p - p(\bar{\mu}))^{1/2}$$

and choose $\mu \in \mathscr{R}_\pm(\mathscr{I})$. The verification of the other assumptions constitutes the

Lemma 8.2. Set $p(\mu) = q(\mu)$ of (4.4). Then for each $\mu \in \mathscr{R}_\pm(\mathscr{I})$, the closures of the following operators are bounded.

$$M(p - q(\mu))^{1/2}R(\mu, L(q(\mu))) \in \mathscr{B}(\mathscr{H}) \tag{8.17}$$

and

$$R(\mu, L(q(\mu)))M(p - q(\mu))^{1/2} \in \mathscr{B}(\mathscr{H}). \tag{8.18}$$

Furthermore, the closure of the following operator is compact, that is to say,

$$M(p - q(\mu))^{1/2}R(\mu, L(q(\mu)))M(p - q(\mu))^{1/2} \in \mathscr{S}_\infty(\mathscr{H}). \tag{8.19}$$

Proof. The kernel of the closure of the operator in (8.17) is given by

$$M(p - q(\mu))^{1/2}R(\mu, L(q(\mu)))(\xi, \eta) = (p - q(\mu))^{1/2}(\xi)K(\mu)(\xi, \eta). \tag{8.20}$$

According to estimate (6.12),

$$|k(\mu)(\xi, \eta)|^2 = 0(1)t^2(\mu)(\xi)t^2(\mu)(\eta) \exp(-2|v(\mu)(\xi) - v(\mu)(\eta)|).$$

As in Section 6, it is easy to see that

$$\sup_\xi \int_{-\infty}^{\infty} t^2(\mu)(\eta) \exp(-2|v(\mu)(\xi) - v(\mu)(\eta)|)\, d\eta < \infty.$$

Similar considerations as in (6.23) gives us the fact that

$$(p - q(\mu))t^2(\mu) \in L_1(R).$$

Combining these three estimates, we get

$$\iint |(p - q(\mu))^{1/2}(\xi)K(\mu)(\xi, \eta)|^2\, d\xi\, d\eta < \infty$$

implying that the operator in (8.17) is Hilbert–Schmidt, in particular bounded. Identical considerations lead to the conclusion (8.18).

Setting

$$B(\mu) = M(p - q(\mu))^{1/2}R(\mu, L(q(\mu)))M(p - q(\mu))^{1/2}, \tag{8.21}$$

we observe that $B(\mu)$ is an integral operator and its kernel is given by

$$B(\mu)(\xi, \eta) = (p - q(\mu))^{1/2}(\xi)K(\mu)(\xi, \eta)(p - q(\mu))^{1/2}(\eta). \tag{8.22}$$

It is elementary to observe that since

$$|(p - q(\mu))^{1/2}(\xi)| < n^{1/2}(\xi),$$

we have

$$|B(\mu)(\xi, \eta)| < |X(\mu)(\xi, \eta)|. \tag{8.23}$$

and therefore by virtue of (5.12) we conclude that

$$\iint |B(\mu)(\xi, \eta)|^2\, d\xi\, d\eta < \infty,$$

i.e. the operator in (8.19) is Hilbert–Schmidt, in particular compact. This completes the proof of Lemma 8.1.

It is useful to observe that while the Hilbert–Schmidt norm of $B(\mu)$ is uniformly bounded in μ for μ in $\mathscr{R}_\pm(\mathscr{I})$, the same is *not* true of the other two operators in Lemma 8.1. In fact, a closer examination of estimates in Section 6 tells us that the Hilbert–Schmidt norm of $M(p - q(\mu))^{1/2}R(\mu, L(q(\mu)))$ behaves like $|\text{Im}\,\mu|^{-1/2}$ as $\text{Im}\,\mu \to 0$.

Now that we have verified all the hypothesis of the proposition 8.1, Condition $A_2(\mathscr{I})$ follows from the conclusion of the proposition as in [21].

9. The Proof of Theorem 2.1

In this section, we collect all the results and return to the proof of Theorem 2.1 and 2.2. Suppose the potential $p = p_1 + p_2$, where p_1 is short-range and p_2 is a Stark-like potential, satisfying either Condition B or simplifying assumptions of Section 2. Then with reference to every compact sub-interval $\mathscr{I}$ of R, we have constructed a family of operators $L(q(\mu))$ approximating $L(p)$ in Sections 4, 5 and 6, where $\mu \in \mathscr{R}_{\pm}(\mathscr{I})$. In Sections 7 and 8, we have verified additional Conditions $A_1(\mathscr{I})$ and $A_2(\mathscr{I})$ for this approximating family $L(q(\mu))$. Therefore, by virtue of Theorem 3.1, we can conclude that for every compact subinterval $\mathscr{I}$ of $\mathbf{R}$, $L(p)(\mathscr{I})$, the part of $L(p)$ over $\mathscr{I}$, is absolutely continuous. This together with the countable additivity of the spectral projections [10] implies that

$$L(p) = L(p)_{\text{a.c.}}.$$

Acknowledgments

The authors thank Prof. M. Guenin for hospitality in the Department of Theoretical Physics and Dr. W. Amrein for useful discussions.

REFERENCES

[1] W. O. Amrein, in *Scattering Theory in Mathematical Physics*, edited by J. A. Lavita and J.-P. Marchand; Nato Advanced Study Institute Series, Series C (D. Reidel Publishing Company 1974).

[2] P. A. Rejto and Kalyan Sinha, *Local Decay in Presence of a Stark Field.* Preprint (University of Geneva 1975).

[3] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (*Nonrelativistic Theory*) (Pergamon Press 1958). See chapter VII; (b) H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One and Two-Electron Atoms* (Springer Verlag 1957). See chapter III(b).

[4] P. A. Rejto, *On a Theorem of Titchmarsh–Neumark–Walter Concerning Absolutely Continuous Operators*, I (to appear in Lett. Math. Phys.).

[5] P. A. Rejto, *On a Theorem of Titchmarsh–Neumark–Walter Concerning Absolutely Continuous Operators*, II (to appear in Lett. Math. Phys.).

[6] J. Walter, *Absolute Continuity of the Essential Spectrum of* $-d^2/dt^2 + q(t)$ *without Monotony of* q, Math. Z. *129*, 83–94 (1972).

[7] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-order Differential Equations*, Part I, second edition (Clarendon Press, Oxford, 1962).

[8] M.-A. Neumark, *Linear Differentialoperatoren* (Akademie-Verlag, Berlin 1960).

[9] P. A. Rejto, *On the Theorem of Titchmarsh–Neumark–Walter Concerning Absolutely Continuous Operators*, I, University of Geneva report, 1975.

[10] N. Dunford and J. I. Schwartz, *Linear Operators*, Parts I and II (Wiley (Interscience) 1963 and 1967).

[11] P. A. Rejto, in *Perturbation Theory and Quantum Mechanics*, edited by C. H. Wilcox (Wiley 1966), pp. 57–95.

[12] B. S. Pavlov and S. V. Petras, *The Singular Spectrum of the Weakly Perturbed Multiplication Operator.* (In Russian) Funk. Anal. Prilozen 1970, pp. 54–61.

[13] K. O. Friedrichs, *Perturbation of Spectra in Hilbert Space* (Amer. Math. Soc., Providence, R.I. 1965).

[14] Edwin C. Kemble, *The Fundamental Principles of Quantum Mechanics* (McGraw Hill 1937), pp. 90–107.

[15] N. Froman and P. O. Froman, *JWKB Approximations* (North-Holland Publishing Company, Amsterdam 1965).

[16] J. P. Eckmann and R. Seneor, *The Maslov-WKB Method for the Harmonic Oscillator* (to appear in Arch. Rat. Mech.).

[17] MARTIN SCHECHTER, *A Unified Approach to Scattering*. Preprint 1974. See proof of Theorem 3.1.
[18] P. A. REJTO, in *Physical Reality and Mathematical Description*, edited by ENZ and MEHRA (Reidel Publishing Co. 1974). See Appendix.
[19] MARTIN SCHECHTER, *Principles of Functional Analysis* (Academic Press 1971).
[20] T. KATO, *Perturbation Theory of Linear Operators* (Springer-Verlag 1966).
[21] P. A. REJTO, *On a Theorem of Titchmarsh–Neumark–Walter Concerning Absolutely Continuous Operators*, II. University of Geneva report, 1975.
[22] BOURBAKI, *Espaces Vectoriels Topologiques*. See Theorem 1, Chap. I.2.