An application of the third order JKWB-approximation method to prove absolute continuity. II, The estimates

Autor(en): **Rejto, P.A.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **50 (1977)**

Heft 4

PDF erstellt am: **09.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-114877

Helvetica Physica Acta, Vol. 50, (1977), Birkhäuser Verlag, Basel

An application of the third order JWKB-approximation method to prove absolute continuity II. The estimates

by **P. A. Rejto**[1])

School of Mathematics, University of Minnesota, 206 Church St., Minneapolis, MN 55455, USA
and
École de Physique, Département de Physique Théorique, Bvd d'Yvoy 32, CH-1211, Genève 4, Switzerland

(24. I. 1977)

Abstract. In this second part of this paper we supply the estimates needed to show that the previously constructed family of operators is an approximating family in the sense of the technical Definition 3.1. Then we employ the abstract theorem of the first part to conclude that the interior of the essential spectrum of our Schrödinger operator is absolutely continuous.

Introduction

In this second part of this paper we supply the estimates needed to prove Theorem 2.1 which was stated in the first half. More specifically we use these estimates to derive Theorem 2.1 from the abstract Theorem 3.1 which was also stated in the first half. For convenience we continue the numbering of sections, however we start anew the numbering of references.

In section 6 we prove that the family of operators $L(q(\mu))$ of Section 5 approximates the operator $L(p)$ of Theorem 2.1 in the technical sense of definition 3.1. We prove this fact by proving that this family of operators satisfies the assumptions of the intermediate Lemma 4.1. To prove that Condition $I(\mathscr{I})$ of this lemma holds for this family we need that to the given interval $\mathscr{I}$ there are open regions of the form $\mathscr{R}_{\pm}(\mathscr{I})$ such that each μ in $\mathscr{R}_{\pm}(\mathscr{I})$ is in the resolvent set of the operator $L(q(\mu))$. This is the statement of Theorem 6.1. We prove this theorem in turn, by showing that in spite of the non-Hermitian character of the operator $L(q(\mu))$ the Weyl construction for the resolvent kernel [1] can be carried out. This is, essentially, the statement of Lemma 6.1. To prove that Condition $II(\mathscr{I})$ holds we need estimates for the resolvent kernels of this family of operators. These estimates are formulated in Lemma 6.2. Having established Conditions $I(\mathscr{I})$ and $II(\mathscr{I})$ it is not difficult to show that this family of operators satisfies all of the assumptions of Lemma 4.1. Therefore we can conclude that this family does approximate the operator $L(p)$.

In section 7 we show that this family also satisfies Condition $A(\mathscr{I})$. This fact is implied by Theorem 7.1. The proof of this theorem is based on a version of the

[1]) Supported by NSF grant MCS 76–06013.

Rellich uniqueness property for the operator $L(p)$. This version is formulated in Lemma 7.1. Let us emphasize that the Rellich uniqueness property has been established for various classes of ordinary and partial differential operators by various authors, [2]–[12]. For a recent and general such class we refer to the report of Agmon–Hörmander [17].

6. The family of operators $L(q(\mu))$ approximates $L(p)$

Recall that in Section 5 to the given compact interval $\mathscr{I}$ and to p_2, the given long range part of the potential p of Theorem 2.1, we constructed a family of approximate potentials $q(\mu)$. At the same time in Lemma 5.2 we have shown that if in addition the interval $\mathscr{I}$ does not contain zero then p admits a long range part p_2, which satisfies Condition $0(\mathscr{I})$. Hence it is no loss of generality to assume that this is the case.

In this section we show that under general circumstances the corresponding family of operators, $L(q(\mu))$, approximates the operator $L(p)$ over such an interval. We shall show this fact by applying Lemma 4.1. to the family of potentials,

$$p(\mu) = q(\mu). \tag{6.1}$$

(*a*) *Condition* $I(\mathscr{I})$. The theorem that follows gives circumstances which ensure that the operator $L(q(\mu))$ satisfies assumption (3.11).

Theorem 6.1. *Let the potential* $q(\mu)$ *be defined by equation* (5.24) *and let the operator* $L(q(\mu))$ *be defined by replacing* p *by* $q(\mu)$ *in relation* (2.9). *Suppose that* $\mathscr{I}$ *is a compact subinterval of* $\mathscr{R}^+$. *Suppose, further, that the potential* p_2 *satisfies Condition* $0(\mathscr{I})$. *Then there are non-empty regions* $\mathscr{R}_\pm(\mathscr{I})$ *of the form* $(3.1)_\pm$ *such that each* μ *in* $\mathscr{R}_\pm(\mathscr{I})$ *is in the resolvent set of this operator; that is,*

$$\mu \in \rho(L(q(\mu))). \tag{6.2}$$

First one might be tempted to derive conclusion (6.2) from the fact that the spectrum of a self-adjoint operator is real [23]. However, the potential $q(\mu)$ is non-real and hence the operator $L(q(\mu))$ is not even Hermitian symmetric. Nevertheless, the special form of this potential allows us to formulate estimates, which in turn, allow us to carry out the basic part of the Weyl construction [1] for the resolvent kernel. These estimates are formulated in the lemma that follows.

Lemma 6.1. *Suppose that the potential* p_2 *satisfies Condition* $0(\mathscr{I})$. *With the aid of the function* $y(\mu)$ *of formula* (5.25) *define*

$$x(\mu)(\xi) = \int_0^\xi \frac{d\sigma}{y(\mu)(\sigma)^2} + 1, \tag{6.3}$$

and

$$z(\mu) = x(\mu)y(\mu). \tag{6.4}$$

Then each of the two functions $y(\mu)$ and $z(\mu)$ satisfies the differential equation (5.2). *Furthermore to each complex number μ there is a positive increasing function $v(\mu)$ and a constant γ such that,*

$$|y(\mu)(\xi)| \leqslant \gamma \exp(-v(\mu)(\xi)) \tag{6.5}$$

and

$$|z(\mu)(\xi)| \leqslant \gamma \exp(+v(\mu)(\xi)). \tag{6.6}$$

We have already seen in Section 5 that the function $y(\mu)$ satisfies equation (5.2). We stated this fact for completeness only.

To prove that the function $z(\mu)$ also satisfies equation (5.2) we need that according to elementary algebra,

$$(\mu - L(q(\mu)))\, z(\mu) = \hbar^2[2y'(\mu)x'(\mu) + y(\mu)x''(\mu)]. \tag{6.7}$$

It is not difficult to show that definition (6.3) implies that the right member is zero. In fact, this property motivated this definition.

To prove conclusion (6.5) we make essential use of the fact that the potential p_2 satisfies Condition $0(\mathscr{I})$. We see from assumption (5.28) of Condition $0(\mathscr{I})$ that the formula,

$$v(\mu)(\xi) = \operatorname{Re} \int_0^\xi (w^+(\mu) - a_0(\mu))\, \sigma\, d\sigma, \tag{6.8}$$

defines such a function. Conclusion $(5.13)_0$ of Lemma 5.1 shows that,

$$\exp\left(\int_0^\xi a_0(\mu)(\sigma)\, d\sigma\right) = \left|\frac{p_2(0) - \mu}{p_2(\xi) - \mu}\right|^{1/4} \tag{6.9}$$

Inserting this relation in definition (5.24) yields,

$$|y(\mu)(\xi)| = \left|\frac{p_2(0) - \mu}{p^2(\xi) - \mu}\right|^{1/4} \left|\exp\left(\int_0^\xi (w^-(\mu) - a_0(\mu))(\sigma)\, d\sigma\right)\right|. \tag{6.10}$$

According to definitions $(5.23)^\pm$,

$$w^+(\mu) - a_0(\mu) = -(w^-(\mu) - a_0(\mu)). \tag{6.11}$$

Relations (6.10), (6.11) and definition (6.8) together show that,

$$|y(\mu)(\xi)| = \left|\frac{p_2(0) - \mu}{p_2(\xi) - \mu}\right|^{1/4} \cdot \exp(-v(\mu)(\xi)). \tag{6.12}$$

According to assumption (5.26) of Condition $0(\mathscr{I})$ the supremum of the first factor is finite. Therefore we see from relation (6.12) the validity of conclusion (6.5).

To prove conclusion (6.6) we formulate an elementary proposition.

Proposition 6.1. *Let $f(\tau)$ and $g(\tau)$ be twice continuously differentiable complex valued functions of the real variable τ and define the function x by*

$$x(\xi) = \int_0^\xi f(\tau)g(\tau)\, d\tau. \tag{6.13}$$

Suppose that the function

$$\tau \to |f(\tau)| \quad \textit{is increasing.} \tag{6.14}$$

Then

$$|x(\xi)| \leqslant |f(\xi)| \left\{ 2 \sup_{0<\tau<\xi} \left| \frac{f(\tau)}{f'(\tau)} g(\tau) \right| + \int_0^\xi \left| \left(\frac{f(\tau)}{f'(\tau)} g(\tau) \right)' \right| d\tau \right\} \tag{6.15}$$

To prove conclusion (6.15) note that according to definition (6.13)

$$x(\xi) = \int_0^\xi f'(\tau) \frac{f(\tau)}{f'(\tau)} g(\tau)\, d\tau. \tag{6.16}$$

Hence an integration by parts yields,

$$x(\xi) = \left[f(\tau) \cdot \frac{f(\tau)}{f'(\tau)} g(\tau) \right]_0^\xi - \int_0^\xi f(\tau) \left(\frac{f(\tau)}{f'(\tau)} g(\tau) \right)' d\tau. \tag{6.17}$$

Inserting assumption (6.14) in this relation and using the triangle inequality we obtain the validity of conclusion (6.15).

Having established this proposition we return to the proof of conclusion (6.6). We apply this proposition to the functions

$$f(\mu)(\tau) = \exp\left(+2 \int_0^\tau (w^+(\mu) - a_0(\mu)(\sigma)\, d\sigma \right) \tag{6.18}$$

and

$$g(\mu)(\tau) = \sqrt{\frac{p_2(\tau) - \mu}{p_2(0) - \mu}}. \tag{6.19}$$

Remembering definition (6.8) we see that

$$|f(\mu)(\tau)| = \exp\left(+2 \int_0^\tau v(\mu)(\sigma)\, d\sigma \right). \tag{6.20}$$

Since the function $v(\mu)$ is increasing so is $|f(\mu)|$; that is to say this function satisfies assumption (6.14).

To estimate the right member of conclusion (6.15) recall relation (5.42). Combining it with definitions (6.18) and (6.19) we see that,

$$\frac{f(\mu)}{f'(\mu)} \cdot g(\mu) = \frac{1}{2b(\mu)} \sqrt{\frac{1}{p_2(0) - \mu}}. \tag{6.21}$$

It is clear from definition (5.41) that,

$$\sup_{\xi \in \mathscr{R}^+} \left| \frac{1}{b(\mu)(\xi)} \right| < \infty. \tag{6.22}$$

At the same time remembering definitions (5.34)[3] and (5.33) we see that,

$$b'(\mu) \in \mathfrak{L}(0^3(p_2), \mathfrak{A}(p_2 - \mu)^{-1/2}).$$

Combining this relation with estimate (5.39)[3] and with assumptions (2.10) and (5.26) we obtain that for each μ,

$$b'(\mu) \in \mathfrak{L}_1(\mathscr{R}^+) \quad \text{and hence} \left(\frac{f(\mu)}{f'(\mu)} \cdot g(\mu) \right)' \in \mathfrak{L}_1(\mathscr{R}^+). \tag{6.23}$$

Inserting estimates (5.22) and (5.23) in conclusion (6.15) of Proposition 6.1 we arrive at the existence of a constant γ, such that,

$$\left|\int_0^\xi f(\mu)(\sigma)g(\mu)(\sigma)\,d\sigma\right| < \gamma|f(\mu)(\xi)|.$$

Inserting definitions (6.3), (6.18), (6.19), and relation (6.20), in turn, in this estimate, we arrive at

$$|x(\mu)(\xi) - 1| \leq \gamma \exp\left(+2\int_0^\xi v(\mu)(\sigma)\,d\sigma\right). \tag{6.24}$$

Finally combining the already established conclusion (6.5) with definition (6.4) and with estimate (6.24) we arrive at the validity of conclusion (6.6). This completes the proof of Lemma 6.1.

Having established Lemma 6.1 we return to the proof of Theorem 6.1. Following the Weyl construction, first we set

$$y^1(\mu) = z(\mu) + \gamma(\mu)y(\mu), \qquad (6.25)^1$$

where the constant $\gamma(\mu)$ is determined from the boundary condition (2.8). We do not claim that to each μ in $\mathscr{R}_\pm(\mathscr{I})$ it is possible to find such a constant. We do claim, however, that there are non-empty regions of this form where this is possible. To prove this claim insert definition $(6.25)^1$ in the boundary condition (2.8). This shows that it is equivalent to the equation,

$$\gamma(\mu)(\cos\alpha - \sin\alpha w^-(\mu)(0)) = \sin\alpha w^+(\mu)(0) - \cos\alpha. \tag{6.26}$$

Next set,

$$w_\pm^+(\omega)(\xi) = \lim_{\epsilon\to+0} w^+(\omega \pm i\varepsilon)(\xi) \quad\text{and}\quad w_\pm^-(\omega)(\xi) = \lim_{\epsilon\to+0} w^-(\omega \pm i\varepsilon)(\xi). \qquad (6.27)_\pm^\pm$$

Then we see from relation (5.42) and from the assumption that $\mathscr{I}$ is a subinterval of $\mathscr{R}^+$ that for each of these four functions,

$$|\mathrm{Im}\; w_\pm^\pm(\omega)(0)| = |\sqrt{p_2(0) - \omega}\; b(\omega)(0)|. \qquad (6.28)_\pm^\pm$$

At the same time it follows from assumption (5.26) of Condition $0(\mathscr{I})$ that,

$$|p_2(0) - \omega| \neq 0. \tag{6.29}$$

Remembering formula (5.41) we see that,

$$p_2(0) = p_2''(0) = 0 \quad\text{implies}\quad b(\omega)(0) = \frac{1}{\hbar}. \tag{6.30}$$

We have not required these assumptions explicitly in Condition $0(\mathscr{I})$. Nevertheless the proof of Lemma 5.2 shows that it is no loss of generality to assume that these assumptions hold. These four relations together show that for μ close enough to the interval $\mathscr{I}$ the coefficient of $\gamma(\mu)$ in equation (6.26) does not vanish. In view of the compactness of $\mathscr{I}$ this, in turn, shows that there are regions $\mathscr{R}_\pm(\mathscr{I})$ of the form $(3.1)_\pm$ such that for each μ in $\mathscr{R}_\pm(\mathscr{I})$,

$$\cos\alpha - \sin\alpha w^-(\mu)(0) \neq 0. \tag{6.31}$$

Thus equation (6.26) admits a solution $\gamma(\mu)$ and the function $y^1(\mu)$ of definition $(6.25)^1$ satisfies the boundary condition (2.8). At the same time according to Lemma 6.1 it also satisfies equation (5.2).

Following the Weyl construction, secondly we set,

$$y^r(\mu) = y(\mu). \tag{6.25^r}$$

Then according to Lemma 6.1 this function also satisfies equation (5.2) and a slightly sharper version of conclusion (6.5) shows that it also satisfies the boundary condition at infinity. That is to say, for each non-real complete number μ,

$$y^r(\mu) \in \mathfrak{L}_2(1, \infty). \tag{6.26^r}$$

Following the Weyl construction thirdly we set,

$$W^{1r}(\mu, \mu) = y^1(\mu)(\xi)y^r(\mu)'(\xi) - y^r(\mu)(\xi)y^1(\mu)'(\xi) \tag{6.32}$$

and claim that

$$W^{1r}(\mu, \mu) = 1. \tag{6.33}$$

Since the differential equation (5.2) does not contain first order terms this Wronskian is independent of the variable ξ, [21]. Hence to prove relation (6.33) it suffices to evaluate the right member of definition (6.32) at $\xi = 0$. For this purpose recall definitions (5.25) and (6.3). They show, after an elementary calculation, that

$$y(\mu)(0)z(\mu)''(0) - y(\mu)'(0)z(\mu)(0) = 1.$$

Combining this relation with definition $(6.25)^1$ we obtain the validity of relation (6.33).

We complete the Weyl construction by defining a kernel by,

$$K(\mu)(\xi, \eta) = \begin{cases} y^1(\mu)(\eta)y^r(\mu)(\xi) & \eta \leq \xi, \\ y^1(\mu)(\eta)y^r(\mu)(\eta) & \eta \geq \xi. \end{cases} \tag{6.34}$$

It is not difficult to show that this kernel defines a bounded operator which is the inverse of the operator $(\mu I - L(q(\mu)))$. In fact, this follows by combining the proof of Proposition 6.1 with a proof used elsewhere [15]. For brevity we do not carry out this combination and consider the proof of Theorem 6.1 complete. We see from this theorem that the family of operators $L(q(\mu)))$ satisfies assumption (3.11).

To see that the family of potentials of definition (6.1) is related to the potential p of Theorem 2.1 by estimate (4.1) recall assumption (2.8). This shows that

$$|(p - q(\mu))(\xi)| \leq |p_1(\xi)| + |(p_2 - q(\mu))(\xi)|.$$

Inserting assumptions (2.2) and (5.27) in this inequality we obtain the validity of estimate (4.1) for the family of potentials of definition (6.1).

To see that for the family of potentials of definition (6.1) assumption (4.2) of Condition $I(\mathscr{I})$ holds, recall conclusions $(5.13)_0$ and $(5.13)_1$ of Lemma 5.1 and assumption (5.26) of Condition $0(\mathscr{I})$. Together they show that for each ω in $\mathscr{I}$ each of the two limits does exist,

$$\lim_{\varepsilon \to +0} a_0(\omega \pm i\varepsilon)(\xi) \quad \text{and} \quad \lim_{\varepsilon \to +0} a_1(\omega \pm i\varepsilon)(\xi).$$

At the same time it follows that these limits are uniform in ξ in any compact subset of $\mathscr{R}^+$. Inserting these facts in definitions (5.21) and (5.24) we obtain the validity of assumption (4.2).

(*b*) *Condition* II($\mathscr{I}$). Remembering definition (6.1) we see that assumptions (4.3) and (4.4) of Condition $II(\mathscr{I})$ are implied by the lemma that follows.

Lemma 6.2. *Let $\mathscr{I}$ be a compact subinterval of $\mathscr{R}^+$ and let the approximate potential $q(\mu)$ be defined by equation (5.24). Suppose that the potential p_2 satisfies Condition $0(\mathscr{I})$. Then the approximate resolvent kernel is such that,*

$$\sup_{\mu \in \mathscr{R}_\pm(\mathscr{I})} \sup_{(\xi,\eta) \in \mathscr{R}^\pm \times \mathscr{R}^\pm} |R(\mu, L(q(\mu)))(\xi, \eta)| < \infty. \tag{6.35}$$

Furthermore as μ converges to the point ω of $\mathscr{I}$ this kernel converges, uniformly in (ξ, η) in any compact subset of $\mathscr{R}^+ \times \mathscr{R}^+$ and ω in $\mathscr{I}$.

To prove conclusion (6.35) note that according to the proof of Theorem 6.1,

$$R(\mu, L(q(\mu)))(\xi, \eta) = K(\mu)(\xi, \eta). \tag{6.36}$$

Inserting conclusions (6.5) and (6.6) of Lemma 6.1 and definitions $(6.25)^{1,r}$ in formula (6.34) and remembering that the function $v(\mu)$ of Lemma 6.1 is increasing, we obtain,

$$|K(\mu)(\xi, \eta)| \leq \gamma(1 + |\gamma(\mu)|). \tag{6.37}$$

A repetition of the proof of relation (6.31) shows that,

$$\sup_{\mu \in \mathscr{R}_\pm(T)} |\gamma(\mu)| < \infty. \tag{6.38}$$

Inserting estimates (6.37) and (6.38), in turn, in relation (6.36) we arrive at the validity of conclusion (6.35).

Having established Conditions $I(\mathscr{I})$ and $II(\mathscr{I})$ we show that the family of potentials of definition (6.1) satisfies the remaining assumptions of Lemma 4.1. Accordingly set

$$T(\mu) = (L(p) - L(q(\mu)))R(\mu), L(q(\mu))). \tag{6.39}$$

To see that the operator $T(\mu)$ satisfies assumption (3.9) note that according to definition (2.1)

$$L(p) - L(q(\mu)) = M(p - q(\mu)). \tag{6.40}$$

Hence

$$T(\mu) = M(p - q(\mu))R(\mu, L(q(\mu))). \tag{6.41}$$

Assumptions (2.7) and (2.10) together with estimate (5.30) show that the potential $p - q(\mu)$ satisfies assumption (2.2). Combining this fact with the estimates of Lemma 6.1 allows us to apply to the operator $T(\mu)$ the compactness considerations in the book of Schechter [27]. This yields,

$$T(\mu) \text{ is compact in } \mathfrak{B}(\mathfrak{L}_2(\mathscr{R}^+)). \tag{6.42}$$

Since such an application was carried out elsewhere [16] for a related class of operators we do not carry it out presently.

To prove that the operator $T(\mu)$ of definition (6.39) satisfies assumption (3.10) first we prove that the set

$$\mathfrak{S} = M(n^{1/2})\mathfrak{C}_0(\mathscr{R}^+) \tag{6.43}$$

is such that

$$\mathfrak{S} \subset \mathfrak{H} = \mathfrak{L}_2(\mathscr{R}^+) \tag{6.44}$$

and

$$\mathfrak{S} \text{ is a dense subset of } \mathfrak{G} \tag{6.45}$$

and for each μ in $\mathscr{R}_{\pm}(\mathscr{I})$

$$T(\mu)\mathfrak{S} \subset \mathfrak{G}. \tag{6.46}$$

To see that relation (6.44) holds recall definition (4.6) and assumption (4.1) of Condition $I(\mathscr{I})$. Together they show that

$$n \in \mathfrak{L}_1(\mathscr{R}^+). \tag{6.47}$$

This, in turn, shows the validity of relation (6.44). To see that relation (6.45) holds recall the remark after Lemma 4.1. This remark together with the facts that $\mathfrak{C}_0(\mathscr{R}^+)$ is a dense subset of $\mathfrak{H}$ and that a unitary transformation maps each dense subset onto a dense subset yields the validity of relation (6.45). To see that relation (6.46) holds recall conclusion (6.35) of Lemma 6.2. It shows that

$$R(\mu, L(q(\mu))(\mathfrak{H} \cap \mathfrak{L}_1(\mathscr{R}^+)) \subset \mathfrak{C}(\mathscr{R}^+).$$

In other words this operator maps functions in this intersection into functions which are bounded and continuous on all of $\mathscr{R}^+$. It is clear from relation (6.47) that

$$M(n^{1/2})\mathfrak{C}_0(\mathscr{R}^+) \subset \mathfrak{H} \cap \mathfrak{L}_1(\mathscr{R}^+).$$

Combining these two relations with definition (6.43) yields,

$$R(\mu, L(q(\mu)))\mathfrak{S} \subset \mathfrak{C}(\mathscr{R}^+). \tag{6.48}$$

According to relation (6.41)

$$M\left(\frac{1}{n}\right)^{1/2} (T(\mu)\mathfrak{S}) = M\left(\frac{p - q(\mu)}{n^{1/2}}\right)R(\mu, L(q(\mu))\mathfrak{S}. \tag{6.49}$$

Definition (4.6) and assumption (4.1) of Condition $I(\mathscr{I})$ together show that

$$\frac{p - q(\mu)}{n^{1/2}} \in \mathfrak{H}. \tag{6.50}$$

Inserting relations (6.48) and (6.50) in relation (6.49) we obtain

$$M\left(\frac{1}{n}\right)^{1/2} (T(\mu)\mathfrak{S}) \subset \mathfrak{H}.$$

Inserting this relation in turn, in definition (4.7) we obtain the validity of relation (6.46).

Having established relations (6.44), (6.45) and (6.46) it is not difficult to show that the operator $T(\mu)$ satisfies assumption (3.10). Conclusion (6.32) of Lemma 6.2 and relations (6.41), (6.47), (6.50) together show that

$$M\left(\frac{1}{n}\right)^{1/2} T(\mu)M(n)^{1/2} \in \mathfrak{B}(\mathfrak{H}),$$

in fact, this operator is Hilbert–Schmidt. This, in turn, in view of the remark after Lemma 4.1 shows that

$$T(\mu)_{\mathfrak{G}} \in \mathfrak{B}(\mathfrak{G}). \tag{6.51}$$

Hence the validity of assumption (3.10) follows from relation (6.46) by closure.

7. The proof of Condition $A(\mathscr{I})$

We have shown in Section 6 that under general circumstances the family of operators $L(q(\mu))$ approximates the operator $L(p)$ over the interval $\mathscr{I}$. In this section we shall show that under the same general circumstances these operators satisfy Condition $A(\mathscr{I})$.

The proof of relation (6.51) shows that

$$T(\mu)_{\mathfrak{G}} \text{ is compact in } \mathfrak{B}(\mathfrak{G}). \tag{7.1}$$

According to Lemma 4.1, Condition $G_2(\mathscr{I})$ holds for this family of operators. Hence for each ω in $\mathscr{I}$,

$$T_{\pm}(\omega)_{\mathfrak{G}} \text{ is compact in } \mathfrak{B}(\mathfrak{G}). \tag{7.2}$$

Therefore each of the two limit operators $(I - T_{\pm}(\omega))_{\mathfrak{G}}$ is Fredholm of index zero [26a]. That is to say the one to one property implies bounded invertibility. This one to one property is the statement of the theorem that follows.

Theorem 7.1. *Let $\mathscr{I}$ be a compact subinterval of $\mathscr{R}^+$ which does not contain zero. Suppose that the potential p satisfies the assumptions of Theorem* 2.1 *and that its long range part, p_2, satisfies* Condition $0(\mathscr{I})$. *Suppose, further, that ω in $\mathscr{I}$ is an exceptional point and h in $\mathfrak{G}$ is a corresponding exceptional vector. That is to say,*

$$(I - T_+(\omega))_{\mathfrak{G}} h = 0 \quad \text{or} \quad (I - T_-(\omega))_{\mathfrak{G}} h = 0. \tag{7.3)$_\pm$}$$

Then

$$h = 0. \tag{7.4}$$

To prove conclusion (7.4) recall definition (4.6). It shows, together with relation (6.4) and the Schwarz inequality, that

$$h \in \mathfrak{G} \quad \text{implies} \quad h \in \mathfrak{L}_1(\mathscr{R}^+). \tag{7.5}$$

Thus we see from conclusion (6.35) of Lemma 6.2 that for each fixed positive number ξ the family of functions, $R(\mu, L(q(\mu)))(\xi, \eta)h(\eta)$ admits an integrable majorant. According to the second conclusion of Lemma 6.2 this family of functions converges as μ converges to the given point ω of $\mathscr{I}$. Hence the following limit does exist,

$$g_+(\omega)(\xi) = \lim_{\varepsilon \to +0} \int_0^\infty R(\omega + i\varepsilon, L(q(\omega + i\varepsilon))(\xi, \eta)h(\eta)d\eta. \tag{7.6}$$

At the same time it follows from formula (6.34) and relation (6.36) that setting

$$y_+^{1,r}(\omega)(\xi) = \lim_{\varepsilon \to +0} y^{1,r}(\omega + i\varepsilon)(\xi), \tag{7.7)^{1r}}$$

we have,

$$g_+(\omega)(\xi) = y_+^r(\omega)(\xi) \int_0^\xi y_+^1(\omega)(\eta)h(\eta)\, d\eta + y_+^1(\omega)(\xi) \int_0^\xi y_+^r(\omega)(\eta)h(\eta)\, d\eta. \tag{7.8}$$

First we show that formula (7.8) implies an asymptotic description of the function $g_+(\omega)$ at infinity. For this purpose recall conclusion $(5.13)_0$ of Lemma 5.1 and definition (5.41). They yield,

$$\lim_{\varepsilon \to +0} a_0(\omega \pm i\varepsilon) = a_0(\omega) = \overline{a_0(\omega)} \tag{7.9}$$

and

$$\lim_{\varepsilon \to +0} b(\omega \pm i\varepsilon) = b(\omega) = \overline{b(\omega)}. \tag{7.10}$$

Inserting these relations in relation (5.42) and remembering definition $(5.27)^{+}_{+}$, we obtain

$$w^{+}_{+}(\omega) - a_0(\omega) = \lim_{\varepsilon \to +0} \sqrt{p_2 - (\omega + i\varepsilon)}\, b(\omega).$$

Inserting assumption (5.26) of Condition $0(\mathscr{I})$, the assumption that ω is positive and definition (5.22), in turn, in this relation we obtain,

$$w^{+}_{+}(\omega) - a_0(\omega) = -i|\sqrt{p_2 - \omega}|\, b(\omega). \tag{7.11}$$

Hence

$$\operatorname{Re}(w^{+}_{+}(\omega) - a_0(\omega)) = 0. \tag{7.12}$$

Thus we see from conclusions (6.5) and (6.6) of Lemma 6.1 and from definitions (6.8) and $(7.7)^{1r}$ that

$$\sup_{\xi \in \mathscr{R}+} |y^{1,r}_{+}(\omega)(\xi)| < \infty. \tag*{$(7.13)^{1r}$}$$

Inserting relations $(7.13)^{1r}$ and (7.5) in formula (7.8) shows that the constant,

$$\gamma = \int_0^\infty y^{1}_{+}(\omega)(\eta)h(\eta)\, d\eta, \tag{7.14}$$

is such that,

$$g_+(\omega)(\xi) \sim \gamma y^{r}_{+}(\omega)(\xi), \quad \text{for} \quad \xi \sim \infty. \tag{7.15}$$

Secondly we show that formula (7.8) implies,

$$g_+(\omega)'' + (\omega - q_+(\omega))g_+(\omega) = h. \tag{7.16}$$

In fact, relations (7.5) and $(7.13)^{1r}$ together allow us to differentiate the terms in formula (7.8) formally. This yields the validity of relation (7.16) if we remember definitions $(6.25)^{1r}$ and that according to Lemma 6.1 each of the two functions $y(\mu)$ and $z(\mu)$ satisfy equation (5.2). Incidentally note that in contrast to the case of the second order JWKB-approximation, in our present case,

$$q_+(\omega) = \lim_{\varepsilon \to +0} q(\omega + i\varepsilon) \neq \lim_{\varepsilon \to +0} q(\omega - i\varepsilon) = q_-(\omega).$$

Thirdly we note that formula (7.8) implies that the function $g_+(\omega)$ satisfies the boundary condition (2.8). Again, this follows by combining formal differentiation with relations (7.5) and $(7.13)^{1r}$. For brevity we omit the details of the proof of this fact.

Next we make essential use of the fact that ω is an exceptional point and h is a corresponding exceptional vector. For brevity assume that assumption $(7.3)_+$ holds. Then we observe that this relation allows us to eliminate the function h from the differential equation (7.16). In fact, inserting definition (7.6) and relation (6.41) in assumption $(7.3)_+$ yields,

$$h = (p - q_+(\omega))g_+(\omega). \tag{7.17}$$

Inserting this relation, in turn, in the differential equation (7.16) yields,

$$g_+(\omega)'' + (\omega - p)g_+(\omega) = 0. \tag{7.18}$$

In the following lemma we show that for this differential equation a version of the Rellich uniqueness property [5] holds. In other words the solution of the improper Cauchy problem near infinity is unique.

Lemma 7.1. *Suppose that the function $g_+(\omega)$ satisfies the differential equation* (7.18). *Suppose, further, that*

$$\lim_{\xi\to\infty} g_+(\omega)(\xi) = 0 \quad \textit{and} \quad \lim_{\xi\to\infty} g_+(\omega)'(\xi) = 0. \tag{7.19}$$

Then $g_+(\omega)$ has compact support.

Following Titchmarsh [19] and Miranker [3] we derive Lemma 7.1 from an integral representation of the function $g_+(\omega)$. To formulate this representation set

$$C_+(\omega)(\xi, \eta) = y_+^1(\omega)(\xi)y_+^1(\omega)(\eta) - y_+^1(\omega)(\eta)y_+^r(\omega)(\xi). \tag{7.20}$$

Then we show that

$$g_+(\omega)(\xi) = \int_\xi^\infty C_+(\omega)(\xi, \eta)((p - q_+(\omega))(\eta)\, d\eta. \tag{7.21}$$

To see that the integral on the right does exist note that according to relations $(7.13)^{1r}$

$$\sup_{\xi\in\mathscr{R}^+}\ \sup_{\xi\in\mathscr{R}^+} |C_+(\omega)(\xi, \eta)| < \infty \tag{7.22}$$

and that according to assumption (4.1) of Condition $I(\mathscr{I})$ and definition (6.1)

$$p - q_+(\omega) \in \mathfrak{L}_1(\mathscr{R}^+). \tag{7.23}$$

Let the transformation $C_+(\omega)$ mapping $\mathfrak{L}_1(\mathscr{R}^+)$ into $\mathfrak{C}^1(\mathscr{R}^+)$ be defined by the kernel of definition (7.20). Specifically let,

$$C_+(\omega)f(\xi) = \int_\xi^\infty C_+(\omega)(\xi, \eta)f(\eta)\, d\eta, \quad f \in \mathfrak{L}_1(\mathscr{R}^+). \tag{7.24}$$

Inserting relations (5.2) and (6.33) in definition (7.24) we obtain,

$$(C_+(\omega)f)'' + (\omega - q_+(\omega))C_-(\omega)f = f. \tag{7.25}$$

Assumption (7.19) and relation (7.23) together show that the function h of relation (7.17) is such that

$$h \in \mathfrak{L}_1(\mathscr{R}^+). \tag{7.26}$$

Hence we may apply relation (7.25) to this function and conclude that

$$(C_+(\omega)h)'' + (\omega - q_+(\omega))C_+(\omega)h = h. \tag{7.27}$$

Combining relations (7.27) and (7.16) we see that each of the two functions $g_+(\omega)$ and $C_+(\omega)h$ satisfies the same inhomogeneous differential equation. Therefore there are constants γ_+^1 and γ_+^r such that

$$g_+(\omega) = C_+(\omega)h + \gamma_+^1 y_+^1(\omega) + \gamma_+^r y_+^r(\omega). \tag{7.28}$$

We claim that each of these two constants is zero,

$$\gamma_+^1 = \gamma_+^r = 0. \tag{7.29^{1r}}$$

To see this note that differentiating relation (7.28) yields the following system of linear equations for these constants,

$$\begin{aligned} \gamma^1_+ y^1_+(\omega) + \gamma^r{}_+ y^r_+(\omega) &= g_+(\omega) - C_+(\omega)h \\ \gamma^1_+ y^1_+(\omega)' + \gamma^r_+ y^r_+(\omega)' &= g_+(\omega)' - (C_+(\omega)h)'. \end{aligned} \tag{7.30}$$

According to estimate (7.22) and relations (7.15), (7.23) and (6.33),

$$\lim_{\xi\to\infty} C_+(\omega)h(\xi) = \lim_{\xi\to\infty} (C_+(\omega)h)'(\xi) = 0. \tag{7.31}$$

Inserting relation (7.31) and assumption (7.19) in relation (7.30) and using relation (6.33) and estimates $(7.13)^{1r}$ again we obtain the validity of relations $(7.29)^{1r}$. Inserting relations $(7.29)^{1r}$, in turn, in relation (7.28) and using definition (7.24) we obtain the validity of relation (7.21).

To derive Lemma 7.1 from relation (7.21) note that according to estimate (7.22) and relation (7.23) there is a number $\tilde{\xi}$ such that

$$\theta = \int_{\tilde{\xi}}^{\infty} \sup_{\xi\in\mathscr{R}^+} \sup_{\eta\in\mathscr{R}^+} |C_+(\omega)(\xi,\eta)((p - q_+(\omega))g_+(\omega))(\eta)|\, d\eta < 1. \tag{7.32}$$

Inserting relation (7.32) in relation (7.21) we arrive at

$$\sup_{\xi>\tilde{\xi}} |g_+(\omega)(\xi)| \leqslant \theta \sup_{\eta>\tilde{\xi}} |g_+(\omega)(\eta)|. \tag{7.33}$$

Since the number θ is strictly positive and strictly less than 1 the support of the function $g_+(\omega)$ is contained in the interval $[0, \tilde{\xi}]$. This completes the proof of Lemma 7.1.

We complete the proof of Theorem 7.1 by showing that the function $g_+(\omega)$ of definition (7.6) satisfies the assumptions of Lemma 7.1. We have already seen that it satisfies the differential equation (7.18). To see that it also satisfies assumption (7.19) we claim that there is a non-zero complex number δ such that the function $\delta g_+(\omega)$ is real; that is,

$$\delta g_+(\omega) = \overline{\delta g_+(\omega)}, \quad \delta \neq 0. \tag{7.34}$$

To prove this recall relation (7.8) which shows that the function $g_+(\omega)$ satisfies the real boundary condition (2.8). Clearly in $\mathscr{C}_2$ the ortho-complement of a one-dimensional subspace which is spanned by a real vector can also be spanned by a real vector. This fact together with the reality of the coefficients of equation (7.18) and the uniqueness of the Cauchy problem for this equation yields the validity of relation (7.34). Inserting relation (7.34) in the asymptotic formula (7.15) yields,

$$\operatorname{Im}(\delta\gamma y^r_+(\omega)(\xi)) \sim 0, \quad \text{for} \quad \xi \sim \infty.$$

In other words,

$$\operatorname{Im}(\delta\gamma)\operatorname{Re}(y^r_+(\omega)(\xi)) + \operatorname{Re}(\delta\gamma)\operatorname{Im}(y^r_+(\omega)(\xi)) \sim 0, \quad \text{for} \quad \xi \sim \infty. \tag{7.35}$$

To analyze the real and imaginary parts of the function $y^r_+(\omega)$ set

$$\delta(\xi) = |y^r_+(\omega)(\xi)| \tag{7.36}$$

and

$$\alpha(\xi) = \arg(y^r_+(\omega)(\xi)), \quad \alpha(0) = 0. \tag{7.37}$$

Definitions $(6.27)^{+}_{+}$, $(6.25)^{r}$ and $(7.7)^{r}$ together show that,

$$y^{r}_{+}(\omega)(\xi) = \exp\left(\int_0^{\xi} w^{+}_{+}(\omega)(\sigma)\, d\sigma\right).$$

Inserting relations (7.9), (7.10), (7.11) and (7.12) in this formula yields,

$$\rho(\xi) = \exp\left(\int_0^{\xi} a_0(\omega)(\sigma)\, d\sigma\right) \tag{7.38}$$

and

$$\alpha(\xi) = -\int_0^{\xi} \sqrt{p_2(\sigma) - \omega}\; \overline{\omega}\; b(\omega)(\sigma)\, d\sigma. \tag{7.39}$$

Combining relations (7.39), $(5.39)^{0,2}$, assumption (5.26) of Condition $0(\mathscr{I})$ and definition (5.41) we see that there is a sequence $\{\xi_n\}$, such that

$$\lim_{n\to\infty} \xi_n = \infty \quad \text{and} \quad \operatorname{Re}(y^{r}{}_{+}(\omega)(\xi_n)) = \rho(\xi_n) \quad \text{and} \quad \operatorname{Im}(y^{r}_{+}(\omega)(\xi_n)) = 0. \tag{7.40}$$

Combining relations (7.38) and $(5.39)^{0}$, in turn, with conclusion $(5.13)_0$ of Lemma 5.1 we see that

$$\lim_{\xi\to\infty} \rho(\xi) = \left|\frac{\omega}{p_2(0) - \omega}\right|^{1/4} \neq 0. \tag{7.41}$$

Inserting relations (7.41) and (7.40) in relation (7.35) and remembering definition (7.36) we obtain

$$\operatorname{Im}(\delta\gamma) = 0. \tag{7.42}$$

Similarly we obtain that

$$\operatorname{Re}(\delta\gamma) = 0. \tag{7.43}$$

Combining relations (7.42), (7.43) and (7.34) we arrive at $\gamma = 0$. Combining this relation, in turn, with the asymptotic formula (7.15) we arrive at the validity of assumption (7.19). Therefore we can conclude from Lemma 7.1 that the function $g_{+}(\omega)$ has bounded support. This fact together with the uniqueness of the backward Cauchy problem shows that this function is identically zero. Inserting this fact in relation (7.17) we arrive at the validity of conclusion (7.4). This completes the proof of Theorem 7.1.

8. The proof of Theorem 2.1

Let $\mathscr{I}$ be a given compact subinterval of $\mathscr{R}^{+}$ which does not contain zero. Then according to Lemma 5.2 the potential p of Theorem 2.1 admits a decomposition of the form $p = p_1 + p_2$, where p_1 is short range and p_2 satisfies Condition $0(\mathscr{I})$. Let $q(\mu)$ be the family of approximate potentials defined by inserting this particular p_2 in definition (5.24). Then according to Section 6 the family of operators $L(q(\mu))$ approximates $L(p)$ over such an interval $\mathscr{I}$. According to Section 7 Condition $A(\mathscr{I})$ holds as well. These facts together with the compactness relation (6.42) allow us to conclude from the abstract Theorem 3.1 that

$$L(p)(\mathscr{I}) = L(p)(\mathscr{I})_{\mathrm{ac}}. \tag{8.1}$$

Relation (8.1), in turn, together with the countable additivity of the spectral projectors yields the validity of conclusion (2.12). This completes the proof of Theorem 2.1.

REFERENCES

[1] H. WEYL, Über gewöhnliche Differential-Gleichungen mit Singularitäten und die zugehörigen Entwicklungen wilkürlicher Funktionen. Math. Ann. *68*, 220–269 (1910).

[2] F. RELLICH, Über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten, Her. Deutsch. Math. Verein *53*, 57–65 (1943).

[3] W. L. MIRANKER, The reduced wave equation in a medium of variable index of refraction. Comm. Pure Appl. Math. *10*, 491–502 (1957). See Theorem 1.

[4] T. KATO, Growth properties of solutions of the reduced wave equation with a variable coefficient. Comm. Pure Appl. Math. *12*, 403–425 (1959).

[5] W. LITTMAN, Decay at infinity of solutions to partial differential equations. Trans. Amer. Math. Soc. *123*, 449–459 (1966). See Theorem 1.

[6] WILLI JÄGER, Zur Theorie der Schwingungs-gleichung mit variablen Koefficienten in Aussengebieten. Math. Z. *101*, 62–88 (1967). See Hilfssatz 2.

[7] B. SIMON, On positive eigenvalues of one-body Schrödinger operators. Comm. Pure Appl. Math. *22*, 531–538 (1969).

[8] W. LITTMAN, Decay at infinity of solutions to higher order partial differential equations: removal of the curvature assumption. Israel J. Math. *8*, 403–407 (1970).

[9] S. AGMON, Lower bounds for solutions of Schrödinger equations. J. D'Analyse Math. *23*, 1–25 (1970). asymptotic expansions and maximal rates of decay along a ray. Annali Scuola Norm. Sup. Pisa. *26*

[10] WILLI JÄGER, Ein gewöhnlicher Differentialoperator zweiter Ordnung für Funktionen mit Werten in einem Hilbertraum. Math. A. *113*, 68–98 (1970).

[11] J. R. SCHULENBERGER and C. H. WILCOX, A Rellich uniqueness theorem for steady-state wave propagation in homogeneous anisotropic media. Arch. Rat. Mech. Anal. *41*, 18–45 (1971).

[12] ORAZIO ARENA and WALTER LITTMAN, Farfield behavior of solutions to partial differential equations: (1972).

[13] VOLKER VOGELSANG, Das Strahlungs-Problem für elliptische Differential-Gleichungen in Gebieten mit unbeschränktem Rand. Math Z. *144*, 101–124 (1975).

[14] K. H. JANSEN and H. KALF, On positive eigenvalues of one-body Schrödinger operators. Remarks on papers by Agmon and Simon. Comm. Pure Appl. Math. *28*, 747–752 (1975).

[15] P. A. REJTO, On the resolvents of a family of non-self-adjoint operators. Lett. Math. Phys. *1*, 111–118 (1976).

[16] P. A. REJTO, On a theorem of Titchmarsh-Kodaira-Weidmann concerning absolutely continuous operators, II. Indiana Univ. Math. J. (1976). See the proof of Lemma 4.3.

[17] S. AGMON and L. HORMANDER, Asymptotic properties of solutions of differential equations with simple characteristics. Institut Mittag-Leffler (1974).

[18] E. A. CODDINGTON and N. LEVINSON, *Theory of Ordinary Differential Equations* (McGraw-Hill 1955). See Section III.8.

[19] E. C. TITMARSH, *Eigenfunction Expansions Associated With Second-Order Differential Equations*, Part I (Oxford, Clarendon Press, second edition 1962). See Section V.10.

[20] P. HARTMANN, *Ordinary Differential Equations* (John Wiley, 1964).

[21] W. WASOW, *Asymptotic Expansions For Ordinary Differential Equations* (Wiley-Interscience, 1965).

[22] W. A. COPPEL, *Stability And Asymptotic Behavior of Differential Equations* (Heath & Co., 1965).

[23] T. KATO, *Perturbation Theory For Linear Operators* (Springer-Verlag, 1966).

[24] G. HELLWIG, *Differentialoperatoren der mathematischen Physik* (Springer-Verlag, 1964). See Subsection V.2.1.

[25] M. A. NEUMARK, *Linear Differential Operators.* (F. Ungar Co.) English translation edited by W. N. Everitt.

[26] N. DUNFORD and J. T. SCHWARTZ, *Linear Operators Part II.* (Wiley-Interscience, 1967). (a) Theorem XIII.3.12; (b) Section VI.11.

[27] Schechter, Martin, *Spectra of Partial Differential Operators* (North-Holland–Amer. Elsevier, 1971). See the proofs of Theorems 3.1, 2.2, and 2.1 in Chapter 5.

ADDENDUM

[28] L. HÖRMANDER, Lower bounds at infinity for solutions of differential equations with constant coefficients. Israel J. Math. *16*, 103–116 (1973).