

Distinguished self-adjoint extension for Dirac operator with potential dominated by multicenter Coulomb potentials

Autor(en): **Nenciu, G.**

Objektyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **50 (1977)**

Heft 1

PDF erstellt am: **10.08.2024**

Persistenter Link: <https://doi.org/10.5169/seals-114843>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Distinguished self-adjoint extension for Dirac operator with potential dominated by multicenter Coulomb potentials

by **G. Nenciu**

Institut of Atomic Physics, Bucharest, Romania

(20.V.1976)

Abstract. The existence and the uniqueness of the distinguished self-adjoint extension of the Dirac operator describing an electron in the field of a finite number of point charges with $Z < 137$ is proved.

In [1] we proved some general results about perturbation of non-semibounded self-adjoint operators by quadratic forms. These results were applied to obtain distinguished self-adjoint extensions for Dirac operators with singular potentials. Let H_m be the free particle Dirac operator and $V(x)$ the 4×4 symmetric matrix valued function which represents the potential. Then, one of the results in [1] is

Theorem 1. If

$$\| \| V(x) \| \| \leq v/|x|, \quad 0 \leq v < 1 \quad (1)$$

(here $\| \cdot \|$ means the usual 4×4 matrix norm and in our system of units $v = 1$ correspond to atomic number $Z = 137$) then there exists a unique self-adjoint operator H such that $f \in D(H)$ implies $f \in D(|H_m|^{1/2})$, and

$$(g, Hf) = (H_m g, f) + (Vg, f); \quad g \in D(H_m), \quad f \in D(H). \quad (2)$$

The operator H has the property

$$\sigma_{\text{ess}}(H) \subset \sigma_{\text{ess}}(H_m). \quad (3)$$

The aim of this letter is to prove the following generalization of the above result.

Theorem 1'. The conclusions of Theorem 1 remain valid if the condition (1) is replaced by

$$V(x) = \sum_{i=1}^N V_i(x); \quad \| \| V_i(x) \| \| \leq v_i/|x - x_i| \quad 0 \leq v_i < 1, \quad x_i \neq x_j, \quad N < \infty, \quad (1')$$



Remarks

1. The result contained in Theorem 1' is relevant to the bound state problem in heavy ion scattering [2].

2. Similar arguments as in the proof of Theorem 1' below, lead to the fact that the Rellich theorem [3, Th. 6.3] gives the essential self-adjointness of $H_m + V$ on $\{C_0^\infty(R^3 \setminus \{\cup_i x_i\})\}^4$ for V satisfying (1') with $0 \leq v_i \leq \frac{1}{2}$ which generalizes some results in [4].

3. Combining the proofs of Theorem 5.1 in [1] and of the Theorem 1'' one can prove a more general form of Theorem 1'. More precisely one has

Theorem 1''. *The conclusions of Theorem 1 remain valid if (1') is replaced by*

$$V(x) = V_1(x) + V_2(x)$$

where $V_1(x)$ satisfies (1') and $V_2(x)$ is nonsingular (see [1, Def. 2.1 and Def. 5.1]).

Proof of Theorem 1'. Let $d = \min_{i \neq j} |x_i - x_j|$, $\varphi(t) \in C^1([0, \infty))$ such that $\varphi(t) = 1$ for $t < d/4$, $\varphi(t) = 0$ for $t > d/3$ and $k_s(t)$ defined by

$$k_s^2(t) = \begin{cases} 1 - (m + s)t + (m^2 + s^2)t^2 & \text{for } 0 \leq t \leq (m + s)/(m^2 + s^2) \\ 1 & \text{for } t > (m + s)/(m^2 + s^2). \end{cases} \quad (4)$$

Let

$$\begin{aligned} W(x) &= V(x) - \sum_{i=1}^N \varphi(|x - x_i|) k_s^2(|x - x_i|) V_i(x) \\ &\equiv V(x) - \sum_{i=1}^N \tilde{V}_i(x) \equiv V(x) - \tilde{V}(x). \end{aligned} \quad (5)$$

From the definition of $W(x)$ it follows that

$$\sup_{x \in R^3} \|W(x)\| < \infty; \quad \sup_{x \in R^3} |x| \cdot \|W(x)\| < \infty. \quad (6)$$

Let

$$\tilde{V}(x) = \tilde{S}(x) |\tilde{V}|(x); \quad V_i(x) = S_i(x) |V_i|(x) \quad (7)$$

be the polar decompositions of $\tilde{V}(x)$ and $V_i(x)$ respectively. Using the definition $\varphi(t)$ one can see that

$$|\tilde{V}|^{1/2}(x) = \sum_{i=1}^N |V_i|^{1/2}(x); \quad \tilde{S}(x) |\tilde{V}|^{1/2}(x) = \sum_{i=1}^N S_i(x) |V_i|^{1/2}(x). \quad (8)$$

One can easily see that $W|H_m|^{-1/2}$ is compact and that \tilde{V} satisfies the conditions of Lemma 5.1 in [1] so that $[|\tilde{V}|^{1/2}(H_m - z_0)^{-1}(H_m - z)^{-1}|\tilde{V}|^{1/2}]$ (here $[\cdot]$ means the extension by continuity) is compact. Then due to the Corollary 2.1 in [1] the only thing we have to prove is that there exist $0 \leq \lambda, s < \infty$ such that

$$\| [|\tilde{V}|^{1/2}(H_m + i\lambda)^{-1}|\tilde{V}|^{1/2}] \| < 1. \quad (9)$$

Let $\Phi \in (C_0^\infty(R^3))^4$. Then from the definition of $\varphi(t)$

$$\begin{aligned} \| |\tilde{V}|^{1/2}(H_m + i\lambda)^{-1}|\tilde{V}|^{1/2}\Phi \|^2 &= \sum_{i=1}^N \| |\tilde{V}_i|^{1/2}(H_m + i\lambda)^{-1}|\tilde{V}_i|^{1/2}\Phi \|^2 \\ &\leq \sum_{i=1}^N (A_i^2 + B_i(2A_i + B_i)) \end{aligned} \quad (10)$$

where

$$A_i = \| |\tilde{V}_i|^{1/2} (H_m + i\lambda)^{-1} |\tilde{V}_i|^{1/2} \Phi \|, \quad (11)$$

$$B_i = \left\| \sum_{j=i}^N |\tilde{V}_j|^{1/2} (H_m + i\lambda)^{-1} |\tilde{V}_j|^{1/2} \Phi \right\|. \quad (12)$$

In order to prove (9) it is sufficient to show that for $\lambda = s$

$$\sum_{i=1}^N A_i^2 \leq \left(\max_i v_i^2 \right) \|\Phi\|^2, \quad (13)$$

$$\lim_{\lambda \rightarrow \infty} \left(\sup_{\Phi} B_i / \|\Phi\| \right) = 0. \quad (14)$$

Now

$$\begin{aligned} A_i^2 &\leq \|k_\lambda(|\cdot - x_i|) |V_i|^{1/2} (H_m + i\lambda)^{-1} |V_i|^{1/2} k_\lambda(|\cdot - x_i|) \varphi^{1/2}(|\cdot - x_i|) \Phi\|^2 \\ &\leq v_i \|k_\lambda(|\cdot - x_i|) |\cdot - x_i|^{-1/2} (H_m + i\lambda)^{-1} |\cdot - x_i|^{-1/2} (|\cdot - x_i| |V_i|^{1/2}) \\ &\quad \cdot \varphi^{1/2}(|\cdot - x_i|) \Phi\|^2 \leq v_i^2 \|\varphi^{1/2}(|\cdot - x_i|) \Phi\|^2 \end{aligned} \quad (15)$$

In the last inequality we have used the Lemma 5.2 in [1] and the translation invariance of H_m . The inequality (13) follows from (15) and the definition of $\varphi(t)$. From the explicit form of the integral kernel of $(H_m + i\lambda)^{-1}$ (see for example [1], Section 3) one can easily see that for $|x - x_i| \leq d/3$, $i \neq j$, $\lambda > 1$

$$|((H_m + i\lambda)^{-1} |\tilde{V}_j|^{1/2} \Phi)_k(x)| \leq K \lambda^2 e^{-\lambda d} d^{-2} \|\Phi\| \quad (16)$$

so that

$$B_i \leq K' \lambda^2 e^{-d\lambda} \|\Phi\| \quad (17)$$

Then (14) is proved and the proof of the Theorem 1' is finished.

REFERENCES

- [1] G. NENCIU, *Self-adjointness for Dirac operators with singular potentials*. Commun. math. Phys. 48, 235 (1976).
- [2] V. S. POPOV, Zh. Exp. Theor. Fiz. 65, 35 (1973).
- [3] W. FARIS, *Self-adjoint operators*. In *Lecture Notes in Mathematics*, Vol. 433, (Springer Berlin-Heidelberg - New York, 1975).
- [4] K. JORGENS, *Perturbation of Dirac operator*. In *Lecture Notes in Mathematics*, Vol. 280, (Springer Verlag, Heidelberg, 1973).

