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Electromagnetic Effects in Low Energy Pion–Nucleon Scattering

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Abstract. Previous work on electromagnetic effects in low energy pion–nucleon scattering is reviewed. In particular, the 'charge independent' analysis by Zimmermann of experiments in the region of the first resonance is discussed in detail. A programme for studying electromagnetic effects on the analysis of $\pi^{\pm}p$ scattering experiments at very low energies is outlined.

1. Introduction

This paper reviews the main studies of electromagnetic effects on low energy pion–nucleon scattering; by low energy we mean a laboratory kinetic energy of the pion of less than about 300 MeV. (All energies given in the paper will be laboratory kinetic energies of the pion.) There is a lot of variation in the symbols used by various authors, and we shall establish a consistent notation of our own as we go along, conforming as far as possible to that of Oades and Rasche [1, 2] and Zimmermann [3, 4] (who are not always mutually consistent as far as notation is concerned). It is possible to distinguish three main sorts of electromagnetic effects, namely, Coulomb effects, mass difference effects, and the influence of the radiative capture channel (γn) on $\pi^- p$ scattering.

In Section 2, which is rather long, we discuss work done within the framework of a nonrelativistic potential model, and consider the relativistic modifications which have been proposed. Coulomb and mass difference effects have been studied within this framework. Section 3 reviews work on dispersion theoretical approaches to Coulomb effects and on the analysis of low energy $\pi^- p$ experiments when radiative capture is taken into account. In Section 4 we discuss the analysis of the accurate experiments of Bugg et al. [5–7] across the first resonance (with $I = \frac{3}{2}$, $J = \frac{3}{2}$, $P = +1$, $W_r \approx 1232$ MeV). In particular we look at the work of Zimmermann [4] who shows how, by the inclusion of mass difference effects, it is possible to make a 'charge independent' phase shift analysis of these experiments. It should be stressed that when electromagnetic effects are taken into account charge independence cannot be stated precisely except within the framework of some model. Zimmermann uses a nonrelativistic potential

model, with relativistic modifications, in which the assumption of charge independence is made for the nuclear *potentials*. Finally, in Section 5 we describe a programme for systematically studying electromagnetic effects in very low energy pion–nucleon scattering (below about 40 MeV). This programme will be developed in a series of further papers.

2. Potential Theory Models of Coulomb and Mass-difference Effects

The study of electromagnetic effects in low energy pion–nucleon scattering began in 1952, when Van Hove [8] discussed the processes $\pi^{\pm}p \to \pi^{\pm}p$ and $\pi^{-}p \to \pi^{0}n$ in terms of a simple nonrelativistic potential model in which the potential was taken for pion–proton separation r greater than some value r_N to be the Coulomb potential of a radially symmetric charge distribution with finite extension less than r_N and for $r < r_N$ to be a purely nuclear potential. He used the radial Schrödinger equation for s- and p-waves, and matched the interior wave function determined by the nuclear interaction for $r \leqslant r_N$ to the Coulomb wave function for $r \geqslant r_N$. In this way he wrote the differential cross-sections for the three processes in terms of the sum of the terms with $l = 0, 1$ of the partial wave expansions of the usual amplitudes $\mathscr{F}$ and $\mathscr{G}$:

$$\mathscr{F}(q, \Theta) = \mathscr{F}^{(p)}(q, \Theta) + \sum_{l=0}^{\infty} P_l(\cos\Theta)[(l+1)\mathscr{F}(l, l+\tfrac{1}{2}; q) + l\mathscr{F}(l, l-\tfrac{1}{2}; q)],$$

$$\mathscr{G}(q, \Theta) = \mathscr{G}^{(p)}(q, \Theta) + \sum_{l=1}^{\infty} P_l^1(\cos\Theta)[\mathscr{F}(l, l+\tfrac{1}{2}; q) - \mathscr{F}(l, l-\tfrac{1}{2}; q)]. \qquad (1)$$

Throughout this paper (and the succeeding ones), q will denote the magnitude of the momentum of either $\pi^{\pm}$ or p in the centre of momentum frame; it is related to the usual Mandelstam variable s by the relativistic equation

$$q^2 = (4s)^{-1}[s - (M+\mu)^2][s - (M-\mu)^2], \qquad (2)$$

where M, μ are the masses of the proton and charged pion respectively. The angle Θ is the usual scattering angle in the same frame. The amplitude $\mathscr{F}^{(p)}$ is the full nonrelativistic amplitude for point charges; it will be different for $\pi^{+}p$ and $\pi^{-}p$ elastic scattering and zero for $\pi^{-}p \to \pi^{0}n$ (charge exchange scattering). The Coulomb amplitude $\mathscr{G}^{(p)}$ is zero for a nonrelativistic calculation; it is included with a view to relativistic modifications later. From now on the processes $\pi^{+}p \to \pi^{+}p$, $\pi^{-}p \to \pi^{-}p$ and $\pi^{-}p \to \pi^{0}n$ will be denoted by using the *subscripts* $+$, $--$ (or very often $-$ only) and $0-$ respectively on amplitudes, scattering lengths, and so on.

The partial wave amplitudes $\mathscr{F}_{+}(l, J; q)$ (with $J = l \pm \frac{1}{2}$) may be written

$$2iq\mathscr{F}_{+}(l, J; q) = \exp[2iv_{+}^{(p)}(l; q)][\exp\{2i(\delta_{+}(l, J; q) + c_{+}(l, J; q))\} - 1]. \qquad (3)$$

Here $v_{+}^{(p)}(l; q)$ are the point charge Coulomb phases, $\delta_{+}(l, J; q)$ the *purely nuclear* phase shifts and $c_{+}(l, J; q)$ the *Coulomb corrections*. In the analysis of experimental data it is assumed that, while the full Coulomb amplitudes $\mathscr{F}^{(p)}$, $\mathscr{G}^{(p)}$ are to be taken, the partial wave expansions of the modified nuclear amplitudes may be truncated.

For multichannel systems two different mass difference effects may be distinguished, though as a matter of principle it is quite arbitrary to consider one but not the other. First there are 'dynamical' effects which arise when the different reduced masses and different channel momenta are taken into account in the Schrödinger equation

from which a partial wave **S**- or **K**-matrix is calculated.[1]) Then there are 'kinematical' effects which are taken into account in relating the **S**-matrix to the matrix $\mathscr{F}$ of partial wave amplitudes and thence to measured cross-sections.[2])

Van Hove ignores mass differences for his dynamical two channel $(\pi^- p)$, $(\pi^0 n)$ calculation and assumes charge independence for the nuclear potentials. The 2×2 nuclear potential matrix is thus

$$\mathbf{U} = \begin{pmatrix} \frac{2}{3}U_1 + \frac{1}{3}U_3 & \sqrt{2}/3(U_3 - U_1) \\ \sqrt{2}/3(U_3 - U_1) & \frac{1}{3}U_1 + \frac{2}{3}U_3 \end{pmatrix}, \tag{4}$$

where U_3, U_1 are the potentials for $I = \frac{3}{2}, \frac{1}{2}$ respectively. Further, U_3 is identified with the potential U_+ which gives rise to the phase shifts δ_+. Van Hove gives the expressions

$$2iq \exp(-2i\nu_-^{(p)})\mathscr{F}_- + 1 = D^{-1}[2(1 + \epsilon_1)\exp\{i(2\delta_1 + \Delta_1)\} + (1 + \epsilon_3)\exp\{i(2\delta_3 + \Delta_3)\}], \tag{5}$$

$$2i\sqrt{qq_0}\exp(-i\nu_-^{(p)})\mathscr{F}_{0-} = D^{-1}[\exp(2i\delta_3) - \exp(2i\delta_1)], \tag{6}$$

where

$$D = 2(1 + \epsilon_1)\exp(-i\Delta_1) + (1 + \epsilon_3)\exp(-i\Delta_3). \tag{7}$$

The quantities $\mathscr{F}_-$, $\mathscr{F}_{0-}$, $\nu_-^{(p)}$, ϵ_α, δ_α, Δ_α $(\alpha = 3, 1)$ are all functions of l, J and q. The amplitude for the process $\pi^0 n \to \pi^0 n$ (which carries the subscript 00) is given by

$$2iq_0\mathscr{F}_{00} + 1 = D^{-1}[2(1 + \epsilon_1)\exp\{i(2\delta_3 - \Delta_1)\} + (1 + \epsilon_3)\exp\{i(2\delta_1 - \Delta_3)\}]. \tag{8}$$

The quantity q_0 in (6) and (8) is the magnitude of the momentum of either π^0 or n in the centre of momentum system; thus

$$q_0 = (4s)^{-1}[s - (M_n + \mu_0)^2][s - (M_n - \mu_0)^2], \tag{9}$$

where M_n, μ_0 are the masses of the neutron and neutral pion respectively.

In (5), (6) and (8) the phase shifts δ_α arise from the nuclear potentials $U_\alpha(\alpha = 3, 1)$ and charge independence asserts that

$$\delta_+ = \delta_3 \tag{10}$$

for each $(l, J; q)$. The Coulomb corrections c_+, Δ_α, ϵ_α are given by Van Hove for $l = 0$, $J = \frac{1}{2}$ to first order in the Coulomb parameter $\eta_\pm(q)$, where

$$\eta_\pm(q) = \pm\beta/2q, \tag{11}$$

$$\beta = 2\alpha m, \tag{12}$$

with $m = M\mu/(M + \mu)$ the reduced mass of the $\pi^\pm p$ systems and α the fine structure constant. (We take $\hbar$, c as basic units throughout.) His results are

$$\begin{aligned} \Delta_\alpha &= \eta_-(q)[\gamma + \ln(2qr_N) - Ci(2qr_N)\cos(2\delta_\alpha) + si(2qr_N)\sin(2\delta_\alpha)], \\ \epsilon_\alpha &= -\eta_-(q)[Ci(2qr_N)\sin(2\delta_\alpha) + si(2qr_N)\cos(2\delta_\alpha)], \\ c_+ &= -\Delta_3. \end{aligned} \tag{13}$$

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[1]) See equation (36) for the s-wave Schrödinger equation for the coupled $(\pi^- p)$, $(\pi^0 n)$ system.

[2]) Note the remarks following equation (14).

Here γ is Euler's constant and

$$Ci(x) = -\int_x^\infty t^{-1}\cos(t)\,dt, \quad si(x) = -\int_x^\infty t^{-1}\sin(t)\,dt.$$

The subscript α will always be an index for total isospin, with $\alpha = 3, 1$ referring to $I = \frac{3}{2}, \frac{1}{2}$ respectively.

Van Hove's model assumes that time-reversal invariance holds and that only the channels $(\pi^- p)$, $(\pi^0 n)$ are open; this implies that the matrix **U** in (4) is symmetric and real. The right sides of (5), (6) and (8) are thus the elements S_{--}, S_{0-}, S_{00} of a 2×2 symmetric unitary matrix **S** for each $(l, J; q)$. The unitarity of the matrix follows from the identity

$$(1 + \epsilon_1)(1 + \epsilon_3)\sin[(\delta_3 - \delta_1) + (\Delta_3 - \Delta_1)] = \sin(\delta_3 - \delta_1). \tag{14}$$

It will also be noted that, while mass differences are not taken into account in the calculation of **S**, they are taken into account in relating the matrix $\mathscr{F}$ of partial wave amplitudes to **S**, by

$$2i\mathbf{Q}^{1/2}\mathbf{S}^{(p)-1}\mathscr{F}\mathbf{S}^{(p)-1}\mathbf{Q}^{1/2} + \mathbf{1}_2 = \mathbf{S}, \tag{15}$$

with

$$\mathbf{Q} = \begin{pmatrix} q & 0 \\ 0 & q_0 \end{pmatrix}, \qquad \mathbf{S}^{(p)} = \begin{pmatrix} \exp(i\nu_-^{(p)}) & 0 \\ 0 & 1 \end{pmatrix}. \tag{16}$$

The momentum matrix **Q** has to appear in (15) in order to obtain the correct threshold behaviour of the reaction cross-sections. Note that the usual spin-averaged differential cross-section for the process $i \to f$ is

$$(d\sigma_{fi}/d\Omega)(q,\Theta) = q_f q_i^{-1}(|\mathscr{F}_{fi}(q,\Theta)|^2 + |\mathscr{G}_{fi}(q,\Theta)|^2). \tag{17}$$

The equations (15)–(17) are quite general; Van Hove's model simply gives particular expressions for the elements of **S**.

Finally, we remark that equations (5), (6) and (8) may be cast into a form involving two eigenphases $(\delta_\alpha + c_\alpha)$ and a mixing parameter C_{31}; to first order in η_-,

$$\begin{aligned} q\mathscr{F}_- &= \exp(2i\nu_-^{(p)})[\tfrac{2}{3}T_1 + \tfrac{1}{3}T_3 + 2\sqrt{2}/3C_{31}(T_3 - T_1)], \\ (qq_0)^{1/2}\mathscr{F}_{0-} &= \exp(i\nu_-^{(p)})\tfrac{1}{3}(T_3 - T_1)(\sqrt{2} + C_{31}), \\ q_0\mathscr{F}_{00} &= \tfrac{1}{3}T_1 + \tfrac{2}{3}T_3 - 2\sqrt{2}/3C_{31}(T_3 - T_1), \end{aligned} \tag{18}$$

where

$$T_\alpha = \sin(\delta_\alpha + c_\alpha)\exp[i(\delta_\alpha + c_\alpha)], \tag{19a}$$

$$c_3 = \tfrac{1}{3}\Delta_3, \quad c_1 = \tfrac{2}{3}\Delta_1, \tag{19b}$$

$$C_{31} = \sqrt{2}\left[\tfrac{1}{3}(2\epsilon_3 + \epsilon_1) + \frac{2\Delta_3 - \Delta_1}{3\tan(\delta_3 - \delta_1)}\right]. \tag{19c}$$

Note particularly that (18) and (19a) are true generally; the particular expressions for c_α and C_{31} given in (19b, c) and (13) are derived from the special model of Van Hove with its simplifying assumptions. We shall consistently refer to the quantities c_+ (for $\pi^+ p$ elastic scattering) and c_α, C_{31} (for the coupled $\pi^- p$, $\pi^0 n$ channels) as the *electromagnetic corrections*. This term will not include any reference to the additive electromagnetic amplitudes in (1) or to the Coulomb phases which appear in the partial wave

amplitudes (3), (5) and (6). The electromagnetic corrections c_α and C_{31} can be expressed (to first order) as the sum of a Coulomb correction and a mass difference correction. As we shall see, it is convenient to choose the strictly nuclear masses so that no mass difference correction is needed in the $\pi^+ p$ case; thus c_+ may also be correctly referred to as a Coulomb correction.

Van Hove was interested in the application of his results to pion scattering experiments in the energy range 30–45 MeV, where it is already necessary to use some sort of relativistic modification. He naturally interprets q as the relativistic momentum and suggests further, on the basis of a heuristic argument from the Klein–Gordon equation, that the Coulomb parameter η defined in (11) should be divided by $(1 - \beta^2)^{1/2}$, β being the relative velocity of charged pion and proton in the centre of momentum frame. This means that η is to be multiplied by

$$[2m(M + \mu)]^{-1}(W^2 - M^2 - \mu^2) \qquad \text{(Van Hove)}, \tag{20}$$

where W is the total energy in the centre of momentum frame. Van Hove does not specify where this modification is to be applied; to be consistent it should be applied to the additive Coulomb amplitude $\mathscr{F}^{(p)}$ in (1), to the point charge Coulomb phases which appear in (3), (5) and (6) and to the first order expressions (13) for the electromagnetic corrections.

Solmitz [9] in 1954 considered the appropriate relativistic generalization of the additive Coulomb point charge amplitude $\mathscr{F}^{(p)}$ in (1). He simply calculated the one-photon exchange contribution to $\pi^\pm p$ elastic scattering, including the Pauli term for the anomalous magnetic moment of the proton in the electromagnetic interaction Hamiltonian, but not including pion or proton form factors. He thus calculates expressions for what we shall call $\mathscr{F}^{(p)}_{R,\text{Born}}$ and $\mathscr{G}^{(p)}_{R,\text{Born}}$ (which is not zero), but only to terms of order $(q/E)^2$, where $E = \sqrt{M^2 + q^2}$. For future reference, we shall denote by $\mathscr{F}^{(p)}_{NR}$ the full nonrelativistic Coulomb point charge amplitude, namely

$$\begin{aligned}\mathscr{F}^{(p)}_{NR}(q, \Theta) = &-\eta(q)[q(1 - \cos\Theta)]^{-1} \\ &\times \exp[-i\eta(q) \ln \tfrac{1}{2}(1 - \cos\Theta) + 2i\nu^{(p)}(0; q)].\end{aligned} \tag{21}$$

The first order nonrelativistic amplitude (namely (21) without the phase factor) will be denoted by $\mathscr{F}^{(p)}_{NR,\text{Born}}$. We have suppressed the subscripts $\pm$.

In 1960, Hamilton and Woolcock [10] pointed out that Van Hove had made only part of the Coulomb correction, which they called the *outer* part, and noted that this outer correction, which was used by Barnes et al. [11, 12] in the analysis of their experiments at 30 and 41.5 MeV, varied appreciably as the radius r_N of the nuclear interaction varied from $0.5\,\mu^{-1}$ to $1.0\,\mu^{-1}$. Hamilton and Woolcock pointed out that Van Hove had neglected the *inner* part of the Coulomb correction. The Coulomb potential does not vanish for $r < r_N$; indeed it increases from its value at $r = r_N$ to a finite value at $r = 0$, since the charged pion and proton have charge distributions. Van Hove neglected this 'inner' Coulomb potential altogether. Using a weak scattering approximation, Hamilton and Woolcock made a very rough estimate of the inner Coulomb correction to the s-wave scattering length a_+ and to the real part of the s-wave scattering length for $\pi^- p \to \pi^- p$. They found in each case that this correction, when applied to the s-wave phase shifts obtained from the experiments of Barnes et al., roughly cancelled the outer Coulomb correction of Van Hove. They also remarked that the variation with r_N of the inner correction should approximately cancel the variation with r_N of the outer correction, but did not consider this point carefully.

Schnitzer [13] made a more careful calculation of the inner Coulomb corrections to the s- and p-wave phases in $\pi^+ p$ elastic scattering. He also gave explicit first order expressions for the outer corrections to the p-wave phases, similar to those given in (13) for the s-wave. He took the Coulomb potential V_+ to be the potential between a point charge and a uniform charge distribution of radius r_c, namely

$$\begin{aligned} V_+(r) &= \alpha/r, & r &\geqslant r_c, \\ V_+(r) &= \tfrac{1}{2}(\alpha/r_c)(3 - r^2/r_c^2), & r &\leqslant r_c. \end{aligned} \tag{22}$$

Schnitzer then gives closed formulae for the inner correction for both s- and p-waves, using the expression (cf. equation (29))

$$c_+^{(\text{inner})}(l, J; q) = -2\int_0^{r_c} dr V_+(r) h_+^2(l, J; r), \tag{23}$$

where $h_+(l, J; r)$ is the radial wavefunction for the partial wave (l, J), corresponding to the nuclear potential only, normalized so that

$$h_+(l, J; r) \underset{r\to\infty}{\sim} \left(\frac{m}{q}\right)^{1/2} \sin\left(qr - \frac{l\pi}{2} + \delta_+(l, J; q)\right). \tag{24}$$

For the p-waves Schnitzer took the nuclear potential to be a square well of radius r_c, whose depth was adjusted at each energy to give the strictly nuclear phase shift δ. For the s-wave he used a hard sphere of radius $-\delta/q(<r_c)$. However, Schnitzer does not discuss the variation of c_+ with energy for the partial waves in question, nor does he consider the dependence on r_c in detail. He also calculated the Coulomb corrections Δ_1 to the $I = \frac{1}{2}$ phases, using $V_-(= -V_+)$ instead of V_+ in (23) and energy dependent square well nuclear potentials. But he does not give a proper treatment of the two-channel problem and does not consider the corrections ϵ_α.

Rasche [14] made a careful study of the inner and outer Coulomb corrections to the s-wave scattering length a_+, using (22) for V_+ and allowing r_c to be different from r_N. With r_N varied from $0.75\,\mu^{-1}$ to $1.5\,\mu^{-1}$ and r_c from $0.5\,\mu^{-1}$ to $1.5\,\mu^{-1}$, he found a rather small variation in the total correction to a_+.

We look next at the way in which electromagnetic effects have been taken into account in most of the phase shift analyses of pion–nucleon scattering data which have been done since 1965. Roper, Wright and Feld [15] give the full one-photon exchange amplitudes $\mathscr{F}^{(p)}_{R,\text{Born}}$ and $\mathscr{G}^{(p)}_{R,\text{Born}}$ for $\pi^\pm p$ elastic scattering, again including the contribution of the Pauli term for the anomalous magnetic moment of the proton. In contrast to the work of Solmitz [9], no further approximations were made in these amplitudes. They suggest that the additive Coulomb point charge amplitudes $\mathscr{F}^{(p)}$, $\mathscr{G}^{(p)}$ which appear in (1) should be

$$\begin{aligned} \mathscr{F}^{(p)} &= (\mathscr{F}^{(p)}_{R,\text{Born}} - \mathscr{F}^{(p)}_{NR,\text{Born}}) \exp[2i\nu^{(p)}(0; q)] + \mathscr{F}^{(p)}_{NR}, \\ \mathscr{G}^{(p)} &= \mathscr{G}^{(p)}_{R,\text{Born}} \exp[2i\nu^{(p)}(0; q)], \end{aligned} \tag{25}$$

where it is understood that, in the 'nonrelativistic' amplitudes in (25), q is the relativistic momentum and $\eta(q)$ as given in (11) is to be multiplied by the factor

$$(2mW)^{-1}(W^2 - M^2 - \mu^2) \qquad \text{(Roper, Wright and Feld).} \tag{26}$$

The relativistic modification given in (26) is also applied by Roper, Wright and Feld to the calculation of the differences

$$\nu^{(p)}(l; q) - \nu^{(p)}(0; q)$$

which appear in the partial wave expansions when an overall phase factor $\exp[2i\nu^{(p)}(0;q)]$ is removed. The prescription (25) in fact enables this factor to be removed from the full amplitudes $\mathscr{F}$, $\mathscr{G}$ of (1). Actually a more reasonable prescription would be to take the full relativistic point-charge amplitudes as

$$\mathscr{F}^{(p)} = \mathscr{F}^{(p)}_{R,\text{Born}} \exp[-i\eta(q) \ln \tfrac{1}{2}(1 - \cos\Theta) + 2i\nu^{(p)}(0;q)], \tag{27}$$

and similarly for $\mathscr{G}^{(p)}$, with $\eta(q)$ modified as in (26). The motivation for (27) is that the effect of taking the whole Born series in the nonrelativistic calculation instead of just the first term is to multiply the first Born approximation by a phase factor. Roper, Wright and Feld do not give a proper argument for the modification (26); a possible argument would be that, when the modification (26) is applied to $(1 - \cos\Theta)\mathscr{F}^{(p)}_{NR,\text{Born}}$, the result agrees with $(1 - \cos\Theta)\mathscr{F}^{(p)}_{R,\text{Born}}$ for small Θ, if the anomalous magnetic moment terms are not taken into account.

Roper, Wright and Feld neglected the electromagnetic corrections altogether. From the phase shift analysis of π^+p elastic scattering experiments they obtained the modified nuclear phase shifts $\delta_+(l,J;q) + c_+(l,J;q)$ as functions of q. But then they neglected the c_+ altogether and identified the modified nuclear phases with the purely nuclear $I = \frac{3}{2}$ phases $\delta_3(l,J;q)$. These phases were then used in the analysis of π^-p experiments. Here too the electromagnetic corrections c_α and C_{31} which appear in (18) and (19a) were taken as zero, and the π^-p elastic and charge exchange scattering experiments analysed in order to obtain the purely nuclear phases $\delta_1(l,J;q)$ as functions of q. It is not clear from [15] whether the kinematical mass difference effects were included in the analysis of charge exchange scattering experiments. Subsequent large scale phase shift analyses of pion–nucleon scattering experiments also neglected electromagnetic corrections and until the accurate experiments of Bugg et al. [5–7] it was possible to achieve a satisfactory fit to the experiments. The new experimental results, with greatly improved accuracy, showed immediately that it was impossible to use the resonant phase $\delta_3(1,\frac{3}{2};q)$, obtained from analysis of the π^+p experiments without electromagnetic correction, in the analysis of π^-p experiments. We return to this in Section 4.

We consider next the paper of Oades and Rasche [16], who studied the scattering of two spin-0 particles with extended charge distributions. They used the Schrödinger equation for the relative motion, the potential being the sum of a short range nuclear potential $U(r)$ ($U(r) = 0, r > r_N$) and a Coulomb potential $V(r)$ which was assumed to be due to the interaction of two uniform spherical charge distributions of finite radius. Thus $V(r)$ was taken to be the point charge Coulomb potential $V^{(p)}(r)$ for $r \geqslant r_c$, where r_c is the sum of the two radii; for $r \leqslant r_c$, $V(r)$ is more complicated than (22), but still readily calculable and finite for $r \to 0$. Using this model Oades and Rasche were able to obtain expressions for the outer Coulomb corrections to the phase shifts to all orders in the parameter η, but for the inner Coulomb corrections they were able to obtain only a first order result. In this case, 'outer' refers to the region $r \geqslant r_0$, 'inner' to the region $0 \leqslant r \leqslant r_0$, where $r_0 = \max\{r_N, r_c\}$. Further, from an argument similar to that of Van Hove involving the Klein–Gordon equation, Oades and Rasche suggest that $\eta(q)$ be multiplied by

$$\left(1 + \frac{q^2}{m^2}\right)^{1/2} = \frac{W^2 - M^2 - \mu^2}{2mW}\left[1 + \frac{4M^2\mu^2\{W^2 - (M+\mu)^2\}}{(M+\mu)^2(W^2 - M^2 - \mu^2)^2}\right]^{1/2}$$

(Oades and Rasche) (28)

in the calculation of the Coulomb phases and the Coulomb corrections. In fact (26) and (28) are almost identical; the extra factor in (28) has a maximum value of 1.027.

Oades and Rasche (in appendix C) also consider the way in which some authors modify the additive Coulomb amplitude and the Coulomb phases in order to take account of extended charge distributions. It might seem that it is not correct to use an additive amplitude and Coulomb phases corresponding to a point charge potential when the *actual* Coulomb potential is modified to take account of the extended charge distributions of the interacting particles. However, the point charge Coulomb potential actually serves as a *reference* potential, whose choice is a matter of convenience. Suppose that $\hat{V}$ is some reference potential and let $\hat{\mathscr{F}}$ and $\hat{\mathscr{G}}$ be the additive Coulomb amplitudes and $\hat{v}(l;q)$ the Coulomb phases corresponding to $\hat{V}$. Suppose the experiments are analysed (in the case of π^+p elastic scattering for definiteness) using (1), (3) but with additive amplitudes $\hat{\mathscr{F}}_+$, $\hat{\mathscr{G}}_+$ and Coulomb phases $\hat{v}_+(l;q)$. The analysis will yield phase shifts

$$\delta_+(l,J;q) + c_+(l,J;q),$$

but the corrections $c_+(l,J;q)$ will depend on both $V_+(r)$, the actual Coulomb potential assumed in the model, and $\hat{V}_+(r)$, the reference potential. If the corrections are calculated properly, they must be such that the sum

$$\hat{v}_+(l;q) + c_+(l,J;q)$$

does *not* depend on the reference potential $\hat{V}_+$. The strictly nuclear phases $\delta_+(l,J;q)$ obtained from the analysis after the corrections have been made will also not depend on $\hat{V}_+$. The important thing is that the additive amplitudes, the Coulomb phases and the corrections must all use the *same* reference potential. The correct first order (in η) formula for the Coulomb corrections is given by Zimmermann [3], who also gives two examples of reference potentials (other than the point charge Coulomb) together with the corresponding additive amplitudes and Coulomb phases. His result is

$$c_+(l,J;q) = -2\int_0^\infty dr\,[V_+(r)h_+^2(l,J;r) - \hat{V}_+(r)mqr^2j_l^2(qr)], \tag{29}$$

where $h_+(l,J;r)$ appears in (23) and (24). Note that, for $l = 0$ and the point charge Coulomb potential as reference potential, it is easy to go from (29) to the generalization of Van Hove's result (13) to include the inner Coulomb correction, namely

$$\begin{aligned} c_+(0,\tfrac{1}{2};q) = -2\int_0^{r_0} & drV_+(r)h_+^2(0,\tfrac{1}{2};r) \\ & + \eta_+(q)[\gamma + \ln(2qr_0) - Ci(2qr_0)\cos(2\delta_+(0,\tfrac{1}{2};q)) \\ & \quad + si(2qr_0)\sin(2\delta_+(0,\tfrac{1}{2};q))]. \end{aligned} \tag{30}$$

To summarize, it is correct to use additive Coulomb amplitudes and Coulomb phases corresponding to a pure point charge potential in the analysis of experiments; use of a reference potential modified to take account of extended charge distributions will not alter the strictly nuclear phases obtained after the Coulomb corrections are made, provided all the calculations are carried out correctly.

The first complete treatment of the coupled (π^-p), (π^0n) system is given in papers

by Oades and Rasche [1] and Auvil [17].[3]) Oades and Rasche outline first the theory of Coulomb corrections to $\pi^+ p$ elastic scattering, and then develop the coupled channel formalism, neglecting the n–p and $\pi^\pm$–π^0 mass differences in their treatment of the electromagnetic corrections. The 2×2 potential matrix was taken to be the sum of the matrix **U** of (4), which assumes charge independence, and the Coulomb potential matrix **V** given by

$$\mathbf{V}(r) = \begin{pmatrix} V_-(r) & 0 \\ 0 & 0 \end{pmatrix}.$$

The nuclear potentials U_3, U_1 were assumed to be of finite range r_N, and V_- was assumed to deviate from the point charge Coulomb potential $V^{(p)}$ only for $r \leqslant r_c$. Their treatment led to results for $\mathscr{F}_-$, $\mathscr{F}_{0-}$ and $\mathscr{F}_{00}$ very similar in form to Van Hove's results (equations (5)–(8)), but complicated by the presence of four quantities labelled χ (with suitable subscripts) which are related to the inner part of the Coulomb potential. They showed that their results went over to those of Van Hove when the inner Coulomb potential was set equal to zero, and that their S-matrix elements formed a symmetric unitary matrix. They also wrote the first order (in η_-) expressions for the four χ's and indicated how similar expressions could be written for ϵ_α and Δ_α. Finally they gave the corrections c_α and C_{31} of (19b, c) in terms of ϵ_α, Δ_α and the four χ's. They do not give formulae for practical calculations; to derive such formulae for the case $l = 0$, we start from the results in (A1.6) and (A1.7) of [16] (which are also found in (18), (22a) and (22b) of Geissler [18]), namely

$$\begin{aligned} F_0(\eta;\rho) &= \sin\rho + \eta[si(2\rho)\sin\rho + Ci(2\rho)\cos\rho - (\ln 2\rho + \gamma)\cos\rho] + O(\eta^2), \\ G_0(\eta;\rho) &= \cos\rho + \eta[-si(2\rho)\cos\rho + Ci(2\rho)\sin\rho + (\ln 2\rho + \gamma)\sin\rho] + O(\eta^2). \end{aligned} \tag{31}$$

Differentiating with respect to ρ,

$$\begin{aligned} F_0'(\eta;\rho) &= \cos\rho + \eta[si(2\rho)\cos\rho - Ci(2\rho)\sin\rho + (\ln 2\rho + \gamma)\sin\rho] + O(\eta^2), \\ G_0'(\eta;\rho) &= -\sin\rho + \eta[si(2\rho)\sin\rho + Ci(2\rho)\cos\rho + (\ln 2\rho + \gamma)\cos\rho] + O(\eta^2). \end{aligned} \tag{32}$$

Using (31) and (32) and many formulae in reference [1], the first order results of Oades and Rasche in the s-wave case are

$$\begin{aligned} c_3 &= \tfrac{1}{3}\eta_-[si(2\rho_0)\sin 2\delta_3 - Ci(2\rho_0)\cos 2\delta_3 + \ln 2\rho_0 + \gamma] - \tfrac{2}{3}\int_0^{r_0} dr V_-(r) h_3^2(r), \\ c_1 &= \tfrac{2}{3}\eta_-[si(2\rho_0)\sin 2\delta_1 - Ci(2\rho_0)\cos 2\delta_1 + \ln 2\rho_0 + \gamma] - \tfrac{4}{3}\int_0^{r_0} dr V_-(r) h_1^2(r), \\ C_{31}\sin(\delta_3 - \delta_1) &= (\sqrt{2}/3)\eta_-[si(2\rho_0)\sin(\delta_3 + \delta_1) - Ci(2\rho_0)\cos(\delta_3 + \delta_1) \\ &\qquad + \cos(\delta_3 - \delta_1)(\ln 2\rho_0 + \gamma)] \\ &\quad - 2\sqrt{2}/3\int_0^{r_0} dr V_-(r) h_3(r) h_1(r), \end{aligned} \tag{33}$$

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[3]) A preliminary report of the work of Oades and Rasche appeared in *Springer Tracts in Modern Physics*, Vol. 55, ed. G. Höhler (Springer-Verlag, Berlin, 1970), p. 61.

where $r_0 = \max\{r_c, r_N\}$, $\rho_0 = qr_0$, and it is assumed that the strictly nuclear radial wave functions h_α are normalized as in (24) and that the reference potential is the point charge Coulomb potential. The results in (33) generalize the results of Van Hove in (13) and (19).

The first part of the work of Auvil [17] is similar to that of Oades and Rasche just described, though his method of using a large cut-off radius which is allowed to go to infinity at the end of the calculation is not necessary and is avoided by Oades and Rasche. Auvil gives first order formulae which generalize (33) to any partial wave; his formulae can be readily generalized further to apply to any reference potential, giving

$$c_3(l, J; q) = -\tfrac{2}{3}\int_0^\infty dr\,[V_-(r)h_3^2(l, J; r) - \hat{V}_-(r)mqr^2 j_l^2(qr)],$$

$$c_1(l, J; q) = -\tfrac{4}{3}\int_0^\infty dr\,[V_-(r)h_1^2(l, J; r) - \hat{V}_-(r)mqr^2 j_l^2(qr)],$$

$$C_{31}(l, J; q)\sin[\delta_3(l, J; q) - \delta_1(l, J; q)]$$
$$= -2\sqrt{2}/3\int_0^\infty dr\,[V_-(r)h_3(l, J; r)h_1(l, J; r)$$
$$- \cos\{\delta_3(l, J; q) - \delta_1(l, J; q)\}\hat{V}_-(r)mqr^2 j_l^2(qr)]. \tag{34}$$

Auvil also has some discussion (which need not concern us here) of the Coulomb corrections in the case when the nuclear potential matrix **U** is an arbitrary symmetric 2×2 matrix function of r. In an appendix he discusses the question of relativistic modifications and proposes the change in η given in (26), arguing from a comparison between the Schrödinger and Klein–Gordon equations and also from a comparison of the nonrelativistic and relativistic point charge Born amplitudes. We remark here that Zimmermann [4] proposes yet another modification of η, namely multiplication by

$$(2mW)^{-1}(M/\sqrt{M^2 + q^2})(W^2 - M^2 - \mu^2) \qquad \text{(Zimmermann)}. \tag{35}$$

We conclude the discussion of relativistic modifications with the remark that the prescription given in (26), as suggested by Roper, Wright and Feld and by Auvil, is the one to be preferred. The prescription of Oades and Rasche in (28) is almost the same, while that of Zimmermann in (33), which would differ greatly from (26) at high energies, will differ very little from it across the first resonance (about 4% at 300 MeV). Even the Van Hove prescription (20) differs from (26) by only 14% at the first resonance position. Thus in the energy region in which such rough relativistic corrections can make sense at all, the various prescriptions are in satisfactory agreement. As we remarked earlier, it is essential for the modified η to be used in the phase factor in (27), in the calculation of $[\nu^{(p)}(l; q) - \nu^{(p)}(0; q)]$ and in the formulae for the Coulomb corrections.

In the appendix of his paper, Auvil looks at the effect of the mass differences on the calculation of the electromagnetic corrections for the two channel case. However, his treatment is incomplete and his conclusion, that the mass difference effects will be unimportant in the energy region covered by the experiments of Bugg et al., is incorrect, as we shall see. The first complete treatment of the effect of mass differences on the coupled $(\pi^- p)$, $(\pi^0 n)$ channels was given by Oades and Rasche [2]. They used a two channel Schrödinger equation, which reads simply

$$(\mathbf{1}_2 d^2/dr^2 + \mathbf{Q}^2 - 2\mathbf{MU}(r) - 2\mathbf{MV}(r))\mathbf{R}(r) = 0 \tag{36}$$

for the radial equation for the s-wave. The matrix $\mathbf{Q}$ was defined in (16),

$$\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & m_0 \end{pmatrix}, \tag{37}$$

m and m_0 being the reduced masses of the $(\pi^- p)$ and $(\pi^0 n)$ channels respectively. The potential matrices $\mathbf{U}$, $\mathbf{V}$ are exactly as in the case with no mass differences.

Oades and Rasche work through the theory of the electromagnetic corrections c_α and C_{31}. One ambiguity which is inherent in any treatment of mass differences needs to be pointed out. In calculating the strictly nuclear phase shifts δ_α from the potentials U_α via the Schrodinger equation it is necessary to insert a 'strictly nuclear' reduced mass $\bar{m}$. But in the absence of any reliable theory of mass differences within isospin multiplets, one cannot know what $\bar{m}$ is. Oades and Rasche (and later Zimmermann [4]) take $\bar{m} = m$ in order to avoid making mass difference corrections in the $\pi^+ p$ case. This is a convenient, but arbitrary, thing to do. All attempts to extract 'charge independent' phases from the results of pion–nucleon scattering experiments, whether by a potential theory model or by a dispersion theory approach, are subject to this uncertainty as to the choice of the 'strictly nuclear' masses.

Oades and Rasche give a first order perturbation treatment of the electromagnetic corrections, first order in the case of the mass differences meaning that only terms proportional to Δm and Δq^2 are kept, where

$$\Delta m = m - m_0, \quad \Delta q^2 = q_0^2 - q^2. \tag{38}$$

However, they do not give explicit formulae for c_α and C_{31}; there is a lot of algebra in going from the formulae in their paper to usable results. Such results are given by Zimmermann [4], using a more general and elegant multichannel formalism.[4]) Zimmermann also makes the generalization that the reference potential $\hat{V}_-$ is not the point charge Coulomb potential. To write his results, we need a little notation. Let

$$h_\alpha(r) = (m/q)^{1/2} \sin(qr + \delta_\alpha), \qquad r \geqslant r_0,$$

and

$$\hat{h}_{i\alpha}(r) = \hat{F}_i(r) \cos\delta_\alpha + \hat{G}_i(r) \sin\delta_\alpha, \qquad r \geqslant r_0,$$

where $i = -, 0$ is the channel index. The functions $\hat{F}_-(r)$, $\hat{G}_-(r)$ are the regular and irregular Coulomb wave functions respectively for $l = 0$, for the reference potential $\hat{V}_-$, with the asymptotic behaviour

$$\hat{F}_-(r) \underset{r\to\infty}{\sim} (m/q)^{1/2} \sin[qr - \eta_-(q) \ln(2qr) + \hat{v}_-],$$

$$\hat{G}_-(r) \underset{r\to\infty}{\sim} (m/q)^{1/2} \cos[qr - \eta_-(q) \ln(2qr) + \hat{v}_-],$$

and

$$\hat{F}_0(r) = (m_0/q_0)^{1/2} \sin q_0 r, \quad \hat{G}_0(r) = (m_0/q_0)^{1/2} \cos q_0 r.$$

With the usual definition of the Wronskian of two differentiable functions, namely

$$W_x[f, g] = f(x)g'(x) - f'(x)g(x),$$

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[4]) A similar formalism was first suggested by G. C. Oades and G. Rasche, Phys. Rev. *D4* 2153 (1971).

Zimmermann gives the following first order results for $l = 0$:

$$c_3 = -X_{33} + (1/3m)W_{r_0}[h_3, \hat{h}_{-3}] + (2/3m)W_{r_0}[h_3, \hat{h}_{03}],$$
$$c_1 = -X_{11} + (2/3m)W_{r_0}[h_1, \hat{h}_{-1}] + (1/3m)W_{r_0}[h_1, \hat{h}_{01}],$$
$$C_{31}\sin(\delta_3 - \delta_1) = X_{31} + (\sqrt{2}/3m)W_{r_0}[h_3, \hat{h}_{-1}] - (\sqrt{2}/3m)W_{r_0}[h_3, \hat{h}_{01}], \quad (39)$$

where

$$X_{33} = 2/3m\int_0^{r_0} dr h_3^2(r)[mV_-(r) - 2\Delta m U_3(r) - \Delta q^2],$$
$$X_{11} = 1/3m\int_0^{r_0} dr h_1^2(r)[4mV_-(r) - 2\Delta m U_1(r) - \Delta q^2],$$
$$X_{31} = \sqrt{2}/3m\int_0^{r_0} dr h_3(r)h_1(r)[-2mV_-(r) - 2\Delta m U_1(r) - \Delta q^2]. \quad (40)$$

Similar results may be written down for partial waves with $l \geqslant 1$. It is easy to show, with the help of (31) and (32), that the expressions of (39) reduce to those of (33) when $\Delta m = 0$, $\Delta q^2 = 0$, $\hat{V}_- = V_-^{(p)}$. The results in (39) and (40) can also be deduced by combining numerous equations in the paper of Oades and Rasche [2].

As we remarked earlier, even when inner Coulomb and mass difference corrections are taken into account, the amplitudes $\mathscr{F}_-$, $\mathscr{F}_{0-}$ and $\mathscr{F}_{00}$ for each partial wave $(l; J)$ continue to be given by (18) and (19a), which involve the strictly nuclear phases δ_α and the corrections c_α and C_{31}; the full amplitudes $\mathscr{F}$, $\mathscr{G}$ are given by (1) and the differential cross-sections by (17). If a reference potential $\hat{V}_-$ different from $V_-^{(p)}$ is used to calculate the corrections, the additive amplitudes $\mathscr{F}_-^{(p)}$ and $\mathscr{G}_-^{(p)}$ and the Coulomb phases $\nu_-^{(p)}$ will be replaced by amplitudes and phases corresponding to $\hat{V}_-$.

There is an important reservation to be made concerning the work of Oades and Rasche [2] and Zimmermann [4] on the two channel $(\pi^- p)$, $(\pi^0 n)$ problem with mass differences, besides the arbitrariness in the choice of the nuclear masses. This is that the equation (36) (and similar equations for $l > 0$) is a model equation which is not Galilei invariant and whose theoretical basis is not clear. One would hope, however, that because of the smallness of Δm the lack of Galilei invariance will not have too serious an effect on the calculation of the corrections. Further, it is not clear whether there should be some relativistic generalization of the difference Δm of the reduced masses (which appears in (40)). Because of these reservations one cannot take the 'charge independent' phases of Zimmermann too seriously. On the other hand, as we shall emphasize at the end of Section 4, the primary aim of Zimmermann was to show that it *is* possible to make a charge independent analysis of the experimental data on $\pi^\pm p$ scattering across the first resonance.

3. Other Work on Electromagnetic Effects

The unsatisfactory features of the potential theory models are all too obvious, and we have pointed out some of them already. In addition, there is the arbitrariness of the shapes of the nuclear potentials U_α and of the Coulomb potentials $V_\pm$ (including the choice of the range parameters r_N and r_c), the possible effect of which will be discussed in Section 4. Thus attempts have been made to put the theory of electromagnetic effects on a more satisfactory basis by using a dispersion theoretical approach.

The first work of this kind is that of Sauter [19, 20]. In the first paper he considers $\pi^+ p$ elastic scattering and looks at the s-wave in the energy region up to 100 MeV. He first calculates the correction c_+ from the potential theory model, using Van Hove's formula (13) for the outer part of the correction and Schnitzer's formula for the inner part (using a hard sphere nuclear potential). Sauter computes the inner, outer and total corrections for a variety of values of r_c. He then considers the dispersion theory method of calculating the Coulomb correction due to Dashen and Frautschi [21]. The first point to note is that dispersion theory can be used only if a small finite mass is ascribed to the photon. In Sauter's presentation the additive Coulomb amplitude is the Born approximation of the scattering amplitude of a screened Coulomb potential of very long range, which is further modified to take account of the form factors at the two vertices. The partial wave expansion of this Born amplitude contains phases which, as the range $\to\infty$, diverge logarithmically. Sauter does not consider what Coulomb phases are to be used in the expressions for the partial wave amplitudes. He defines an s-wave amplitude

$$(2iq)^{-1}[\exp\{2i(\delta_+ + c_+)\} - \exp(2i\delta_+)] \tag{41}$$

which does not diverge as the photon mass approaches zero, and identifies c_+ with the Coulomb correction of the potential model. According to Dashen and Frautschi the amplitude (41) satisfies a dispersion relation; to evaluate this dispersion relation one requires knowledge of the discontinuity of the amplitude across its left-hand cut. Further, the Dashen–Frautschi dispersion relation involves the representation of the strictly nuclear partial wave amplitude as the usual ratio N/D of two analytic functions. There are many difficulties connected with this decomposition which we do not wish to discuss here; such questions are discussed in [22] and references to further work on the N/D method are given in [23].

The special problem in Sauter's work is that he uses an effective range formula for the s-wave amplitude which is decomposed into a constant D-function and an N-function with two complex poles which are certainly not singularities of the exact amplitude. Further, the Dashen–Frautschi dispersion relation then gives a correction c_+ which is complex; Sauter simply takes its real part. His final results for c_+ agree well with those calculated from the potential theory model, but his treatment is open to objections as serious as those which can be brought against the potential theory method. The problems concerning the additive Coulomb amplitude and Coulomb phases remain, and the strong interaction dynamics are not treated in a more satisfactory way. Moreover, although a dispersion relation is used, Sauter's work is nonrelativistic.

In his second paper [20] Sauter discusses the complex $(\pi^- p)$, $(\pi^0 n)$ problem *without* mass differences. He first makes calculations of the corrections c_α and C_{31} for the s-wave amplitudes, using (33) with Schnitzer's prescriptions for the nuclear wave functions h_α in the inner region. He then develops the Dashen–Frautschi dispersion theory method for obtaining the same corrections. Compared with his earlier work, Sauter modifies the $I = \frac{1}{2}$ and $I = \frac{3}{2}$ D-functions to have the correct phase on the physical cut, which makes the corrections real but gives the N-functions a very complicated singularity structure. Again Sauter finds good agreement between the two methods of calculating the corrections.

To remedy the obvious defects in Sauter's work, Hamilton and his collaborators [24–25] have made a more careful dispersion theoretical study of electromagnetic corrections. Again they use a small photon mass λ and carefully separate out the terms

involving ln λ. This is a delicate procedure which they develop first [24] in the framework of a nonrelativistic S-matrix model derived from a potential model and then extend to a relativistic situation [25]. The nonrelativistic treatment is first carried through in the case of Coulomb repulsion for any l. The result of this analysis is that each Coulomb correction obeys a dispersion relation which can be rigorously justified, with the limit $\lambda \downarrow 0$ properly taken. Hamilton et al. then discuss model calculations of s- and p-wave corrections, comparing exact results obtained for a Yukawa potential with the results obtained by solving the dispersion relation. This relation contains a term which is directly calculable from the hadronic phase shift, a second term which involves an integral over the physical cut (for which an iteration procedure can be used) and a third term which involves an integral over the left-hand cut. It is in the evaluation of this third term that significant uncertainties can arise in some cases. Thus for the case of a strong attractive Yukawa potential, sufficient to produce a p-wave resonance, Hamilton et al. are unable to estimate this term accurately because the power series expansion in the coupling constant fails. This is not a large part of the total correction in the case they consider, but there is no guarantee that this is so in a relativistic calculation.

Hamilton et al. then consider the one-channel case when there is Coulomb attraction, in which case the dispersion relation for the Coulomb correction includes a contribution from the Coulomb bound states. They check the agreement between calculations of the s-wave correction for a weak attractive Yukawa potential using the Schrödinger equation and the dispersion relation separately. Finally they consider the two channel $(\pi^- p)$, $(\pi^0 n)$ problem *without* mass differences and set up dispersion relations for c_α and C_{31}.

It is difficult to do justice to the long paper of Tromborg and Hamilton [25] in a brief review. A general survey of the problems involved in calculating electromagnetic corrections to hadron scattering within a dispersion theory framework, including a discussion of the main results of [25], is given in the review of Hamilton [26].[5]) Tromborg and Hamilton begin by studying the elastic scattering of two spin-0 particles, using S-matrix methods, with a finite photon mass λ. Their treatment takes account of the emission of photons by charged particles (including the infinity of soft photons) and the radiative corrections coming from the virtual emission and reabsorption of photons. By looking closely at the cross-sections which experimenters actually measure (in which the total energy loss due to photon emission is below some finite energy resolution ΔE), they are able to derive a finite (as $\lambda \downarrow 0$) amplitude in terms of which the experimental differential cross-section is expressed. This amplitude has the same analytic structure as the pure hadronic amplitude, apart from radiative cuts (associated with processes like $\pi^- p \to \gamma n$) and Coulomb bound state poles. It has the usual crossing properties and the same asymptotic behaviour as the hadronic amplitude. Tromborg and Hamilton extend their treatment to the situation where inelastic processes compete with elastic scattering, and show how it should be possible to calculate the electromagnetic correction to the inelasticity parameter η, at least up to energies where electromagnetic effects in the inelastic channels themselves become

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[5]) Further study of the electromagnetic corrections to $\pi^+ p$ scattering within a dispersion theoretical framework is given in Nordita preprint 75/14 by B. Tromborg, S. Waldenstrøm and I. Øverbø, who work with a finite (as $\lambda \downarrow 0$) scattering amplitude which is slightly different from that of [25]. They estimate the corrections to the $\pi^+ p$ s- and p-wave inelasticities due to bremsstrahlung and find them to be negligible at low energies.

important. They then write, for each l, a relativistic dispersion relation for the Coulomb correction to the (real) phase shift, which has the same general characteristics as the nonrelativistic dispersion relation. The approximation procedures for the evaluation of the correction are similar to those for the nonrelativistic case and the major uncertainty is again in the calculation of the left-hand cut contribution.

Next, Tromborg and Hamilton consider the modifications introduced by the inclusion of form factors for the hadrons. They find that the partial wave amplitude whose deviation from the pure hadronic amplitude gives the Coulomb correction has to be carefully chosen to ensure convergence of the dispersion relation. Finally, the relatively straightforward extension to π^+p elastic scattering is considered. The left-hand cut contribution to the Coulomb correction for the resonant partial wave in π^+p elastic scattering is considered in a preliminary way in [25], with further discussion in [26]. It appears from this discussion that the contributions to the t- and u-channel exchanges which have been evaluated so far are about the same size as the rest of the Coulomb correction which can be reliably calculated. Further, since nothing is known about the very short range effects, it seems that for relativistic problems the dispersion theory method contains incalculable uncertainties.

The formulation of the dispersion theoretical approach to the two channel (π^-p), (π^0n) case has not yet appeared. If mass differences are taken into account (and it is likely that they are crucially important) this will be a formidable undertaking. We therefore regard the study of π^+p and π^-p scattering processes using the potential theory model, for all its obvious deficiencies, as a worthwhile undertaking in indicating very roughly the sort of Coulomb corrections to expect, in showing the importance of mass difference effects and in suggesting that the breaking of $SU(2)$ symmetry in pion–nucleon scattering processes may well be due entirely to electromagnetic effects.

In this section also we mention briefly work on the effect of the radiative capture process $\pi^-p \to \gamma n$ on the analysis of low energy π^-p experiments. This was considered in some detail by Rasche and Woolcock [27], who set up the general framework for analysis of such experiments. They showed that, even if one uses one's approximate knowledge of the multipole amplitudes for $\gamma n \to \pi^-p$, it is still necessary in analysing experiments on π^-p elastic and charge exchange scattering to use three parameters for each partial wave. Even if some sort of 'charge independent' model is used to reduce the number of parameters required for each partial wave, it is still necessary to take account of the γn channel in analysing π^-p experiments. Problems related to the presence of this channel will be considered in a later paper. We also mention here the work of Waldenstrøm [28], who considers the parametrization of a 3×3 unitary, symmetric matrix and gives explicitly the unitarity bounds on the free parameters; his work applies in particular to the s-wave S-matrix for the channels (π^-p), (π^0n), (γn).

4. Analysis of $\pi^\pm p$ Experiments Across the First Resonance

We consider now the analysis of the experiments of Bugg et al. [5–7]. It was already clear from the total cross-section data [5] that a detailed consideration of electromagnetic corrections was necessary. When the cross-sections $\sigma_\pm$ had the contributions of the small partial waves removed, and electromagnetic corrections were neglected, it was found that the resonant phases obtained from σ_+ and from σ_- were significantly different. We shall label these 'phases' as $\delta_{3\pm}$; they are extracted from $\sigma_\pm$ in the crude way just discussed. On comparing the positions and widths of the

'resonances' represented by these phases, it was found that

$$W_r(\pi^- p) - W_r(\pi^+ p) = 1.4 \pm 0.4 \text{ MeV},$$
$$\Gamma(\pi^- p) - \Gamma(\pi^+ p) = 10.3 \pm 1.3 \text{ MeV}. \quad (42)$$

For definiteness, the position is where the phase is 90°, and the width is the difference in the total energies (in the centre-of-momentum system) where the phase is 45° and 135°. Equation (42) means roughly that the phases agreed at about 175 MeV; above that the $\pi^- p$ phase was the smaller, below that the larger.

In the phase shift analysis of the full set of experiments, Carter, Bugg and Carter [29] made electromagnetic corrections which are discussed in a paper by Bugg [30]. Bugg used the formulae (34) of Auvil [17], choosing a complicated nuclear potential whose first Born approximation reproduces the main Born terms deduced from partial wave dispersion relations. For the resonant wave Bugg's correction $c_+(1, \frac{3}{2}; q)$ is negative at low energies, reaches a minimum of about $-1.1°$ at around 155 MeV, goes through zero at about 260 MeV and is positive thereafter. This decreases the $\pi^+ p$ resonance position a little and increases its width. Since mass differences are not taken into account in Bugg's corrections,

$$c_3 = -\tfrac{1}{3} c_+$$

and the effect of this correction on the analysis of $\pi^- p$ experiments is to increase the position of the $\pi^- p$ resonance a little and to decrease its width. The influence of C_{31} also needs to be taken into account; if δ_{3-} is the phase obtained from the analysis of $\pi^- p$ experiments without any electromagnetic corrections, then, well away from the resonance position,

$$\delta_3 \approx \delta_{3-} - c_3 - \sqrt{2}\, C_{31} \tan \delta_3. \quad (43)$$

Now C_{31} without mass differences is about $+0.2°$ and varies little with energy. This also has the effect of reducing the width of the $\pi^- p$ resonance. The result of these corrections in Bugg's analysis is that

$$W_r(\pi^- p) - W_r(\pi^+ p) = 2.6 \text{ MeV},$$
$$\Gamma(\pi^- p) - \Gamma(\pi^+ p) = 6.4 \text{ MeV}. \quad (44)$$

The discrepancy in the resonance positions is made worse, while that in the widths is reduced, but only by a third. The calculation by Tromborg and Hamilton [25] of c_+ for the resonant wave, discussed in Section 3, with the left hand cut contribution neglected, is very close to that of Bugg, so the discrepancy remains.

As mentioned earlier, Zimmermann [3, 4] has been able to obtain a 'charge independent' potential theory model. He showed that, by a suitable choice of the parameters r_c and r_N which is physically reasonable, it is possible to find a charge independent resonant phase which, with Coulomb corrections applied, is able to provide good fits to *both* the $\pi^+ p$ and $\pi^- p$ data. The important result is that *the inclusion of mass difference effects is essential* in calculating the electromagnetic corrections. It should be emphasized that Zimmermann analysed the full set of data from both $\pi^+ p$ and $\pi^- p$ experiments simultaneously. Zimmermann shows in [3] that the correction $c_+(1, \frac{3}{2}; q)$ varies considerably as r_c and r_N are varied, so that one cannot fix these parameters by analysing the $\pi^+ p$ data alone. Only the analysis of the combined $\pi^+ p$ and $\pi^- p$ data determines r_c and r_N, and thus the electromagnetic corrections and the charge independent phases.

Zimmermann's first paper [3] deals with $\pi^+ p$ scattering. He points out an error (which in practice is not very important) in the calculations of [29] and [30], namely that Bugg [30] uses a certain reference potential to calculate the Coulomb corrections, while Carter et al. [29] use additive Coulomb amplitudes and Coulomb phases corresponding to a *different* reference potential. Zimmermann looks at the variation of $c_+(1, \frac{3}{2}; q)$ when r_c and r_N are varied over physically reasonable ranges. He also shows that the correction is insensitive to details of the shape of the nuclear potential. He finds that the general shape of the curve for $c_+(1, \frac{3}{2}; q)$ is the same, whatever the values of r_c and r_N chosen, but that the minimum value can vary from about $-0.6°$ to $-1.4°$, and the position of the subsequent zero from about 190 MeV to 260 MeV. The position of the minimum seems to remain fairly constant at around 150 MeV.

The second paper of Zimmermann [4] is a much more extensive study of low energy pion–nucleon scattering. He gives a multi-channel formalism which would be readily applicable to a problem with more than two channels. He also considers the case where the Coulomb potentials V_i (i indexes the channels) which are assumed to exist in the various channels may differ from the reference potentials $\hat{V}_i$ from which the additive Coulomb amplitudes and Coulomb phases used in the analysis of experiments are derived. For his analysis of $\pi^\pm p$ experiments he uses additive Coulomb amplitudes with the form factor modification of Carter et al. [29], and the corresponding Coulomb phases. The Coulomb potential V_+ is that of (22) and $V_- = -V_+$.

Zimmermann gives his results for $r_c = 1.42\,\mu^{-1}$, $r_N = 1.10\,\mu^{-1}$; with this choice he is able to get good fits to the data of Bugg et al. [5–7]. His work therefore indicates that the observed $SU(2)$ breaking in low energy pion–nucleon scattering *may* well be due solely to the direct electromagnetic effects treated in this paper. For the resonant wave, the correction c_+ has a minimum of $-0.6°$ at 140 MeV and goes through zero at about 195 MeV. The resonance position for $\pi^+ p$ is thus unaltered, whilst its width is increased. The shape and size of c_3 are drastically altered by the mass difference effects. At 80 MeV, c_3 is just over 0.4°; it goes through zero at just above 150 MeV and has a minimum of $-0.4°$ at just about 210 MeV. It will be seen that this reduces W_r for the $\pi^- p$ resonance and substantially reduces its width, as required. The correction C_{31} to the mixing angle is also greatly changed by the mass differences; it is $-0.2°$ at 80 MeV, goes through zero at 145 MeV and increases to 0.1° at around 250 MeV. This gives, using (43), a decrease in the resonance width. All the effects work in the correct way to remove the differences given in (42).

We emphasize that the phrase '$\pi^- p$ resonance' is descriptive only, and is not to be taken precisely. Statements like (42) and (44) are in fact based on an analysis of the $\pi^- p$ data which is incorrect in principle. If $\pi^- p$ experiments could be analysed quite separately from $\pi^+ p$ experiments, and two eigenphases and a mixing angle obtained at each energy for each partial wave, then for $l = 1$, $J = \frac{3}{2}$ one eigenphase would go through 90° and we could in that sense speak of a '$\pi^- p$ resonance'. But such an analysis cannot be done. The analysis of Carter et al. [29] is wrong in principle; they make their Coulomb corrections with a 'charge independent' model but then introduce two different $l = 1$, $J = \frac{3}{2}$ phases in the analysis of $\pi^\pm p$ experiments. For this reason also we do not know what value to put on Pilkuhn's attempt [31] to obtain the width difference in [42] by means of a penetration factor model which is a relativistic generalization of the well known potential theory model. His model is supposed to take into account Coulomb and mass difference effects, but it fails completely to reproduce the width difference of (42). The situation as we see it is this. Since it is not possible (and will not be possible for the foreseeable future) to analyse $\pi^- p$ experiments

independently of $\pi^+ p$ experiments, either one makes a successful analysis of $\pi^- p$ experiments using some sort of 'charge independent' model or one can make no meaningful analysis at all. Zimmermann has made such a charge independent analysis; it is easy to point out its unsatisfactory features, but the onus is on others to produce a more satisfactory alternative analysis.

5. The Study of Electromagnetic Effects on Very Low Energy Pion–Nucleon Scattering

In this section we outline the programme which we shall develop in further papers. With a view to the analysis of experiments at very low energies (below about 40 MeV) which will be carried out at pion factories with much higher statistics than previously, we shall consider first the parametrization of such experiments, neglecting the effect of the (γn) channel. We shall, for both the $I = \frac{1}{2}$ and $I = \frac{3}{2}$ s-wave amplitudes, use the model of an energy dependent short range potential of radius r_0. We shall give a justification for the use of an effective range expansion for fitting these amplitudes at very low energies, and then derive the low energy behaviour of the s-wave amplitudes for $\pi^\pm p$ elastic scattering and for $\pi^- p \to \pi^0 n$. We shall write the theory of the two-channel problem in a matrix form which generalizes readily to more than two channels. Both Coulomb and mass difference effects will be included, and the resulting corrections to the scattering lengths and curvatures (by curvature we mean the coefficient of q^2 in an expansion in powers of q^2) will be calculated as functions of r_0. The s-wave amplitudes for the three processes mentioned above will be parametrized in a form which will be useful for the analysis of experiments below about 40 MeV.

We plan further to consider the effect of the radiative capture channel (γn) on low energy $\pi^- p$ scattering. One then has to consider a three-channel problem and look at the low energy behaviour of the various processes which can occur. We shall look closely at the theory of the Panofsky ratio (the ratio of the cross-sections for $\pi^- p$ charge exchange scattering and radiative capture), both in flight at very low energies and for capture from a $1s$ state, and consider the uncertainties in calculating it indirectly from pion photoproduction and $\pi^- p$ charge exchange scattering data. Finally, we will deal with the proper statement of the final state theorem and the analysis of pion photoproduction experiments.

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