On the Koba-Nielsen scaling behaviour for the high-energy multiplicity distributions

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On the Koba-Nielsen-Olesen scaling behaviour for the high-energy multiplicity distributions

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Summary. For a large class of distribution functions – the so-called infinitely divisible distribution functions – the KNO scaling is interpreted as a particular manifestation of a general property of this class, namely the first-passage time behaviour. This allows a rationale for the early onset of the KNO scaling and provides a simple recipe to determine the KNO-scaling functions.

1. Introduction

Recently [1], using duality considerations of the Huan Lee-Veneziano-Chan-type [2], we studied a simple bootstrap model for the multiplicity distribution functions and showed that the corresponding solution is the class of the so-called infinitely divisible distribution functions (IDDF)[1]) [3, 4]. Using general properties of the IDDF's combined with some additional ingredients, we proved that this class of distribution functions displays the Koba-Nielsen-Olesen (KNO)-scaling behaviour [5].

The purpose of this paper is to show that for the IDDF's (which – as we pointed out in [1] – contain almost all the distribution functions used to study the high energy multiplicities) the KNO scaling may be interpreted as a particular manifestation of a general property of the class, namely the so-called first-passage time (FPT) behaviour [6].

The next section will be devoted to the presentation of basic results, some special questions being relegated to appendices.

2. The basic formalism

We start with the observation that any IDDF may be generated by a stochastic process with stationary independent increments (or is the limit of a sequence of such stochastic processes) [3, 4] and we shall work in what follows with a (one-dimensional) stationary jump process. Let $x(t)$ be the random variable corresponding to such a

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[1]) A distribution function $f(x)$ is called infinitely divisible [3, 4] if the corresponding random variable x may be written for each positive n, as

$$x = x_1 + x_2 + \cdots + x_n$$

where x_i are independent random variables. If $\varphi(t)$ ($\varphi_n(t)$) is the characteristic function of $x(x_n)$ then $\varphi(t) = [\varphi_n(t)]^n$.

process and $F(\xi, t)$ its distribution function. The 'time' τ when the process crosses for the first time the value $x = \xi$ is called the first passage time. More precisely

$$T(\xi) = \min (\tau | x(\tau) = \xi) \tag{1}$$

As $F(\xi)$ is the probability that $x(\tau)$ never crosses $x = \xi$ in the time interval $0 \leqslant \tau \leqslant t$, one result is that (we neglect here a possible dependence on the initial state of the system)

$$F(\xi, t) = \text{Prob}\,(T(\xi) > t) \tag{2}$$

From (1) and (2) one can calculate the moments of $T(\xi)$, i.e.,

$$T_q = q \int_0^\infty dt\, t^{q-1}\, F(\xi, t), \quad q = 1, 2, 3, \ldots \tag{3}$$

Now, the first passage time (FPT) behaviour is a specific approximation of $F(\xi, t)$ in terms of the first moment T_1 for ξ and t large. Let us explain more deeply this approximation. In this respect we shall utilise a method due in essence to Gusak and Korolyuk [7]. This method bears on the observation (see, e.g. [1]) that any IDDF can be considered as a compound Poisson process, i.e., its characteristic function may be written as

$$\varphi(t) = e^{\lambda[\psi(t)-1]} \tag{4}$$

where $\psi(t)$ is the characteristic function of the mixing distribution and λ the first moment of the initial (Poisson) distribution. In [7] it has been proved that the first-passage time moment T_1 is determined from the equation

$$\psi(-i\, T_1) = 0 \tag{5}$$

and the FPT approximation is nothing but the mixed distribution written in terms of T_1 (T_1 large).

To illustrate this technique we shall work a simple example, namely the well-known Polya-distribution [8]. It is known that Polya-distribution is a compound Poisson-distribution with the Gamma-distribution as mixer, i.e.

$$P_n(\alpha, x) = \int \frac{d\lambda}{\lambda}\, e^{-\lambda y}\, \frac{(\lambda y)^\alpha}{(\alpha - 1)!}\, e^{-\lambda}\, \frac{\lambda^n}{n!} \tag{6}$$

$$= (1 - x)^\alpha x^n\, \frac{(n + \alpha - 1)!}{n!\,(\alpha - 1)!}, \quad n = 1, 2, 3, \ldots \tag{6'}$$

where

$$y = \frac{1 - x}{x}, \qquad 0 < x < 1, \qquad \langle n \rangle = \frac{x}{1 - x}\,\alpha \tag{7}$$

The characteristic function of the Gamma distribution may be easily calculated, namely

$$\psi(t) = \left(1 - \frac{it}{c}\right)^{-\alpha} \tag{8}$$

where $c = \langle n \rangle / \alpha$ and c is the constant which enters in the Wroblewski's relation [9] ($D^2 = \langle n^2 \rangle - \langle n \rangle^2$)

$$D^2 \simeq \frac{1}{\alpha} \langle n \rangle^\alpha, \quad \langle n \rangle \to \infty \tag{9}$$

Using Equation (5) one gets

$$T_1 = \langle n \rangle / \alpha \tag{10}$$

and the first passage time approximation of the Polya-distribution is (see Equation (6))

$$\tilde{P}_n = \frac{e^{-n/T_1}\,(n/T_1)^{\alpha-1}}{T_1(\alpha-1)!} \tag{11}$$

$$= \frac{\alpha}{\langle n \rangle} \frac{e^{-n\alpha/\langle n \rangle}\,(n\alpha/\langle n \rangle)^{\alpha-1}}{(\alpha-1)!} \tag{12}$$

From these relations one observes that the KNO approximation of the Polya distribution is nothing but its first passage time approximation. One can prove that a similar result holds for all the distribution functions from the IDDF class.

For the simple jump process we started with, we can obtain the following expression for the first passage time approximation, namely

$$F(\xi, t) \simeq \frac{1}{T_1} e^{-t/T_1} \tag{13}$$

i.e., a particular case of KNO scaling. Let us justify the above relation. For a jump process (with stationary independent increments) one can, obviously, write

$$F(\xi, t) \simeq (1 - \lambda(\xi) \cdot \Delta t)^k + O(\Delta t) \tag{14}$$

where k is the number of observations performed on the system. When $k \to \infty$, $\Delta t \to 0$, $k \cdot \Delta t = \text{const}$, one has

$$F(\xi, t) \simeq e^{-\lambda(\xi) t} \tag{15}$$

If we insert this result in Equation (3) we obtain

$$T_q(\xi) \simeq \frac{q!}{\lambda q(\xi)}, \quad q = 1, 2, 3, \ldots \tag{16}$$

The distribution (13) may be immediately derived from (15) and (16), taking into account the normalization condition. As for a stationary jump process $\lambda(\xi)$ is connected to the jump frequency, which, in turn, is proportional to the average number of particles of the system, one can conclude that the approximation (13) is a KNO scaling function (t – proportional with the particles number).

At the end of this Section several observations are in order.

i) From the Equation (13)–(15) one can observe that a system which displays the FPT behaviour possesses certain ergodic properties. It is interesting to note that in [10] a stochastic system whose distribution function $F(n), v)$ may be written as $\simeq f(n, v)/v$ (where v is the stationary jump frequency, $v \simeq \langle n \rangle$) is called an ergodic one.

ii) Recent papers due to Mandl (6), Newell [11], and Stone [12] have pointed out that the FPT behaviour is precocious, i.e., it is valid even at non-asymptotic values of the variables. This result may explain the observed precocious behaviour of the KNO scaling.

iii) A natural problem which arises in connection with the FPT behaviour is how one can determine the possible corrections to the FPT approximation. To get an answer to this question one must proceed as follows. One attaches

a Sturm-Lionville system (SLS) to the corresponding stochastic process [13, 14] and the problem is reduced to the determination of the SLS set of eigenfunctions and eigenvalues (i.e., T_q defined by Equation (3)). The method is obviously rather laborious. However, it has the advantage to permit (in principle) the calculation of all T_q's. (For further details see Appendix A.)

3. Conclusions

At the end of the paper we shall try to recapitulate the principal results! In essence, our purpose has been to provide a rationale for the KNO-scaling behaviour of the multiplicity distribution functions. We showed that for a large class of distribution functions (the IDDF class) it may be interpreted as a limiting behaviour inherent to this class. At the same time our paper provides a simple recipe to determine the KNO scaling function.

It is necessary to point out that these considerations do not exhaust neither other possible forms for KNO scaling functions,[2]) nor the mathematical content of the KNO scaling. (For a discussion of this second question see Appendix B.)

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Appendix A

Here we shall present the spectral theory for the differential equations corresponding to a stochastic process [13, 14]. It will provide a method to determine the moments of the first passage time, i.e., corrections to the first passage time approximation.

For the sake of definiteness we shall consider a simple birth and death process (a slight generalization of the Polya process). The corresponding differential equations are

$$\begin{aligned}
\frac{\partial}{\partial t} P_s(t) &= -\lambda_s P_s(t) + \mu_1 P_1(t) \\
\frac{\partial}{\partial t} P_n(t) &= -(\lambda_n + \mu_n) P_n(t) + \lambda_{n-1} P_n(t) + \mu_{n+1} P_{n+1}(t), \quad 1 \leqslant n \leqslant N-1 \\
\frac{\partial}{\partial t} P_N(t) &= -\mu_N P_N(t) + \lambda_{N-1} P_{N-1}(t)
\end{aligned} \tag{A.1}$$

[2]) The reader may provide as a counter example the Buras-Koba [15] KNO-function

$$\langle n \rangle P_n = \frac{\pi n}{2\langle n \rangle} \exp\left[-\frac{\pi}{4} \left(\frac{n}{\langle n \rangle} \right)^2 \right]$$

To some extent, this function and the FPT approximation are complementary in the sense that the first one is valid in the domain where the Central Limit Theorem [3] applies, while the second gives the tail of the actual distribution. In principle, it is possible to give an ad hoc statistical interpretation for the Buras-Koba function, too.

Here P_i, $i = 0, 1, 2, \ldots, N$ are the transition probabilities and λ_i, p_i are appropriately defined real parameters.

Let us now define

$$\pi_s = 1, \quad \pi_n = \frac{\lambda_{n-1}}{\mu_n}\pi_{n-1}, \quad n \geqslant 0 \tag{A.2}$$

and write

$$P_n(t) = \pi_n H_n(t), \quad n = 0, 1, 2, \ldots, N \tag{A.3}$$

The system (A.1) becomes

$$\pi_n \frac{\partial}{\partial t} H_n = \begin{cases} \mu_1\pi_1\, \Delta H_0(t), & n = 0 \\ \Delta(\mu_n\pi_n\, \Delta H_{n-1}(t), & 1 \leqslant n \leqslant N-1 \\ -\mu_N\pi_N\, \Delta H_{N-1}(t), & n = N \end{cases} \tag{A.4}$$

where $\Delta f_n = f_{n+1} - f_n$. The new system (A.4) may be recast in a matricial form

$$\pi \frac{\partial}{\partial t}\mathscr{H}(t) = L\mathscr{H}(t) \tag{A.5}$$

where L is a linear difference operator.

The corresponding Sturm-Liouville system is

$$L\psi = \beta\pi\psi \tag{A.6}$$

where $\beta(\psi)$ denotes the corresponding eigenvalues (eigenfunctions). Using the orthogonality condition

$$(\psi^{(J)}, \pi\psi^{(K)}) = \delta_{JK} \tag{A.7}$$

the Green's function of (A.5) (i.e., the transition probability) is

$$G_{mn} = \pi_n \sum_{J=0}^{N} \psi_m^{(J)}\psi_n^{(J)}\, e^{\beta_J t} \tag{A.8}$$

The term with $J = 0$ is the FPT approximation and the other terms are successive corrections to it.

Appendix B

This Appendix is devoted to a short discussion of some mathematical aspects of the KNO scaling relations

$$\frac{\sigma_n}{\sigma} \xrightarrow[p\to\infty]{} \frac{1}{\langle n\rangle}\psi\left(\frac{n}{\langle n\rangle}\right) \tag{B.1}$$

and

$$\langle n^q\rangle/\langle n\rangle^q \underset{J\to\infty}{\simeq} C_q, \quad q = 1, 2, 3, \ldots \tag{B.2}$$

Here σ_n is the cross-section for the process $a + b \to n$(charged) + any neutral, $\sigma = \sum_n \sigma_n$, $\langle n^q\rangle = \sum_n n^q P(n, s)$ is the q-th order momentum of the multiplicity distribution $P(n, s) = \sigma_n/\sigma$ and s is the c.m. energy squared. The C_q's are assumed energy independent.

As we have observed in [16] Equation (B.1) may be formally written as

$$P(n, \langle n\rangle) = \int \psi(x)\, \delta(n - \langle n\rangle x)\, \partial x, \quad x = n/\langle n\rangle \tag{B.3}$$

i.e., $P(n, \langle n\rangle)$ may be interpreted as the Radon-transform of $\psi(x)$, [17]. Now, in order to converge the integral on the right-hand side of Equation (B.3), it is usually assumed that (x) is an infinitely differentiable rapidly decreasing function. (A less stringent condition is summatibility of $\psi(x)$.) Under this assumption one can prove that the necessary and sufficient conditions for $P(n, \langle n\rangle$ to satisfy Equation (B.3) are:

i) $P(\alpha n, \alpha\langle n\rangle) = \alpha^{-1} P(n, \langle n\rangle)$, i.e., $P(n, \langle n\rangle)$ is homogeneous of degree -1.
ii) $P(n, \langle n\rangle)$ is infinitely differentiable with respect to both the variables ($\langle n\rangle \neq 0$) and rapidly decreasing with respect to n.
iii) For any $q > 0$, the integral $\int n^q P(n, \langle n\rangle)\, \partial n$, $q = 1, 2, 3, \ldots$ is a polynomial in $\langle n\rangle$, homogenous of degree q.

Let us now shortly comment on these conditions. It is easy to see that the condition (iii) is essentially equivalent with the Equation (B.2). We point out that there have been some attempts to identify the KNO scaling (i.e., Equation (B.1) with the property (i), see, e.g., [18], [19]). From the above theorem one can conclude that the content of the KNO scaling is richer than the homogeneity property (i).

In a recent paper [20] the KNO-scaling is obtained from a so-called renormalization-group equation (RGE) for the distribution $P(n, \langle n\rangle)$. In essence, one utilises the conditions (ii), (the RGE property is nothing but the differentiality of $P(n, \langle n\rangle)$) and (iii), the result being obviously a particular case of KNO scaling. As a matter of fact, the KNO function obtained in [20] is just the Gamma-distribution (Equations (11)–(12)). In this paper we considered the IDDF class which also satisfies the properties (ii) and (iii) (approximately).

We observed that for these distribution functions the KNO scaling is a particular manifestation of a general property of this class and we pointed out a simple method to determine the KNO-scaling function.

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