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The Parametrization of Threshold $\pi^{\pm}p$ Scattering Experiments

by **G. Rasche**

Institut für Theoretische Physik der Universität,
Schönberggasse 9, CH-8001 Zürich, Switzerland

and **W. S. Woolcock**

Research School of Physical Sciences,
The Australian National University, Canberra, Australia

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Abstract. Convenient parametric forms are obtained for the behaviour at very low energies of the s-wave amplitudes for the processes $\pi^{\pm}p \to \pi^{\pm}p$ and $\pi^{-}p \to \pi^{0}n$, taking account of Coulomb and mass difference effects.

1. Introduction

In a previous paper [1] we reviewed work which has been done on electromagnetic effects in low energy pion–nucleon scattering. In particular, we discussed in detail the charge independent analysis by Zimmermann [2] of experiments in the region of the first resonance and emphasized the crucial importance of mass difference effects on the calculation of the electromagnetic corrections.

With three pion factories now coming into operation, it will not be long before there is a new generation of high statistics $\pi^{\pm}p$ scattering experiments performed at laboratory kinetic energies of the pion of less than 40 or 50 MeV. In analysing these experiments it will be necessary to put in p-wave amplitudes obtained from dispersion relations; since the p-wave phases are small in this energy region it will not matter that they are not known very accurately or that their electromagnetic corrections can only be roughly estimated. The experiments will yield information about the s-wave amplitudes for the processes $\pi^{\pm}p \to \pi^{\pm}p$ and $\pi^{-}p \to \pi^{0}n$, hopefully down to energies well below 15 MeV. It will be important to have convenient parametric forms for these amplitudes, which take account of the Coulomb interaction and of the $\pi^{\pm}$–π^{0} and p–n mass differences. It will then be possible to analyse experiments in which the measured events are initiated by degraded pions with a spread in kinetic energy, and the whole body of experimental data will be able to be analysed to yield values of the strictly nuclear s-wave scattering lengths and curvatures for total isospin $I = \frac{3}{2}, \frac{1}{2}$.

The purpose of this paper is to obtain convenient parametric forms for these s-wave amplitudes. We shall obtain these parametric forms on the assumption that the nuclear potentials are of finite range r_0. This assumption can be avoided; the parametrizations can be justified provided the nuclear potentials are exponentially bounded ($V(r) = O(e^{-\mu r})$ for large r, with $\mu > 0$), but the arguments then become much more

elaborate. The scattering length and curvature parameters which appear in the parametric forms will not be strictly nuclear ones but will be modified by the effects of the Coulomb interaction and the mass differences. We shall estimate the corrections which must be applied to these modified nuclear parameters in order to obtain strictly nuclear parameters. For this estimate we shall use for the nuclear potentials energy dependent square wells of radius r_0. The work of Zimmermann [2] has shown that the electromagnetic corrections in the first resonance region are quite insensitive to the shape of the nuclear potential, and his final analysis in that paper uses a nuclear potential which is very close to a square well. We therefore feel justified in using an energy dependent square well for our analysis at very low energies. From a practical point of view, it greatly simplifies the calculation of electromagnetic corrections. In the same spirit, we take a constant inner Coulomb potential which is an average of that given in (1-22).[1]) To summarize,

$$U_\alpha(q^2; r) = \begin{cases} U_\alpha(q^2), & r < r_0, \\ 0, & r > r_0, \end{cases} \tag{1}$$

where $\alpha = 3, 1$ is again the isospin index, and

$$V_+(r) = \begin{cases} \alpha/r, & r > r_0, \\ 1.2\alpha/r_0, & r < r_0, \end{cases} \tag{2}$$

$$V_-(r) = -V_+(r). \tag{3}$$

For the calculations in this paper we do not distinguish between r_c and r_N. We shall study later the way in which the electromagnetic corrections to the scattering lengths and curvatures vary with r_0.

Throughout this paper we shall neglect the effect of the (γn) channel on $\pi^- p$ scattering processes at very low energies. Thus we treat only the two-channel $(\pi^- p)$, $(\pi^0 n)$ case and defer consideration of the (γn) channel to another paper. In Section 2 we consider one-channel problems; we treat these in some detail as a preparation for our method of dealing with the two-channel case. First we consider the case where no Coulomb interaction is present and look at the conditions under which an effective range expansion (of either q cot δ or tan δ/q) might be expected to be reliable, for the s-wave amplitude for scattering by an energy dependent square well of radius r_0. This analysis will apply to strictly nuclear pion–nucleon scattering and to np scattering. After that we shall consider the case when the Coulomb interaction is switched on (for example, in $\pi^+ p$ and pp scattering) and look at the modified effective range expansion and its validity. Section 3 will deal specifically with the $(\pi^- p)$, $(\pi^0 n)$ coupled channel case, though the formalism will be written in a matrix form which generalizes readily to the case of more than two channels. Effective range expansions will be written for the s-wave amplitudes for the processes $\pi^- p \to \pi^- p$, $\pi^- p \to \pi^0 n$ and $\pi^0 n \to \pi^0 n$; these expansions will involve three scattering length parameters and three curvature parameters which differ slightly from the strictly nuclear (charge independent) parameters. For the two-channel case the 2×2 nuclear potential matrix $\mathbf{U}$ will be

$$\mathbf{U} = \begin{pmatrix} \frac{2}{3}U_1 + \frac{1}{3}U_3 & \sqrt{2}/3(U_3 - U_1) \\ \sqrt{2}/3(U_3 - U_1) & \frac{1}{3}U_1 + \frac{2}{3}U_3 \end{pmatrix}, \tag{4}$$

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[1]) We refer to equations in [1] by inserting the prefix 1- before the equation number. We also use the same notation as in [1].

while the Coulomb potential matrix will be

$$\mathbf{V} = \begin{pmatrix} V_- & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

so that charge independence is built in as an assumption on the nuclear potential matrix. The matrix of the total potential is of course $\mathbf{U} + \mathbf{V}$. Finally, in Section 4 we shall look at the results of numerical calculations of the electromagnetic corrections to the scattering length and curvature parameters for pion–nucleon scattering.

2. One Channel Problems

Consider the s-wave amplitude for scattering by an attractive energy dependent square well $U(q^2)(<0)$ of radius r_0. The regular wave function is then

$$R(r) = c \sin(\omega(q^2)r), \qquad r < r_0,$$
$$R(r) = (m/q)^{1/2}(\kappa \sin qr + \sigma \cos qr), \qquad r > r_0,$$

where m is the reduced mass of the two-body system, q the magnitude of the momentum of either particle in the centre-of-momentum frame, and

$$\omega(q^2) = [q^2 - 2mU(q^2)]^{1/2}. \tag{6}$$

The constant c is arbitrary; the constants κ, σ are to be determined by matching $R(r)$, $R'(r)$ at $r = r_0$, a procedure which gives

$$\kappa = cm^{-1/2}[q^{1/2} \sin qr_0 \sin(\omega(q^2)r_0) + q^{-1/2}\omega(q^2) \cos qr_0 \cos(\omega(q^2)r_0)],$$
$$\sigma = cm^{-1/2}[q^{1/2} \cos qr_0 \sin(\omega(q^2)r_0) - q^{-1/2}\omega(q^2) \sin qr_0 \cos(\omega(q^2)r_0)].$$

If $\delta(q)$ is the phase shift, then

$$K(q) = \tan \delta(q) = \sigma\kappa^{-1} = \rho\left[\frac{\cos \rho - (\rho^{-1} \sin \rho)\, d(q^2)}{\rho \sin \rho + \cos \rho\, d(q^2)}\right],$$

where $\rho = qr_0$ and

$$d(q^2) = \omega(q^2)r_0 \cot(\omega(q^2)r_0). \tag{7}$$

Let

$$r_0^{-1}A(q^2) = \frac{\cos \rho - (\rho^{-1} \sin \rho)\, d(q^2)}{\rho \sin \rho + \cos \rho\, d(q^2)}; \tag{8}$$

then

$$\rho^{-1}K(q) = r_0^{-1}A(q^2) \tag{9}$$

and the partial wave amplitude $\mathscr{F}(q)$ is given by

$$\mathscr{F}(q) = \frac{K(q)}{q[1 - iK(q)]} = \frac{A(q^2)}{1 - iqA(q^2)}. \tag{10}$$

The function $d(q^2)$ of (7) has the expansion

$$d(q^2) = d^{(0)} + d^{(1)}\rho^2 + \cdots, \tag{11}$$

where

$$d^{(0)} = \omega(0)r_0 \cot(\omega(0)r_0), \tag{12}$$

$$d^{(1)} = \tfrac{1}{2}[1 - 2mU'(0)][(\omega(0)r_0)^{-1} \cot(\omega(0)r_0) - 1 - \cot^2(\omega(0)r_0)], \tag{13}$$

the prime in $U'(0)$ denoting differentiation with respect to q^2.[2]) Then, using (8), the function $A(q^2)$ has the expansion

$$A(q^2) = a + \alpha q^2 + \cdots, \tag{14}$$

where

$$r_0^{-1}a = (d^{(0)})^{-1}(1 - d^{(0)}), \tag{15}$$

$$r_0^{-3}\alpha = (d^{(0)})^{-2}[-1 - \tfrac{1}{3}(d^{(0)})^2 + d^{(0)} - d^{(1)}]. \tag{16}$$

The constants a, α are the scattering length and curvature respectively. There is no standard designation for α in the literature; we have chosen to call it the curvature.

In the case of a repulsive energy dependent square well $U(q^2)(>0)$ of radius r_0, the equations (8)–(11) and (14)–(16) remain, but now

$$d(q^2) = \lambda(q^2)r_0 \coth(\lambda(q^2)r_0), \tag{17}$$

where

$$\lambda(q^2) = [2mU(q^2) - q^2]^{1/2}. \tag{18}$$

Thus

$$d^{(0)} = \lambda(0)r_0 \coth(\lambda(0)r_0), \tag{19}$$

$$d^{(1)} = -\tfrac{1}{2}[1 - 2mU'(0)][(\lambda(0)r_0)^{-1} \coth(\lambda(0)r_0) + 1 - \coth^2(\lambda(0)r_0)]. \tag{20}$$

It is also common to use an expansion of $[A(q^2)]^{-1}$ in powers of q^2:

$$[A(q^2)]^{-1} = a^{-1} + \tfrac{1}{2}r_{\text{eff}}q^2 + \cdots, \tag{21}$$

where r_{eff}, the effective range, is given by

$$r_{\text{eff}} = -2\alpha/a^2.$$

We need criteria to decide whether (12) or (21) will have the greater range of validity (that is, will more accurately represent the expanded function over a larger energy range). To get these criteria a little more notation is needed; from (8),

$$r_0^{-1}A(q^2) = B(q^2)/\Omega(q^2), \tag{22}$$

where

$$B(q^2) = \cos\rho - (\rho^{-1}\sin\rho)\, d(q^2),$$
$$\Omega(q^2) = \rho\sin\rho + \cos\rho\, d(q^2), \tag{23}$$

Now for the rapid convergence of either (12) or (19) we require that

$$\rho \ll \pi/2 \tag{24}$$

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[2]) In this paper many functions will be expanded in powers of ρ^2. The superscripts (0), (1) will always be used to label the coefficients in the first and second terms of such an expansion, as in (11).

to ensure rapid convergence of the expansions of the trigonometric functions. We require further that the expansion (11) of $d(q^2)$ converges rapidly; for this it is possible to give only a rough criterion. Using (6) for $\omega(q^2)$ and (18) for $\lambda(q^2)$ we would expect such a criterion to be

$$\rho^2 \ll [1 - 2mU'(0)]^{-1} 2m|U(0)|r_0^2. \tag{25}$$

While (24), (25) apply to both (12) and (21), the third criterion is different in the two cases. If $A(q^2)$ is expanded, then $1/\Omega(q^2)$ must be expanded and we require

$$|\Omega^{(1)}|\rho^2 \ll |\Omega^{(0)}|. \tag{26a}$$

Similarly, if $[A(q^2)]^{-1}$ is expanded the criterion is

$$|B^{(1)}|\rho^2 \ll |B^{(0)}|. \tag{26b}$$

The criteria (26a), (26b) may be very different, as we shall see in a moment.

It is convenient to look immediately at the criteria (24)–(26a, b) in the cases of *np* elastic scattering and strictly nuclear πN scattering. For *np* scattering in the state with $l = 0, J = 0$ we have approximately

$$a = 23.7 \text{ fm}, \quad r_{\text{eff}} = 2.5 \text{ fm}.$$

For this case we can find an energy independent square well with

$$r_0 = 2.4 \text{ fm}, \quad U(0) = -16.42 \text{ Mev}, \quad \omega(0)r_0 = 1.51.$$

Then

$$B^{(0)} = 0.908, \quad \Omega^{(0)} = 0.092, \quad d^{(0)} = 0.092,$$
$$B^{(1)} = -0.003, \quad \Omega^{(1)} = 0.472, \quad d^{(1)} = -0.482.$$

Criterion (26a) becomes $\rho \ll 0.44$, while criterion (26b) is $\rho \ll 17$. Thus the expansion of $[A(q^2)]^{-1}$, that is of $q \cot \delta(q)$, is overwhelmingly superior. The limiting criterion is in fact (25), which gives

$$\rho \ll \omega(0)r_0 \approx 1.5,$$

or $q \ll 123$ Mev, $T_{\text{lab}} \ll 32$ Mev. The reason for the great superiority of the expansion of $q \cot \delta(q)$ is that there is a large positive scattering length due to the presence of a virtual state.

For strictly nuclear πN scattering the situation is different. We do not give all the numbers here, as detailed numerical calculations will be given in Section 4. We give the criteria for the case $r_0 = 1.0\ \mu^{-1}$. Then we find that, for $I = \frac{3}{2}$, (26a) gives $\rho \ll 2.3$, while (26b) gives $\rho \ll 1.2$. Similarly, for $I = \frac{1}{2}$, (26a) gives $\rho \ll 2.1$, while (26b) gives $\rho \ll 2.3$. However, these differences do not matter since in both cases it is criterion (25) which is the limiting one. For $I = \frac{3}{2}$ it gives $\rho \ll 0.62$, while for $I = \frac{1}{2}$ it gives $\rho \ll 0.64$. Thus it does not matter whether $A_\alpha(q^2)$ or $[A_\alpha(q^2)]^{-1}$ is expanded; in the rest of this paper we shall expand amplitudes like $A_\alpha(q^2)$. For $I = \frac{1}{2}$ or $I = \frac{3}{2}$ we would then expect the expansion of $A_\alpha(q^2)$ to converge rapidly for

$$q \ll 0.64\mu, \quad T_{\text{lab}} \ll 35 \text{ MeV}.$$

This estimate changes rapidly with the value of r_0 assumed; it is therefore not a very reliable guide. In practice one will have to determine empirically, by fitting the *s*-wave amplitudes obtained from experiments, at what energy the deviations from a two

parameter fit become apparent. But the rough estimate just given indicates that there should be a reasonable range of energies over which such a two parameter fit should be satisfactory.

We turn now to the case of s-wave $\pi^+ p$ elastic scattering; the potential is the sum of a repulsive energy dependent square well $U_3(q^2)(>0)$ of radius r_0 and the repulsive Coulomb potential (2). The regular wave function is

$$R(r) = c \sinh(\lambda_+(q^2)r), \qquad r < r_0,$$
$$R(r) = (m/q)^{1/2}[\kappa F_0(\eta_+ ; qr) + \sigma G_0(\eta_+ ; qr)], \qquad r > r_0,$$

where

$$\lambda_+(q^2) = [2mU_3(q^2) + 2m\overline{V}_+ - q^2]^{1/2}, \tag{27}$$
$$\overline{V}_+ = 1.2\alpha/r_0.$$

The rest of the notation is taken directly from [1]. On matching $R(r)$ and $R'(r)$ at $r = r_0$ and solving the resulting equations, using

$$G_0F_0' - G_0'F_0 = q, \tag{28}$$

where the prime denotes differentiation with respect to r, we obtain

$$[K_+(q)]^{-1} = \cot \delta_+(q) = \kappa\sigma^{-1}$$
$$= \frac{-r_0G_0'(\eta_+ ; qr_0) + G_0(\eta_+ ; qr_0)d_+(q^2)}{r_0F_0'(\eta_+ ; qr_0) - F_0(\eta_+ ; qr_0)d_+(q^2)},$$

$$C_0^2(\eta_+)q \cot \delta_+(q)$$
$$= \frac{-C_0(\eta_+)r_0G_0'(\eta_+ ; qr_0) + C_0(\eta_+)G_0(\eta_+ ; qr_0)d_+(q^2)}{[C_0(\eta_+)]^{-1}q^{-1}r_0F_0'(\eta_+ ; qr_0) - [C_0(\eta_+)]^{-1}q^{-1}F_0(\eta_+ ; qr_0)d_+(q^2)}, \tag{29}$$

where

$$C_0^2(\eta) = \frac{2\pi\eta}{\exp(2\pi\eta) - 1} \tag{30}$$

and $d_+(q^2)$ is given in terms of $\lambda_+(q^2)$ by equation (17).

To proceed further we use the definitions of auxiliary Coulomb wave functions given by Breit and Bouricius [3]. With the prime still denoting differentiation with respect to r, but now considering r_0 to be fixed and defining the new variable $\rho = qr_0$, we have

$$[C_0(\eta_+)]^{-1}F_0(\eta_+ ; qr_0) = \rho\Phi_0(\beta r_0 ; \rho^2), \tag{31}$$

$$[C_0(\eta_+)]^{-1}r_0F_0'(\eta_+ ; qr_0) = \rho\Phi_0^*(\beta r_0 ; \rho^2), \tag{32}$$

$$C_0(\eta_+)G_0(\eta_+ ; qr_0) = \Psi_0(\beta r_0 ; \rho^2)$$
$$+ \beta r_0(h(\eta_+) + 2\gamma - 1 + \ln \beta r_0)\Phi_0(\beta r_0 ; \rho^2), \tag{33}$$

$$C_0(\eta_+)r_0G_0'(\eta_+ ; qr_0) = \Psi_0^*(\beta r_0 ; \rho^2) + \beta r_0\Phi_0(\beta r_0 ; \rho^2)$$
$$+ \beta r_0(h(\eta_+) + 2\gamma - 1 + \ln \beta r_0)\Phi_0^*(\beta r_0 ; \rho^2), \tag{34}$$

where

$$h(\eta) = -\ln|\eta| + \operatorname{Re} \psi(1 + i\eta), \qquad \eta \neq 0,$$

γ is Euler's constant and $\beta = 2\alpha m$. Here ψ is the psi (digamma) function as defined for example in [4], so that

$$\operatorname{Re}\psi(1 + i\eta) = -\gamma + \eta^2 \sum_{k=1}^{\infty} k^{-1}(k^2 + \eta^2)^{-1}.$$

For large $|\eta|$ (small q) we have the asymptotic expansion

$$h(\eta) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k-1}B_{2k}}{2k\eta^{2k}}. \tag{35}$$

It is also convenient to define two further functions (our notation), namely

$$X_0(\beta r_0; \rho^2) = \Psi_0(\beta r_0; \rho^2) + \beta r_0(2\gamma - 1 + \ln \beta r_0)\Phi_0(\beta r_0; \rho^2), \tag{36}$$

$$X_0^*(\beta r_0; \rho^2) = -[\Psi_0^*(\beta r_0; \rho^2) + \beta r_0 \Phi_0(\beta r_0; \rho^2) + \beta r_0(2\gamma - 1 + \ln \beta r_0)\Phi_0^*(\beta r_0; \rho^2)]. \tag{37}$$

Note that, from the Wronskian relation (28),

$$\Phi_0 X_0^* = 1 - \Phi_0^* X_0. \tag{38}$$

The arguments of the functions just defined have been carefully indicated, because we want to expand each of them in a power series in ρ^2. Thus, if f denotes any one of the six functions Φ_0, Φ_0^*, Ψ_0, Ψ_0^*, X_0, X_0^* we have the expansions

$$f(\beta r_0; \rho^2) = f^{(0)}(\beta r_0) + f^{(1)}(\beta r_0)\rho^2 + \cdots.$$

The coefficients, to second order in βr_0, are

$$\Phi_0^{(0)}(\beta r_0) = 1 + \tfrac{1}{2}\beta r_0 + \tfrac{1}{12}(\beta r_0)^2 + \cdots,$$
$$\Phi_0^{(1)}(\beta r_0) = -\tfrac{1}{6}(1 + \tfrac{1}{3}\beta r_0 + \tfrac{1}{24}(\beta r_0)^2 + \cdots), \tag{39}$$

$$\Phi_0^{*(0)}(\beta r_0) = 1 + \beta r_0 + \tfrac{1}{4}(\beta r_0)^2 + \cdots,$$
$$\Phi_0^{*(1)}(\beta r_0) = -\tfrac{1}{2}(1 + \tfrac{4}{9}\beta r_0 + \tfrac{5}{72}(\beta r_0)^2 + \cdots), \tag{40}$$

$$\Psi_0^{(0)}(\beta r_0) = 1 - \tfrac{3}{4}(\beta r_0)^2 + \cdots,$$
$$\Psi_0^{(1)}(\beta r_0) = -\tfrac{1}{2}(1 - \tfrac{1}{9}\beta r_0 - \tfrac{43}{216}(\beta r_0)^2 + \cdots), \tag{41}$$

$$\Psi_0^{*(0)}(\beta r_0) = -\tfrac{3}{2}(\beta r_0)^2 + \cdots,$$
$$\Psi_0^{*(1)}(\beta r_0) = -(1 - \tfrac{1}{6}\beta r_0 - \tfrac{43}{108}(\beta r_0)^2 + \cdots). \tag{42}$$

Note also that

$$\Phi_0(0; \rho^2) = \rho^{-1}\sin\rho, \quad \Phi_0^*(0; \rho^2) = \cos\rho,$$
$$X_0(0; \rho^2) = \cos\rho, \quad X_0^*(0; \rho^2) = \rho\sin\rho. \tag{43}$$

Returning now to (29) and using the functions defined in (31)–(34) and (36)–(37), we have

$$C_0^2(\eta_+)q\cot\delta_+(q) + \beta h(\eta_+) = 1/A_+(q^2), \tag{44}$$

where

$$r_0^{-1}A_+(q^2) = B_+(\rho^2)/\Omega_+(\rho^2), \tag{45}$$

$$B_+(\rho^2) = \Phi_0^*(\beta r_0; \rho^2) - \Phi_0(\beta r_0; \rho^2)d_+(q^2),$$
$$\Omega_+(\rho^2) = X_0^*(\beta r_0; \rho^2) + X_0(\beta r_0; \rho^2)d_+(q^2). \tag{46}$$

Noting (43), it is clear how (44)–(46) generalize (9), (22) and (23). When βr_0 is small compared with unity, and when the average inner Coulomb potential is small compared with the average nuclear potential, as is the case for $\pi^+ p$ elastic scattering, the conditions under which the two-parameter expansion of $A_+(q^2)$ is reliable will be the same as in the strictly nuclear case. The s-wave amplitude $\mathscr{F}_+(q)$ is given in terms of $A_+(q^2)$ by

$$\mathscr{F}_+(q) = \frac{K_+(q)}{q[1 - iK_+(q)]} = \frac{C_0^2(\eta_+)A_+(q^2)}{1 - \beta h(\eta_+)A_+(q^2) - iqC_0^2(\eta_+)A_+(q^2)}, \tag{47}$$

where we have omitted the phase factor $\exp[2i\nu_+^{(p)}(0;q)]$, which we assume to be factored out of the total scattering amplitudes $\mathscr{F}_+(q;\Theta)$ and $\mathscr{G}_+(q;\Theta)$. If this is done in the analysis of very low energy $\pi^+ p$ experiments, the amplitude $\mathscr{F}_+(q)$ will be the quantity to be determined from the experiments as a function of q. Approximate values of the p-wave amplitudes and the value of $\nu_+^{(p)}(l;q) - \nu_+^{(p)}(0;q)$ will be required for the analysis. Equation (47) gives the correct low energy behaviour of $\mathscr{F}_+(q)$ and in practice up to say 35 or 40 MeV one will try to approximate $A_+(q^2)$ by the first two terms in its expansion in powers of q^2:

$$A_+(q^2) = a_+ + \alpha_+ q^2 + \cdots, \tag{48}$$

and to determine from the experiments the parameters a_+, α_+. We note here the low energy behaviour of the phase shift $\delta_+(q)$. From (44), (48), (30) and (35) we have

$$\delta_+(q) \sim \pi\beta a_+ \exp(-\pi\beta/q).$$

The final step in the analysis will be to calculate from the phenomenological parameters a_+, α_+ the strictly nuclear scattering length a_3 and curvature α_3. In Section 4 we shall estimate the corrections which need to be made to go from a_+, α_+ to a_3, α_3. For these calculations we shall fix the value of r_0 and assume certain input values of a_3, α_3. The precise values are not important; neither correction varies much as a_3, α_3 are varied within the limits set by our present rather rough knowledge of these quantities. What is of primary interest to us is the variation of $(a_+ - a_3)$ and $(\alpha_+ - \alpha_3)$ with r_0. As we shall see, the correction $(a_+ - a_3)$ is very small and, though it varies considerably with r_0, the uncertainty in the correction is much smaller than the present uncertainty in a_3. On the other hand, the correction $(\alpha_+ - \alpha_3)$ varies very little with r_0.

We give here the formulae with which the calculations in Section 4 are made. From a_3, α_3 (with r_0 fixed) we obtain $d_3^{(0)}$, $d_3^{(1)}$ from the expressions

$$r_0^{-1}a_3 = B_3^{(0)}/\Omega_3^{(0)}, \quad r_0^{-3}\alpha_3 = (\Omega_3^{(0)})^{-2}(\Omega_3^{(0)}B_3^{(1)} - \Omega_3^{(1)}B_3^{(0)}), \tag{49}$$

where B_3, Ω_3 are given in terms of d_3 by (23), so that

$$\begin{aligned} &B_3^{(0)} = 1 - d_3^{(0)}, \quad \Omega_3^{(0)} = d_3^{(0)}, \\ &B_3^{(1)} = -\tfrac{1}{2} + \tfrac{1}{6}d_3^{(0)} - d_3^{(1)}, \quad \Omega_3^{(1)} = 1 - \tfrac{1}{2}d_3^{(0)} + d_3^{(1)}. \end{aligned} \tag{50}$$

From $d_3^{(0)}$ and $d_3^{(1)}$ the quantities $U_3(0)$, $U_3'(0)$ are obtained via (18)–(20) with the subscript 3 on the various quantities. For a_+, α_+ we have expressions similar to (49), namely

$$r_0^{-1}a_+ = B_+^{(0)}/\Omega_+^{(0)}, \quad r_0^{-3}\alpha_+ = (\Omega_+^{(0)})^{-2}(\Omega_+^{(0)}B_+^{(1)} - \Omega_+^{(1)}B_+^{(0)}), \tag{51}$$

where

$$B_+^{(0)} = \Phi_0^{*(0)} - \Phi_0^{(0)} d_+^{(0)}, \quad \Omega_+^{(0)} = X_0^{*(0)} + X_0^{(0)} d_+^{(0)},$$
$$B_+^{(1)} = \Phi_0^{*(1)} - \Phi_0^{(1)} d_+^{(0)} - \Phi_0^{(0)} d_+^{(1)},$$
$$\Omega_+^{(1)} = X_0^{*(1)} + X_0^{(1)} d_+^{(0)} + X_0^{(0)} d_+^{(1)}. \tag{52}$$

The quantities $d_+^{(0)}$, $d_+^{(1)}$ are obtained from (18)–(20) with the subscript + on the various quantities, where

$$U_+(0) = U_3(0) + \overline{V}_+, \quad U'_+(0) = U'_3(0).$$

The coefficients in the expansions of the Coulomb functions, which appear in (52), are given by (39)–(42) and (36)–(37). In practice it is sufficient to take terms up to the first order only in (39)–(42), neglecting terms involving $(\beta r_0)^2$. The corrections $(a_+ - a_3)$, $(\alpha_+ - \alpha_3)$ have an obvious separation into inner and outer corrections. The inner corrections come from the change from $d_3^{(0)}$, $d_3^{(1)}$ to $d_+^{(0)}$, $d_+^{(1)}$ which results from the addition of $\overline{V}_+$ to $U_3(q^2)$. The outer corrections come from the nonzero value of βr_0. The inner and outer corrections are additive to first order. It is important to note that the expressions (8)–(11), (14)–(16) for the case where the potential is zero for $r > r_0$ and (44)–(48), (51)–(52) for the case where the potential for $r > r_0$ is the Coulomb potential of a spherically symmetric charge distribution of radius $\leqslant r_0$, are true generally. In setting out the theory we have taken the potential for $r < r_0$ to be constant, but this restriction is not necessary for the expressions listed above to be valid. In general, the potential $U(q^2; r)$ for $r < r_0$ will yield a regular radial wave function $R(q^2; r)$ for $r \leqslant r_0$ from which the quantity

$$d(q^2) = r_0 R'(q^2; r_0)/R(q^2; r_0)$$

is calculated, and it is this quantity $d(q^2)$ which appears in (8) or (46). Taking $U(q^2; r)$ to be constant for $r < r_0$ merely gives the simple expressions (7) or (18) for $d(q^2)$, which make practical calculations much easier.

Our final remarks in this section concern *pp* scattering. The theory given for $\pi^+ p$ scattering applies also to *pp* scattering, with only the obvious change that the nuclear potential is attractive. Using the parameters given earlier for *np* scattering we have

$$\beta r_0 = 0.083, \quad \overline{V} = 0.72 \text{ MeV}.$$

Assuming that the *np* potential given earlier is strictly nuclear, and adding to it the Coulomb potential, we find, using (51) and (52), that

$$a_{pp} = 9.55 \text{ fm}, \quad r_{\text{eff}} = 2.35 \text{ fm}.$$

The measured values are 7.68 fm, 2.65 fm respectively. Our crude model does very well in eliminating almost 90% of the difference between a_{np} and a_{pp}. The very large change is of course due to the inner Coulomb potential $\overline{V}$, which has a crucial effect since $\omega(0) r_0$ is so close to $\pi/2$. We are not interested here in refinements to our simple model.[3]) Its success leads us to hope that a similar model for the *s*-wave amplitudes of low energy $\pi^{\pm}p$ scattering processes, though very rough, will serve to provide a reasonable estimate of the (much smaller) electromagnetic corrections to the scattering lengths and curvatures which appear in the very low energy parametrizations of these amplitudes.

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[3]) It is well known that for *np* and *pp* scattering a more refined theory exists.

3. The Two Channel $(\pi^- p)$, $(\pi^0 n)$ System

The notation for this case was already established in [1]. We recall the Schrödinger equation (1-36) for the s-wave:

$$(\mathbf{1}_2 d^2/dr^2 + \mathbf{Q}^2 - 2\mathbf{M}\mathbf{U}(r) - 2\mathbf{M}\mathbf{V}(r))\mathbf{R}(r) = \mathbf{0}, \tag{53}$$

where $\mathbf{Q}$ is given in (1-16), $\mathbf{M}$ in (1-37) and $\mathbf{U}$, $\mathbf{V}$ in (4), (5) respectively, with U_α, V_- given by (1), (3) respectively. We shall label the channels by $-$, 0 as in [1].

Now consider a pair of linearly independent regular solutions $\mathbf{R}^{(1)}(r)$, $\mathbf{R}^{(2)}(r)$ of (53). They satisfy

$$\mathbf{R}^{(j)}(0) = \mathbf{O}, \quad j = 1, 2; \quad \det(\mathbf{P}'(0)) \neq 0; \quad \det(\mathbf{P}(r)) \neq 0, \quad r \neq 0 \tag{54}$$

where $\mathbf{P}(r)$ is the 2×2 matrix formed by writing $\mathbf{R}^{(1)}(r)$ and $\mathbf{R}^{(2)}(r)$ as its two columns. The matrix

$$\mathbf{\Phi}(r) = \mathbf{P}'(r)[\mathbf{P}(r)]^{-1} \tag{55}$$

does not depend on the particular pair of regular solutions $\mathbf{R}^{(j)}(r)$ chosen. For the transformation to any other pair of linearly independent solutions is given by

$$\mathbf{P}(r) \mapsto \mathbf{P}(r)\mathbf{N},$$

where $\mathbf{N}$ is a constant nonsingular 2×2 matrix. We prove next that

$$\mathbf{M}^{-1}\mathbf{\Phi}(r) = \mathbf{\Phi}^t(r)\mathbf{M}^{-1}, \tag{56}$$

where the superscript t denotes the transposed matrix. To save some writing we put

$$\mathbf{W} = \mathbf{U} + \mathbf{V}.$$

It then follows from (53) that

$$\mathbf{P}^t\mathbf{M}^{-1}\mathbf{P}'' = \mathbf{P}^t(2\mathbf{W} - \mathbf{M}^{-1}\mathbf{Q}^2)\mathbf{P}.$$

Since $\mathbf{W}^t = \mathbf{W}$ it follows from the transpose of (53) that

$$\mathbf{P}''^t\mathbf{M}^{-1}\mathbf{P} = \mathbf{P}^t(2\mathbf{W} - \mathbf{M}^{-1}\mathbf{Q}^2)\mathbf{P}.$$

Thus

$$\mathbf{0} = \mathbf{P}^t\mathbf{M}^{-1}\mathbf{P}'' - \mathbf{P}''^t\mathbf{M}^{-1}\mathbf{P} = (\mathbf{P}^t\mathbf{M}^{-1}\mathbf{P}' - \mathbf{P}'^t\mathbf{M}^{-1}\mathbf{P})'.$$

Since $(\mathbf{P}^t\mathbf{M}^{-1}\mathbf{P}' - \mathbf{P}'^t\mathbf{M}^{-1}\mathbf{P})$ vanishes for $r = 0$ by (54), it vanishes for all r and (56) is proved. We now define

$$\mathbf{d} = r_0\mathbf{M}^{-1/2}\mathbf{\Phi}(r_0)\mathbf{M}^{1/2}. \tag{57}$$

The matrix $\mathbf{d}$ is a function of q^2 (we shall expand it later in a power series in ρ^2) and, by (56), it is symmetric:

$$\mathbf{d}^t = \mathbf{d}. \tag{58}$$

The matrix function $\mathbf{P}(r)$ for the internal region $r < r_0$ joins smoothly at $r = r_0$ to the exterior function, which has the form

$$\mathbf{M}^{1/2}\mathbf{Q}^{-1/2}(\mathbf{f}(r)\boldsymbol{\varkappa} + \mathbf{g}(r)\boldsymbol{\sigma}), \qquad r > r_0, \tag{59}$$

where $\boldsymbol{\varkappa}$, $\boldsymbol{\sigma}$ are constant 2×2 matrices and

$$\mathbf{f}(r) = \begin{pmatrix} f_-(r) & 0 \\ 0 & f_0(r) \end{pmatrix}, \quad \mathbf{g}(r) = \begin{pmatrix} g_-(r) & 0 \\ 0 & g_0(r) \end{pmatrix}, \tag{60}$$

with

$$f_-(r) = F_0(\eta_-; qr), \quad g_-(r) = G_0(\eta_-; qr), \tag{61}$$

$$f_0(r) = \sin q_0 r, \quad g_0(r) = \cos q_0 r. \tag{62}$$

Note that

$$\mathbf{g}(r)\mathbf{f}'(r) - \mathbf{g}'(r)\mathbf{f}(r) = \mathbf{Q}. \tag{63}$$

On joining the function (59) and $\mathbf{P}(r)$ smoothly at $r = r_0$ and solving for $\boldsymbol{\varkappa}$, $\boldsymbol{\sigma}$ using (63), we have

$$\boldsymbol{\sigma} = \mathbf{M}^{-1/2}\mathbf{Q}^{-1/2}[\mathbf{f}'(r_0)\mathbf{P}(r_0) - \mathbf{f}(r_0)\mathbf{P}'(r_0)],$$
$$\boldsymbol{\varkappa} = \mathbf{M}^{-1/2}\mathbf{Q}^{-1/2}[-\mathbf{g}'(r_0)\mathbf{P}(r_0) + \mathbf{g}(r_0)\mathbf{P}'(r_0)],$$

so that, using (55) and (57),

$$\begin{aligned}\mathbf{K}^{-1} &= \boldsymbol{\varkappa}\boldsymbol{\sigma}^{-1} \\ &= \mathbf{Q}^{-1/2}[-r_0\mathbf{g}'(r_0) + \mathbf{g}(r_0)\mathbf{d}][r_0\mathbf{f}'(r_0) - \mathbf{f}(r_0)\mathbf{d}]^{-1}\mathbf{Q}^{1/2}.\end{aligned} \tag{64}$$

To see that $\mathbf{K}$ is symmetric, one uses (63) to write

$$\mathbf{K}^{-1} = -\mathbf{g}(r_0)[\mathbf{f}(r_0)]^{-1} + r_0\mathbf{Q}^{1/2}[r_0\mathbf{f}'(r_0)\mathbf{f}(r_0) - \mathbf{f}(r_0)\mathbf{d}\mathbf{f}(r_0)]^{-1}\mathbf{Q}^{1/2},$$

so that (58) implies that $\mathbf{K}^t = \mathbf{K}$.

Up to this point the formalism has been put in a form which applies equally to a system consisting of $n(>2)$ coupled two-body channels (for example, $(K^-p, K^0n, \pi^0\Lambda, \pi^+\Sigma^-, \pi^0\Sigma^0, \pi^-\Sigma^+)$) in which the potential matrix for $r < r_0$ is an arbitrary real symmetric one (with suitable restrictions on the behaviour of its elements at $r = 0$), while the potential matrix for $r > r_0$ is diagonal, each diagonal element being either zero or the Coulomb potential of a spherically symmetric charge distribution of radius $\leqslant r_0$. We now write explicit results for the (π^-p), (π^0n) system by expressing the Coulomb wave functions $F_0(\eta_-; qr_0)$ and $G_0(\eta_-; qr_0)$ and their derivatives in terms of the auxiliary functions introduced in (31)–(34) and (36)–(37). The change from η_+ to η_- will mean that the first argument in the functions Φ_0, Φ_0^*, ..., will be $-\beta r_0$ instead of βr_0. On the right sides of (33)–(34) and (36)–(37), $\beta r_0 \to -\beta r_0$ and $\ln \beta r_0$ remains unchanged. With the definition $\rho_0 = q_0 r_0$, equation (64) may be written explicitly, using (60)–(62), as

$$\begin{aligned} & r_0\begin{pmatrix} C_0(\eta_-) & 0 \\ 0 & 1 \end{pmatrix}\mathbf{Q}^{1/2}\mathbf{K}^{-1}\mathbf{Q}^{1/2}\begin{pmatrix} C_0(\eta_-) & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} X_0^*(-\beta r_0; \rho^2) + \beta r_0 h(\eta_-) & \\ \times\ \Phi_0^*(-\beta r_0; \rho^2) + d_{--}(\rho^2)\{X_0(-\beta r_0; \rho^2) & d_{0-}(\rho^2)\{X_0(-\beta r_0; \rho^2) \\ \qquad -\beta r_0 h(\eta_-)\Phi_0(-\beta r_0; \rho^2)\} & \qquad -\beta r_0 h(\eta_-)\Phi_0(-\beta r_0; \rho^2)\} \\ d_{0-}(\rho^2)\cos\rho_0 & \rho_0\sin\rho_0 + d_{00}(\rho^2)\cos\rho_0 \end{pmatrix} \\ &\times \begin{pmatrix} \Phi_0^*(-\beta r_0; \rho^2) - d_{--}(\rho^2)\Phi(-\beta r_0; \rho^2) & -d_{0-}(\rho^2)\Phi_0(-\beta r_0; \rho^2) \\ -d_{0-}(\rho^2)\rho_0^{-1}\sin\rho_0 & \cos\rho_0 - d_{00}(\rho^2)\rho_0^{-1}\sin\rho_0 \end{pmatrix}^{-1}. \end{aligned}$$

Thus

$$\begin{pmatrix} C_0(\eta_-) & 0 \\ 0 & 1 \end{pmatrix} \mathbf{Q}^{1/2}\mathbf{K}^{-1}\mathbf{Q}^{1/2} \begin{pmatrix} C_0(\eta_-) & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\beta h(\eta_-) & 0 \\ 0 & 0 \end{pmatrix} = [\mathbf{A}(q^2)]^{-1}, \tag{65}$$

where

$$\begin{aligned} r_0^{-1}\mathbf{A}(q^2) &= \begin{pmatrix} \Phi_0^*(-\beta r_0;\rho^2) - d_{--}(\rho^2)\Phi(-\beta r_0;\rho^2) & -d_{0-}(\rho^2)\Phi_0(-\beta r_0;\rho^2) \\ -d_{0-}(\rho^2)\rho_0^{-1}\sin\rho_0 & \cos\rho_0 - d_{00}(\rho^2)\rho_0^{-1}\sin\rho_0 \end{pmatrix} \\ &\times \begin{pmatrix} X_0^*(-\beta r_0;\rho^2) + d_{--}(\rho^2)X_0(-\beta r_0;\rho^2) & d_{0-}(\rho^2)X_0(-\beta r_0;\rho^2) \\ d_{0-}(\rho^2)\cos\rho_0 & \rho_0\sin\rho_0 + d_{00}(\rho^2)\cos\rho_0 \end{pmatrix}^{-1}. \end{aligned} \tag{66}$$

Equation (66), when written out in full, gives

$$r_0^{-1}\mathbf{A}(q^2) = \mathbf{B}(\rho^2)/\Omega(\rho^2), \tag{67}$$

where

$$\begin{aligned} B_{--}(\rho^2) &= \rho_0\sin\rho_0\Phi_0^*(-\beta r_0;\rho^2) - \rho_0\sin\rho_0\Phi_0(-\beta r_0;\rho^2)d_{--}(\rho^2) \\ &\quad + \cos\rho_0\Phi_0^*(-\beta r_0;\rho^2)d_{00}(\rho^2) - \cos\rho_0\Phi_0(-\beta r_0;\rho^2)D(\rho^2), \end{aligned} \tag{68}$$

$$B_{0-}(\rho^2) = -d_{0-}(\rho^2), \tag{69}$$

$$\begin{aligned} B_{00}(\rho^2) &= \cos\rho_0 X_0^*(-\beta r_0;\rho^2) + \cos\rho_0 X_0(-\beta r_0;\rho^2)d_{--}(\rho^2) \\ &\quad - \rho_0^{-1}\sin\rho_0 X_0^*(-\beta r_0;\rho^2)d_{00}(\rho^2) - \rho_0^{-1}\sin\rho_0 X_0(-\beta r_0;\rho^2)D(\rho^2), \end{aligned} \tag{70}$$

$$\begin{aligned} \Omega(\rho^2) &= \rho_0\sin\rho_0 X_0^*(-\beta r_0;\rho^2) + \rho_0\sin\rho_0 X_0(-\beta r_0;\rho^2)d_{--}(\rho^2) \\ &\quad + \cos\rho_0 X_0^*(-\beta r_0;\rho^2)d_{00}(\rho^2) + \cos\rho_0 X_0(-\beta r_0;\rho^2)D(\rho^2), \end{aligned} \tag{71}$$

and

$$D(\rho^2) = \det[\mathbf{d}(\rho^2)].$$

To calculate the matrix $\mathscr{F}$ of s-wave amplitudes we use the relation

$$\mathbf{Q}^{1/2}\mathscr{F}\mathbf{Q}^{1/2} = \mathbf{K}(\mathbf{1}_2 - i\mathbf{K})^{-1}; \tag{72}$$

since

$$\mathbf{S} = (\mathbf{1}_2 + i\mathbf{K})(\mathbf{1}_2 - i\mathbf{K})^{-1},$$

(72) is the same as (1-15) if the matrix $\mathbf{S}^{(p)}$ is omitted. Again we are assuming that, in the analysis of $\pi^- p$ experiments, the overall factor $\exp[2i\nu_-^{(p)}(0;q)]$ is omitted from the amplitudes $\mathscr{F}_-(q;\Theta)$, $\mathscr{G}_-(q;\Theta)$ and the factor $\exp[i\nu_-^{(p)}(0;q)]$ from the amplitudes $\mathscr{F}_{0-}(q;\Theta)$, $\mathscr{G}_{0-}(q;\Theta)$. From (65) and (72),

$$\begin{aligned} \mathscr{F}^{-1} &= \mathbf{Q}^{1/2}\mathbf{K}^{-1}\mathbf{Q}^{1/2} - i\mathbf{Q} \\ &= \begin{pmatrix} C_0(\eta_-) & 0 \\ 0 & 1 \end{pmatrix}^{-1}\left[\{\mathbf{A}(q^2)\}^{-1} - i\begin{pmatrix} C_0^2(\eta_-)q & 0 \\ 0 & q_0 \end{pmatrix} + \begin{pmatrix} \beta h(\eta_-) & 0 \\ 0 & 0 \end{pmatrix}\right] \\ &\quad \times \begin{pmatrix} C_0(\eta_-) & 0 \\ 0 & 1 \end{pmatrix}^{-1}. \end{aligned}$$

When the elements of $\mathscr{F}$ are written out in full, we have

$$\mathscr{F}_{-}(q) = [\Delta(q)]^{-1}C_0^2(\eta_-)[A_{--}(q^2) - iq_0 \det\{\mathbf{A}(q^2)\}], \tag{73}$$

$$\mathscr{F}_{0-}(q) = \mathscr{F}_{-0}(q) = [\Delta(q)]^{-1}C_0(\eta_-)A_{0-}(q^2), \tag{74}$$

$$\mathscr{F}_{00}(q) = [\Delta(q)]^{-1}[A_{00}(q^2) - iqC_0^2(\eta_-)\det\{\mathbf{A}(q^2)\} + \beta h(\eta_-)\det\{\mathbf{A}(q^2)\}], \tag{75}$$

where

$$\begin{aligned}\Delta(q) = 1 &+ \beta h(\eta_-)A_{--}(q^2) - iqC_0^2(\eta_-)A_{--}(q^2)\\ &- iq_0[A_{00}(q^2) + \beta h(\eta_-)\det\{\mathbf{A}(q^2)\}] - C_0^2(\eta_-)qq_0\det\{\mathbf{A}(q^2)\}.\end{aligned} \tag{76}$$

Throughout this section so far we have *not* used the assumption that the potential matrix $\mathbf{W} = \mathbf{U} + \mathbf{V}$ in the inner region ($0 \leqslant r \leqslant r_0$) has constant elements. Our results therefore apply generally to any well-behaved inner potential matrix $\mathbf{W}$. The matrix $\mathbf{W}(q^2; r)$ for $r < r_0$ determines the matrix $\mathbf{d}(q^2)$ via (55) and (59), and all the amplitudes are ultimately expressed in terms of $\mathbf{d}(q^2)$. The key assumption used so far is (5) for $\mathbf{V}$, with $V_-(r) = -\alpha/r$ for $r > r_0$.

We now calculate $\mathbf{d}(q^2)$ for the special case when $\mathbf{W}(q^2; r)$ has constant elements $\mathbf{W}(q^2)$ for $r \leqslant r_0$. This is needed for the calculations to be presented in Section 4. We recall (53):

$$(\mathbf{1}_2 d^2/dr^2 + \mathbf{Q}^2 - 2\mathbf{MW})\mathbf{R}(r) = \mathbf{0},$$

and assume that $\mathbf{W}$ is constant for *all* $r \geqslant 0$. The solution we seek has

$$\mathbf{R}(0) = \mathbf{0}, \quad \mathbf{R}'(0) = \mathbf{e},$$

where $\mathbf{e}$ is an arbitrary constant column vector. Taking the Laplace transform we have the algebraic equation

$$(p^2\mathbf{1}_2 + \mathbf{Q}^2 - 2\mathbf{MW})\mathbf{f}(p) = \mathbf{e},$$

which gives

$$\mathbf{f}(p) = [(p^2 + \omega^2)(p^2 - \lambda^2)]^{-1}\begin{pmatrix} p^2 + q_0^2 - 2m_0W_{00} & 2mW_{0-} \\ 2m_0W_{0-} & p^2 + q^2 - 2mW_{--}\end{pmatrix}\mathbf{e},$$

where

$$\begin{aligned}\omega^2 = &\tfrac{1}{2}(q^2 - 2mW_{--}) + \tfrac{1}{2}(q_0^2 - 2m_0W_{00})\\ &+ \tfrac{1}{2}[\{(q^2 - 2mW_{--}) - (q_0^2 - 2m_0W_{00})\}^2 + 16mm_0W_{0-}^2]^{1/2},\end{aligned} \tag{77}$$

$$\begin{aligned}\lambda^2 = &-\tfrac{1}{2}(q^2 - 2mW_{--}) - \tfrac{1}{2}(q_0^2 - 2m_0W_{00})\\ &+ \tfrac{1}{2}[\{(q^2 - 2mW_{--}) - (q_0^2 - 2m_0W_{00})\}^2 + 16mm_0W_{0-}^2]^{1/2}.\end{aligned} \tag{78}$$

Let

$$\begin{aligned}\cos^2\phi &= \frac{1}{2}\left\{1 + \frac{(q^2 - 2mW_{--}) - (q_0^2 - 2m_0W_{00})}{[\{(q^2 - 2mW_{--}) - (q_0^2 - 2m_0W_{00})\}^2 + 16mm_0W_{0-}^2]^{1/2}}\right\},\\ \sin^2\phi &= \frac{1}{2}\left\{1 - \frac{(q^2 - 2mW_{--}) - (q_0^2 - 2m_0W_{00})}{[\{(q^2 - 2mW_{--}) - (q_0^2 - 2m_0W_{00})\}^2 + 16mm_0W_{0-}^2]^{1/2}}\right\}.\end{aligned} \tag{79}$$

Then

$$\mathbf{f}(p) = \mathbf{M}^{1/2}\begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}(p^2+\omega^2)^{-1} & 0\\ 0 & (p^2-\lambda^2)^{-1}\end{pmatrix} \times \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\mathbf{M}^{-1/2}\mathbf{e},$$

$$\mathbf{R}(r) = \mathbf{M}^{1/2}\begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\omega^{-1}\sin\omega r & 0\\ 0 & \lambda^{-1}\sinh\lambda r\end{pmatrix} \times \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\mathbf{M}^{-1/2}\mathbf{e}, \quad r \geqslant 0.$$

Using (55) and (57),

$$\mathbf{d} = \begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\omega r_0\cot\omega r_0 & 0\\ 0 & \lambda r_0\coth\lambda r_0\end{pmatrix}\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}. \tag{80}$$

The above theory has been written out with the specific case of the $(\pi^- p)$, $(\pi^0 n)$ system in mind. Thus ω is close to $\omega(q^2)$ for $I = \frac{1}{2}$, λ is close to $\lambda(q^2)$ for $I = \frac{3}{2}$ and ϕ is close to the charge independent mixing angle $\arccos\sqrt{\frac{2}{3}}$. The deviations are due to the electromagnetic (inner Coulomb and mass difference) effects which have been incorporated into the model. If we write

$$d_\alpha = \omega r_0\cot\omega r_0, \quad d_\beta = \lambda r_0\coth\lambda r_0, \tag{81}$$

then (80) gives

$$\begin{aligned} d_{--} &= d_\alpha\cos^2\phi + d_\beta\sin^2\phi,\\ d_{0-} &= \sin\phi\cos\phi(d_\beta - d_\alpha),\\ d_{00} &= d_\alpha\sin^2\phi + d_\beta\cos^2\phi,\\ D &= d_\alpha d_\beta. \end{aligned} \tag{82}$$

The equations (73)–(75) (with the supplementary definition (76)) give the required parametrization of the s-wave amplitudes for the processes $\pi^- p \to \pi^- p$, $\pi^- p \leftrightarrow \pi^0 n$ and $\pi^0 n \to \pi^0 n$. To fit data up to 35 or 40 MeV, one will try to use a two term expansion of $\mathbf{A}(q^2)$:

$$\mathbf{A}(q^2) = \mathbf{a} + \boldsymbol{\alpha} q^2 + \cdots, \tag{83}$$

giving six parameters in all. However, in the charge independent limit (no mass differences, no Coulomb interaction),

$$a_{--} \to \tfrac{1}{3}(2a_1 + a_3), \quad a_{0-} \to \sqrt{2}/3(a_3 - a_1), \quad a_{00} \to \tfrac{1}{3}(a_1 + 2a_3), \tag{84}$$

and similarly with α replacing a. In Section 4 we shall calculate the differences between a_{--}, a_{0-}, a_{00} and the charge independent quantities given in (84) from which they differ by small electromagnetic corrections. Similarly we shall calculate the electromagnetic corrections to α_{--}, α_{0-}, α_{00}. The analysis of data from very low energy $\pi^\pm p$ experiments will then yield values of a_3, α_3, a_1, α_1. The connection of these parameters with actual $\pi^\pm p$ differential cross-section data is established via equations (1-17), (1-1), (47), (48), (73)–(76) and (83), together with the electromagnetic corrections discussed

in this section and the last. Approximate values of the p-wave amplitudes and the values of $[\nu_{\pm}^{(p)}(l;q) - \nu_{\pm}^{(p)}(0;q)]$ will be required for the analysis.

The computation of the electromagnetic corrections for the two channel case uses quite complicated formulae. However, the numerical work involved is straightforward, since no differential equations have to be solved numerically. Just as in the discussion of the π^+p case in Section 2, we shall fix the value of r_0 and start from assumed values of a_1, α_1. Again the exact values of these quantities are not important. From these we obtain $d_1^{(0)}$ and $d_1^{(1)}$ using (15) and (16) and $U_1(0)$, $U_1'(0)$ using (6), (12) and (13). Then $\mathbf{U}(0)$ and $\mathbf{U}'(0)$ are calculated from (4) and V_- for $r < r_0$ from (2) and (3). The quantities ω, λ and ϕ of (77)–(79) are functions of q^2 and from these equations (with $\mathbf{W} = \mathbf{U} + \mathbf{V}$) we obtain the coefficients $\omega^{(k)}$, $\lambda^{(k)}$ and $\phi^{(k)}$ ($k = 0, 1$) of the usual power series expansions in ρ^2. Note that, in obtaining the coefficients with $k = 1$, we require the second term in the expansion of q_0^2:

$$q_0^2(q^2) = q_0^2(0) + vq^2 + \cdots,$$

where

$$v = [4\mu M(\mu + M)^2]^{-1}[(M + \mu)^4 - (M_n^2 - \mu_0^2)^2].$$

Next, using (81) and (82), the values of $\mathbf{d}^{(k)}$ ($k = 0, 1$) are calculated from $\omega^{(k)}$, $\lambda^{(k)}$ and $\phi^{(k)}$. At this stage one goes back to (67)–(71) to compute $\mathbf{B}^{(k)}$, $\Omega^{(k)}$ ($k = 0, 1$) and thence $\mathbf{A}$ and $\boldsymbol{\alpha}$ from (83). The expressions for $\mathbf{B}^{(1)}$, $\Omega^{(1)}$ are very complicated, and it is not necessary to reproduce them here. The expressions for $\mathbf{B}^{(k)}$, $\Omega^{(k)}$ involve the coefficients $\Phi_0^{(k)}$, $\Phi_0^{*(k)}$, ..., in the expansions of the auxiliary Coulomb functions in powers of ρ^2; these are given by (39)–(42) and (36)–(37), but with $\beta r_0 \to -\beta r_0$ and $\ln \beta r_0$ unchanged. This completes the method of calculation of the electromagnetic corrections to a_{--}, a_{0-}, a_{00} and α_{--}, α_{0-}, α_{00}.

The very low energy behaviour of the amplitudes $\mathscr{F}_-$, $\mathscr{F}_{0-}$, $\mathscr{F}_{00}$ is interesting, since we shall consider in another paper the effect of the (γn) channel and will need the low energy behaviour of the differential cross-sections for π^-p charge exchange scattering and radiative capture. Within the context of the present model, which takes account of Coulomb and mass difference effects, it follows from (73)–(76), (30) and (35) that, for every small q,

$$\begin{aligned} \mathscr{F}_- &\sim [\Delta(0)]^{-1}\pi\beta(a_{--} - iq_0(0)\det \mathbf{a})q^{-1}, \\ \mathscr{F}_{0-} &\sim [\Delta(0)]^{-1}\sqrt{\pi\beta}a_{0-}q^{-1/2}, \\ \mathscr{F}_{00} &\approx [\Delta(0)]^{-1}(a_{00} - i\pi\beta \det \mathbf{a}), \end{aligned} \tag{85}$$

where

$$\Delta(0) = 1 - i\pi\beta a_{--} - \pi\beta q_0(0)\det \mathbf{a} - iq_0(0)a_{00}.$$

From (1-17) the differential cross-section for $\pi^-p \to \pi^0 n$, for very small q, has the asymptotic behaviour

$$\left(\frac{d\sigma}{d\Omega}\right)_{0-} \sim \pi\beta q_0(0)a_{0-}^2|\Delta(0)|^{-2}q^{-2}. \tag{86}$$

4. Numerical Calculations

In Sections 2 and 3 we have outlined the way in which the calculations to be presented in this section were done. The input values of a_3, α_3, a_1 and α_1 were taken

from the work of Samaranayake and Woolcock [5, 6], namely

$$a_3 = -0.0892\,\mu^{-1}, \quad a_1 = 0.1814\,\mu^{-1},$$
$$\alpha_3 = -0.0454\,\mu^{-3}, \quad \alpha_1 = -0.0055\,\mu^{-3}. \tag{87}$$

These values are not the most up to date values available for a_3, a_1, nor are they strictly nuclear quantities; the papers themselves make it clear that electromagnetic corrections were not made. We used the values (87) because they come from a single consistent analysis of pion–nucleon scattering data. More recent values of a_3, a_1 are given by Bugg et al. [7], namely

$$a_3 = -0.091\,\mu^{-1}, \quad a_1 = 0.168\,\mu^{-1}. \tag{88}$$

But these are not really strictly nuclear quantities either, and there are unsatisfactory features about the calculation which are discussed in Section 4 of [1] and also by Woolcock [8]. In fact the values (88) include the modification to $(a_1 - a_3)$ suggested in [8]. We shall give later for one value of r_0 the scattering length corrections which are obtained if the input values of a_α are those of (88) rather than (87). In fact the corrections are practically unchanged; the variation of the corrections with r_0 is far more important.

For the calculations we took three values of r_0, namely $r_0 = 0.7\,\mu^{-1}$, $r_0 = 1.0\,\mu^{-1}$ and $r_0 = 1.3\,\mu^{-1}$. This is the range of values of r_0 considered by Zimmermann [2] in his charge independent calculation in the first resonance region. Calculations by Clayton and Cook [9] of energy independent potentials which reproduce the s- and p-wave pion–nucleon scattering amplitudes from threshold up to several hundred MeV also indicate that for the s-waves the potential has a range of 1.0–1.5 μ^{-1}. This is some indication that, for the sort of estimates of the electromagnetic corrections which we are considering in our potential model, it is most reasonable to assume that r_0 lies somewhere between $0.7\,\mu^{-1}$ and $1.5\,\mu^{-1}$.

The method of calculation was fully explained in the previous sections. In Table I we give some parameters which appear in the calculation. The fixed input numbers are the parameters (87) and $q_0(0) = 0.2012$, $v = 0.9895$, $m = 0.87051\,\mu$, $m_0 = 0.84555\,\mu$.

In Table II we list the corrections as functions of r_0. With the scattering lengths (88) and with $r_0 = 0.7\,\mu^{-1}$, the corrections are

$$a_+ - a_3 = -0.0007\,\mu^{-1},$$
$$a_{--} - \tfrac{1}{3}(2a_1 + a_3) = -0.0020\,\mu^{-1},$$
$$a_{0-} - (\sqrt{2}/3)(a_3 - a_1) = +0.0020\,\mu^{-1},$$
$$a_{00} - \tfrac{1}{3}(a_1 + 2a_3) = +0.0009\,\mu^{-1},$$

Table I

Parameters appearing in the calculation of the electromagnetic corrections to the scattering lengths and curvatures for s-wave $\pi^\pm p$ scattering

	$r_0 = 0.7\,\mu^{-1}$	$r_0 = 1.0\,\mu^{-1}$	$r_0 = 1.3\,\mu^{-1}$
$U_3(0)$	$0.5288\,\mu$	$0.1721\,\mu$	$0.0762\,\mu$
$1 - 2mU_3'(0)$	0.3755	0.7772	0.8860
$U_1(0)$	$-0.6947\,\mu$	$-0.2566\,\mu$	$-0.1218\,\mu$
$1 - 2mU_1'(0)$	1.1211	1.0956	1.0774
$\overline{V}_+$	$0.0125\,\mu$	$0.0088\,\mu$	$0.0067\,\mu$
βr_0	0.0089	0.0127	0.0165

Table II
Electromagnetic corrections to the scattering lengths and curvatures for s-wave $\pi^{\pm}p$ scattering as functions of r_0

	$r_0 = 0.7\ \mu^{-1}$	$r_0 = 1.0\ \mu^{-1}$	$r_0 = 1.3\ \mu^{-1}$
$a_+ - a_3$	$-0.0006\ \mu^{-1}$	$-0.0003\ \mu^{-1}$	$+0.0002\ \mu^{-1}$
$a_{--} - \frac{1}{3}(2a_1 + a_3)$	$-0.0021\ \mu^{-1}$	$-0.0030\ \mu^{-1}$	$-0.0043\ \mu^{-1}$
$a_{0-} - (\sqrt{2}/3)(a_3 - a_1)$	$+0.0021\ \mu^{-1}$	$+0.0027\ \mu^{-1}$	$+0.0034\ \mu^{-1}$
$a_{00} - \frac{1}{3}(a_1 + 2a_3)$	$+0.0008\ \mu^{-1}$	$+0.0007\ \mu^{-1}$	$+0.0007\ \mu^{-1}$
$\alpha_+ - \alpha_3$	$-0.0008\ \mu^{-3}$	$-0.0010\ \mu^{-3}$	$-0.0012\ \mu^{-3}$
$\alpha_{--} - \frac{1}{3}(2\alpha_1 + \alpha_3)$	$-0.0001\ \mu^{-3}$	$+0.0002\ \mu^{-3}$	$+0.0004\ \mu^{-3}$
$\alpha_{0-} - (\sqrt{2}/3)(\alpha_3 - \alpha_1)$	$+0.0001\ \mu^{-3}$	$+0.0001\ \mu^{-3}$	$+0.0002\ \mu^{-3}$
$\alpha_{00} - \frac{1}{3}(\alpha_1 + 2\alpha_3)$	$+0.0013\ \mu^{-3}$	$+0.0014\ \mu^{-3}$	$+0.0016\ \mu^{-3}$

almost the same as in Table II. We conclude that the corrections are quite insensitive to the input values of a_3, a_1, α_3, α_1. The variation of the corrections with r_0 is significant, and it is important to compare the corrections with the present uncertainties in the determination of the parameters. Taking the rather wide range of values of r_0 discussed earlier, we give in Table III suggested values of the corrections and estimates of the probable uncertainties in these corrections; we also give for comparison the present experimental uncertainties in the values of the parameters. In the calculations of both Samaranayake and Woolcock [5, 6] and Bugg et al. [7] there are systematic uncertainties which are not taken into account in Table III; they may increase substantially the errors given in the last two columns of the table.

What emerges from Table III is that the corrections to the curvatures are small, and that the uncertainties in these corrections are extremely small, indeed negligible compared with the experimental accuracies likely to be achieved in the foreseeable future. The situation with respect to the scattering length corrections is somewhat different; in the case of $a_{--} - \frac{1}{3}(2a_1 + a_3)$ and $a_{0-} - (\sqrt{2}/3)(a_3 - a_1)$, the uncertainties which we have quoted for these corrections and the errors on the parameters quoted by Bugg et al. [7] are not greatly different.

Table III
Suggested values of the electromagnetic corrections to the scattering lengths and curvatures for s-wave $\pi^{\pm}p$ scattering, probable uncertainties in these corrections and the present experimental errors on the parameters

	Suggested values of the corrections	Probable uncertainties in the corrections	Experimental errors on the parameters	
			Refs. [5, 6]	Ref. [7]
$a_+ - a_3$	$-0.0001\ \mu^{-1}$	$0.0005\ \mu^{-1}$	$0.0045\ \mu^{-1}$	$0.0021\ \mu^{-1}$
$a_{--} - \frac{1}{3}(2a_1 + a_3)$	$-0.0034\ \mu^{-1}$	$0.0013\ \mu^{-1}$	$0.0047\ \mu^{-1}$	$0.0022\ \mu^{-1}$
$a_{0-} - (\sqrt{2}/3)(a_3 - a_1)$	$+0.0029\ \mu^{-1}$	$0.0008\ \mu^{-1}$	$0.0053\ \mu^{-1}$	$0.0019\ \mu^{-1}$
$a_{00} - \frac{1}{3}(a_1 + 2a_3)$	$+0.0007\ \mu^{-1}$	$0.0001\ \mu^{-1}$	$0.0027\ \mu^{-1}$	$0.0017\ \mu^{-1}$
$\alpha_+ - \alpha_3$	$-0.0011\ \mu^{-3}$	$0.0003\ \mu^{-3}$	$0.012\ \mu^{-3}$	
$\alpha_{--} - \frac{1}{3}(2\alpha_1 + \alpha_3)$	$+0.0002\ \mu^{-3}$	$0.0003\ \mu^{-3}$	$0.014\ \mu^{-3}$	
$\alpha_{0-} - (\sqrt{2}/3)(\alpha_3 - \alpha_1)$	$+0.0001\ \mu^{-3}$	$0.0001\ \mu^{-3}$	$0.011\ \mu^{-3}$	
$\alpha_{00} - \frac{1}{3}(\alpha_1 + 2\alpha_3)$	$+0.0015\ \mu^{-3}$	$0.0002\ \mu^{-3}$	$0.010\ \mu^{-3}$	

In practice we would expect the analysis of $\pi^{\pm}p$ experiments at very low energies to be carried out in the following way. Since it will be difficult to obtain good values of the four parameters a_3, a_1, α_3, α_1 from the analysis of very low energy $\pi^{\pm}p$ data alone, even with greatly improved experimental accuracy, it will be necessary to have an independent determination of a_3, a_1. This will come from a dispersion theoretical analysis of experimental data obtained across the first resonance (say from 80 MeV to 300 MeV). Higher statistics experiments in this region will yield scattering lengths with smaller errors than those of Bugg et al. [7], provided of course that it is still possible to make a 'charge independent' phase shift analysis of the more accurate experiments. From these values of a_3, a_1 input values of a_+, a_{--}, a_{0-} will be obtained by applying the corrections suggested in the first column of Table III. These input values should have quite small errors due to the errors on the experimental data in the first resonance region, but will have (perhaps larger) systematic errors arising from uncertainties in the 'charge independent' phase shift analysis of the data, in the subsequent dispersion theoretical analysis and in the electromagnetic corrections (these uncertainties being given in the second column of Table III). Using the input values of a_+, a_{--} and a_{0-}, the parametric forms given in this paper and the corrections given in the lower half of the first column of Table III, the analysis of the very low energy $\pi^{\pm}p$ experiments will then yield values of α_3, α_1.

There are two further points of interest. First, the value of a_{0-} differs from $(\sqrt{2}/3)(a_3 - a_1)$ by a significant amount. With the scattering lengths given in (88) we have

$$(\sqrt{2}/3)(a_3 - a_1) = -0.1221\,\mu^{-1}.$$

With the suggested correction in Table III,

$$a_{0-} = -0.1192\,\mu^{-1},$$

so that

$$\frac{2(a_3 - a_1)^2}{9a_{0-}^2} = 1.049 \pm 0.014. \tag{89}$$

We shall see in another paper that this correction is important in computing the value of the Panofsky ratio indirectly from $\pi^- p$ charge exchange data and pion photoproduction data; it makes a difference of around 5% in the calculated value of the Panofsky ratio.

The second point is that the corrections in Table III are small because of the numerical cancellation between the various components of the corrections. Thus in the case of $a_+ - a_3$, $\alpha_+ - \alpha_3$ one can separate the total correction into inner and outer corrections, as remarked in Section 2. For $r_0 = 0.7\,\mu^{-1}$ we have

$$(a_+ - a_3)^{(\text{inner})} = -0.0018\,\mu^{-1}, \quad (a_+ - a_3)^{(\text{outer})} = 0.0012\,\mu^{-1},$$
$$(\alpha_+ - \alpha_3)^{(\text{inner})} = +0.0005\,\mu^{-3}, \quad (\alpha_+ - \alpha_3)^{(\text{outer})} = -0.0013\,\mu^{-3}.$$

The cancellation between the variations of the inner and the outer corrections also gives a much reduced variation of the total correction with r_0. In the case of the corrections for the coupled $(\pi^- p)$, $(\pi^0 n)$ system four components can be distinguished: inner and outer Coulomb corrections and inner and outer mass difference corrections. The outer corrections come from the nonzero value of $-\beta r_0$ and from the difference between q and q_0; the inner corrections come from the modifications to **d** arising from

Table IV
The components of the electromagnetic corrections to a_{--}, a_{0-} and a_{00}.

Correction	$a_{--} - \frac{1}{3}(2a_1 + a_3)$	$a_{0-} - (\sqrt{2}/3)(a_3 - a_1)$	$a_{00} - \frac{1}{3}(a_1 + 2a_3)$
Outer Coulomb	$-0.0044\ \mu^{-1}$	$+0.0006\ \mu^{-1}$	$+0.0007\ \mu^{-1}$
Inner Coulomb	$+0.0034\ \mu^{-1}$	$-0.0006\ \mu^{-1}$	$+0.0001\ \mu^{-1}$
Outer mass difference	$-0.0005\ \mu^{-1}$	$+0.0013\ \mu^{-1}$	$-0.0046\ \mu^{-1}$
Inner mass difference	$-0.0006\ \mu^{-1}$	$+0.0008\ \mu^{-1}$	$+0.0047\ \mu^{-1}$
Total	$-0.0021\ \mu^{-1}$	$+0.0021\ \mu^{-1}$	$+0.0008\ \mu^{-1}$

$\overline{V}_-$ and from the differences between m and m_0 and between q and q_0. In Table IV we give the four components of the corrections to a_{--}, a_{0-} and a_{00}, for $r_0 = 0.7\,\mu^{-1}$. Again there is considerable cancellation of the components. One point of interest is that for the correction to a_{0-} the Coulomb corrections cancel completely while the mass difference corrections add to give the whole correction. The inclusion of mass differences in the model is thus essential to the result (89). We shall consider the effect of the radiative capture channel on very low energy $\pi^- p$ scattering in another paper.

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