Helvetica Physica Acta
Vol. 50 (1977).

Birkhäuser Verlag, Basel

Distinguished self-adjoint extension for Dirac operator with potential dominated by multicenter Coulomb potentials

by **G. Nenciu**

Institut of Atomic Physics, Bucharest, Romania

(20.V.1976)

Abstract. The existence and the uniqueness of the distinguished self-adjoint extension of the Dirac operator describing an electron in the field of a finite number of point charges with $Z < 137$ is proved.

In [1] we proved some general results about perturbation of non-semibounded self-adjoint operators by quadratic forms. These results were applied to obtain distinguished self-adjoint extensions for Dirac operators with singular potentials. Let H_m be the free particle Dirac operator and $V(x)$ the 4×4 symmetric matrix valued function which represents the potential. Then, one of the results in [1] is

Theorem 1. If

$$|||V(x)||| \leqslant v/|x|, \quad 0 \leqslant v < 1 \tag{1}$$

(here $|||\cdot|||$ means the usual 4×4 matrix norm and in our system of units $v = 1$ correspond to atomic number $Z = 137$) then there exists a unique self-adjoint operator H such that $f \in D(H)$ implies $f \in D(|H_m|^{1/2})$, and

$$(g, Hf) = (H_m g, f) + (Vg, f); \qquad g \in D(H_m), \quad f \in D(H). \tag{2}$$

The operator H has the property

$$\sigma_{\mathrm{ess}}(H) \subset \sigma_{\mathrm{ess}}(H_m). \tag{3}$$

The aim of this letter is to prove the following generalization of the above result.

Theorem 1′. *The conclusions of Theorem 1 remain valid if the condition (1) is replaced by*

$$V(x) = \sum_{i=1}^{N} V_i(x); \qquad |||V_i(x)||| \leqslant v_i/|x - x_i| \quad 0 \leqslant v_i < 1, x_i \neq x_j, N < \infty, \tag{1'}$$

Remarks

1. The result contained in Theorem 1′ is relevant to the bound state problem in heavy ion scattering [2].

2. Similar arguments as in the proof of Theorem 1′ below, lead to the fact that the Rellich theorem [3, Th. 6.3] gives the essential self-adjointness of $H_m + V$ on $\{C_0^\infty(R^3\backslash\{\bigcup_i x_i\})\}^4$ for V satisfying (1′) with $0 \leqslant v_i \leqslant \frac{1}{2}$ which generalizes some results in [4].

3. Combining the proofs of Theorem 5.1 in [1] and of the Theorem 1″ one can prove a more general form of Theorem 1′. More precisely one has

Theorem 1″. *The conclusions of Theorem 1 remain valid if (1′) is replaced by*

$$V(x) = V_1(x) + V_2(x)$$

where $V_1(x)$ satisfies (1′) and $V_2(x)$ is nonsingular (see [1, Def. 2.1 and Def. 5.1]).

Proof of Theorem 1′. Let $d = \min_{i \neq j}|x_i - x_j|$, $\varphi(t) \in C^1([0, \infty))$ such that $\varphi(t) = 1$ for $t < d/4$, $\varphi(t) = 0$ for $t > d/3$ and $k_s(t)$ defined by

$$k_s^2(t) = \begin{cases} 1 - (m + s)t + (m^2 + s^2)t^2 & \text{for } 0 \leqslant t \leqslant (m + s)/(m^2 + s^2) \\ 1 & \text{for } \quad t > (m + s)/(m^2 + s^2). \end{cases} \tag{4}$$

Let

$$\begin{aligned} W(x) &= V(x) - \sum_{i=1}^{N} \varphi(|x - x_i|)k_s^2(|x - x_i|)V_i(x) \\ &\equiv V(x) - \sum_{i=1}^{N} \tilde{V}_i(x) \equiv V(x) - \tilde{V}(x). \end{aligned} \tag{5}$$

From the definition of $W(x)$ it follows that

$$\sup_{\chi \in R^3} |||W(x)||| < \infty; \qquad \sup_{\chi \in R^3} |x| \cdot |||W(x)||| < \infty. \tag{6}$$

Let

$$\tilde{V}(x) = \tilde{S}(x)|\tilde{V}|(x); \qquad V_i(x) = S_i(x)|V_i|(x) \tag{7}$$

be the polar decompositions of $\tilde{V}(x)$ and $V_i(x)$ respectively. Using the definition $\varphi(t)$ one can see that

$$|\tilde{V}|^{1/2}(x) = \sum_{i=1}^{N} |\tilde{V}_i|^{1/2}(x); \qquad \tilde{S}(x)|\tilde{V}|^{1/2}(x) = \sum_{i=1}^{N} S_i(x)|\tilde{V}_i|^{1/2}(x). \tag{8}$$

One can easily see that $W|H_m|^{-1/2}$ is compact and that $\tilde{V}$ satisfies the conditions of Lemma 5.1 in [1] so that $[|\tilde{V}|^{1/2}(H_m - z_0)^{-1}(H_m - z)^{-1}|\tilde{V}|^{1/2}$ (here $[\cdot]$ means the extension by continuity) is compact. Then due to the Corollary 2.1 in [1] the only thing we have to prove is that there exist $0 \leqslant \lambda, s < \infty$ such that

$$\|[|\tilde{V}|^{1/2}(H_m + i\lambda)^{-1}|\tilde{V}|^{1/2}]\| < 1. \tag{9}$$

Let $\Phi \in (C_0^\infty(R^3))^4$. Then from the definition of $\varphi(t)$

$$\begin{aligned} \||\tilde{V}|^{1/2}(H_m + i\lambda)^{-1}|\tilde{V}|^{1/2}\Phi\|^2 &= \sum_{i=1}^{N} \||\tilde{V}_i|^{1/2}(H_m + i\lambda)^{-1}|\tilde{V}|^{1/2}\Phi\|^2 \\ &\leqslant \sum_{i=1} (A_i^2 + B_i(2A_i + B_i)) \end{aligned} \tag{10}$$

where

$$A_i = \| |\tilde{V}_i|^{1/2}(H_m + i\lambda)^{-1}|\tilde{V}_i|^{1/2}\Phi\|, \tag{11}$$

$$B_i = \| \sum_{j=i}^{N} |\tilde{V}_i|^{1/2}(H_m + i\lambda)^{-1}|\tilde{V}_j|^{1/2}\Phi\|. \tag{12}$$

In order to prove (9) it is sufficient to show that for $\lambda = s$

$$\sum_{i=1}^{N} A_i^2 \leqslant \left(\max_i v_i^2\right)\|\Phi\|^2, \tag{13}$$

$$\lim_{\lambda\to\infty}\left(\sup_{\Phi} B_i/\|\Phi\|\right) = 0. \tag{14}$$

Now

$$\begin{aligned} A_i^2 &\leqslant \|k_\lambda(|\cdot - x_i|)|V_i|^{1/2}(H_m + i\lambda)^{-1}|V_i|^{1/2}k_\lambda(|\cdot - x_i|)\varphi^{1/2}(|\cdot - x_i|)\Phi\|^2 \\ &\leqslant v_i\|k_\lambda(|\cdot - x_i|)|\cdot - x_i|^{-1/2}(H_m + i\lambda)^{-1}|\cdot - x_i|^{-1/2}(|\cdot - x_i||V_i|^{1/2}) \\ &\quad \cdot\varphi^{1/2}(|\cdot - x_i|)\Phi\|^2 \leqslant v_i^2\|\varphi^{1/2}(|\cdot - x_i|)\Phi\|^2 \end{aligned} \tag{15}$$

In the last inequality we have used the Lemma 5.2 in [1] and the translation invariance of H_m. The inequality (13) follows from (15) and the definition of $\varphi(t)$. From the explicit form of the integral kernel of $(H_m + i\lambda)^{-1}$ (see for example [1], Section 3) one can easily see that for $|x - x_i| \leqslant d/3$, $i \neq j$, $\lambda > 1$

$$|((H_m + i\lambda)^{-1}|\tilde{V}_j|^{1/2}\Phi)_k(x)| \leqslant K\lambda^2 e^{-\lambda d} d^{-2}\|\Phi\| \tag{16}$$

so that

$$B_i \leqslant K'\lambda^2 e^{-d\lambda}\|\Phi\| \tag{17}$$

Then (14) is proved and the proof of the Theorem 1′ is finished.

References

[1] G. Nenciu, *Self-adjointness for Dirac operators with singular potentials*. Commun. math. Phys. *48*, 235 (1976).
[2] V. S. Popov, Zh. Exp. Theor. Fiz. *65*, 35 (1973).
[3] W. Faris, *Self-adjoint operators*. In *Lecture Notes in Mathematics*, Vol. 433, (Springer Berlin-Heidelberg – New York, 1975).
[4] K. Jorgens, *Perturbation of Dirac operator*. In *Lecture Notes in Mathematics*, Vol. 280, (Springer Verlag, Heidelberg, 1973).