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Autor(en): **Aaberge, T.**

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On the description of free relativistic particles

by **T. Aaberge**

Départment de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland

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Abstract. Starting with a manifestly covariant classical symplectic structure, we construct the *unique* classical counterpart to Wigner's theory for free relativistic quantal particles of spin O. It is argued that the Wigner theories for quantal particles with spin different from O can not be considered as quantizations of this classical theory, however, that there exist theories for particles with 'spin'. We show how these 'spin' theories can be constructed, and that a certain class of them 'contains' the Wigner theories of spin.

1. Construction of the classical theory

Let (Ω, ω) be the symplectic manifold given by

$$\Omega = \{(p^\mu, q^\mu) \in \mathbb{R}^8 \mid p^{02}c^2 - \mathbf{p}^2 > 0 \;\&\; p^0 > 0\}$$

$$\omega = dp_\mu \wedge dq^\mu = \sum_i dp^i \wedge dq^i - c^2\, dp^0 \wedge dq^0$$

and let a symplectic action of the inhomogeneous Lorentz group $\mathscr{P}$ be defined by

$$(p^\mu, q^\mu) \mapsto (\Lambda^\mu{}_\nu p^\nu, \quad \Lambda^\mu{}_\nu q^\nu + a^\mu)$$

for Λ being the usual representation of the homogeneous Lorentz group $SO(1, 3)$ on $\mathbb{R}^4$, isometric with respect to the Minkowski-metric.

Let us then consider the following change of coordinates[1])

$$(p^\mu, q^\mu) \mapsto (y^\mu, x^\mu)$$

$$y^\mu = \begin{pmatrix} m \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \frac{1}{c}\sqrt{p^{02}c^2 - \mathbf{p}^2} \\ \mathbf{p} \end{pmatrix}$$

$$x^\mu = \begin{pmatrix} x^0 \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{p^{02}c^2 - \mathbf{p}^2}}{p^0 c}\, q^0 \\ \mathbf{q} - \dfrac{\mathbf{p}}{p^0}\, q^0 \end{pmatrix} \tag{1}$$

One verifies without difficulties that it is a canonical transformation,

$$\sum_i dp^i \wedge dq^i - c^2\, dp^0 \wedge dq^0 = \sum_i dp^i \wedge dx^i - c^2\, dm \wedge dx^0.$$

[1]) The observable $\mathbf{x}$ has been considered in [1].

The action of $\mathscr{P}$ on these new coordinates is easily determined,

$$m \mapsto m$$

$$p^i \mapsto \Lambda^i{}_\nu \left(\frac{1}{c} \sqrt{\mathbf{p}^2 + m^2 c^2}, \mathbf{p} \right)^\nu$$

(We will in the following use the notation $\Lambda\mathbf{p}$ and $\mathbf{p}'$ for the transformed of $\mathbf{p}$.)

$$x^\mu \mapsto \sigma(m, \mathbf{p}, \Lambda)^\mu_\nu x^\nu$$

where

$$\sigma(\mathbf{m}, \mathbf{p}, \Lambda(\boldsymbol{\theta})) = \Lambda(\boldsymbol{\theta}) \qquad (\Lambda(\boldsymbol{\theta}) \text{ rotation})$$

$$\sigma(m, \mathbf{p}, \Lambda(\mathbf{u}))^0_0 = 1$$

$$\sigma(m, \mathbf{p}, \Lambda(\mathbf{u}))^0_i = \frac{m}{\sqrt{\mathbf{p}'^2 + m^2 c^2}} \gamma \frac{u_i}{c}$$

$$\sigma(m, \mathbf{p}, \Lambda(\mathbf{u}))^i_0 = 0$$

$$\sigma(m, \mathbf{p}, \Lambda(\mathbf{u}))^i_j = \delta^i_j + \frac{\gamma^2}{\gamma + 1} \frac{u_i u_j}{c^2} - \frac{\gamma p_i'^2 u_j}{c\sqrt{\mathbf{p}'^2 + m^2 c^2}}$$

$$\gamma = \left(1 - \frac{\mathbf{u}^2}{c^2} \right)^{-1/2}.$$

Moreover, for a translation we find

$$x^0 \mapsto x^0 + \frac{mc}{\sqrt{\mathbf{p}^2 + m^2 c^2}} a^0$$

$$\mathbf{x} \mapsto \mathbf{x} + \mathbf{a} - \frac{\mathbf{p}c}{\sqrt{\mathbf{p}^2 + m^2 c^2}} a^0.$$

Let $\sim$ be defined as

$$(y^\mu, x^\mu) \sim (y'^\mu, x'^\mu) \Leftrightarrow$$

$$y'^\mu = y^\mu, (x'^\mu - x^\mu) = (b, \mathbf{o}), \quad b \in \mathbb{R}.$$

It is a Lorentz-invariant equivalence relation; $x^\mu - x'^\mu$ is invariant under the translations, and moreover $x^\mu - x'^\mu = (b, \mathbf{o})$ implies $(\sigma(x - x'))^\mu = (b', \mathbf{o})$.

$$\Omega \xrightarrow{\Pi_0} \Omega_0 = \Omega/\sim$$

is thus a Lorentz-invariant fibration of Ω, i.e. $\mathscr{P}$ acts by morphisms on $\Omega \xrightarrow{\Pi_0} \Omega_0$,

$$\Pi_0(m, \Lambda\mathbf{p}, (\sigma(m, \mathbf{p}, \Lambda)x)^\mu) = (m, \Lambda\mathbf{p}, \bar{\sigma}(m, \mathbf{p}, \Lambda)\mathbf{x})$$

where $\bar{\sigma}$ is given by

$$\bar{\sigma}^i_j = \sigma^i_j.$$

We proceed by defining a fibration of Ω_0,

$$\Omega_0 \xrightarrow{\Pi} \mathscr{B} = \{m\}$$

for

$$\Pi:(m, \mathbf{p}, \mathbf{x}) \mapsto m.$$

It obviously has the property of being invariant, and moreover each of the fibres $\Gamma_m = \Pi^{-1}(m)$ is a surface of transitivity for the given action of $\mathscr{P}$. Moreover, each Γ_m inherits a symplectic structure from Ω given by the form

$$\bar{\omega} = \sum_i dp^i \wedge dx^i.$$

One should notice that $\bar{\omega}$ is invariant under $\mathscr{P}$.

The free classical relativistic particle of rest-mass m is by definition described by the phase space $(\Gamma_m, \bar{\omega})$, on which the observables momentum and position are represented by the functions

$$p^i(\mathbf{p}, \mathbf{q}) = p^i$$

$$x^i(\mathbf{p}, \mathbf{q}) = x^i$$

We assume that the evolution of such a particle leaves invariant the symplectic structure. It is thus governed by the Hamilton equations. As usual, we let the associated Hamiltonian be of the form

$$\kappa = c\sqrt{\mathbf{p}^2 + m^2c^2}$$

the equations of motion are then

$$\dot{p}^i = -\frac{\partial \kappa}{\partial x^i} = 0$$

$$\dot{x}^i = \frac{\partial \kappa}{\partial p^i} = \frac{p^i c}{\sqrt{\mathbf{p}^2 + m^2c^2}} \tag{1}$$

If we denote by $\mathbf{v}$ the velocity of the particle, we can write down the following solutions

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}}$$

$$\mathbf{x} = \mathbf{v}t + \mathbf{a}, \tag{2}$$

and on the orbit,

$$\kappa = \frac{mc^2}{\sqrt{1 - \mathbf{v}^2/c^2}}$$

This is, by construction, a covariant description of the free relativistic particle. To complete the interpretation, we notice that for $\mathbf{v} = \mathbf{0}$, $\mathbf{x}$ transforms according to

$$\mathbf{x} \mapsto \mathbf{x} - \gamma \frac{\mathbf{u} \cdot \mathbf{x}}{c^2} \mathbf{u} + \frac{\gamma^2}{\gamma + 1} \frac{\mathbf{u} \cdot \mathbf{x}}{c^2} \mathbf{u}$$

for $\gamma = (1 - \mathbf{u}^2/c^2)^{-1/2}$, or

$$\mathbf{x}_\perp \mapsto \mathbf{x}_\perp, \quad \mathbf{x}_{\parallel} \mapsto \mathbf{x}_{\parallel}\sqrt{1 - \mathbf{u}^2/c^2}$$

under a special Lorentz transformation.

If one looks at the corresponding dynamics on Ω, one finds in addition to (1) the equations of motion

$$\dot{m} = 0$$

$$\dot{x}^0 = -\frac{mc}{\sqrt{\mathbf{p}^2 + m^2c^2}}$$

with the solutions

$$m = \text{const}$$

$$x^0 = -\frac{1}{\gamma}t, \qquad \gamma = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2} \tag{3}$$

In the original coordinates these solutions (2) and (3) read

$$p^\mu = m(\gamma, \gamma\mathbf{v})$$

$$q^\mu = (-t, \mathbf{a}).$$

This should give a certain basis for judging the theory; in particular, we notice that the above classical theory is different from the 'usual' one [2].

2. The quantal particle of spin *O*

Before considering the quantum model we give the abstract definition of a free relativistic particle.

Definition. A free relativistic particle of mass m is an irreductible realisation of the following 'system of imprimitivity'

$$S(\Lambda, a^\mu)\bar{\mathbf{p}}(\Delta) = \bar{\mathbf{p}}(\Lambda \cdot \Delta)$$

$$S(\Lambda, a^\mu)\bar{\mathbf{x}}(\Delta) = \bar{\mathbf{x}}\left(\bar{\sigma}(\mathbf{p}, \Lambda) \cdot \Delta + \mathbf{a} - \frac{\mathbf{p}c}{\sqrt{\mathbf{p}^2 + m^2c^2}}a^0\right) \tag{4}$$

for $\quad \Lambda \cdot \Delta = \{\Lambda\mathbf{p} \,/\, \mathbf{p} \in \Delta\} \qquad$ etc.,

for $\bar{\mathbf{p}}$ and $\bar{\mathbf{x}}$ denoting the observables momentum and position. $\Delta \in \mathscr{P}(\mathbb{R}^3)$ or $\mathscr{B}(\mathbb{R}^3)$, i.e. is a subset or a borel set of $\mathbb{R}^3$ representing the points of the spectra of $\bar{\mathbf{p}}$ and $\bar{\mathbf{x}}$.

The classical particle described above is clearly an irreducible realization of the systems of imprimitivity. Moreover, a quantum particle is easily constructed; in fact, the following realization satisfies the 'imprimitivity systems'.

$$(U(\Lambda, a^\mu)f)(\mathbf{p}) = e^{(i/\hbar)p^\mu a_\mu}f(\Lambda^{-1}\mathbf{p})$$

in

$$L^2\left(\mathbb{R}^3, \frac{d^3p}{\sqrt{\mathbf{p}^2 + m^2c^2}}\right),$$

$$p^\mu = \left(\frac{1}{c}\sqrt{\mathbf{p}^2 + m^2c^2}, \mathbf{p}\right);$$

and the observables $\overline{\mathbf{p}}$ and $\overline{\mathbf{x}}$ are represented by

$$(\mathbf{p}f)(\mathbf{p}) = \mathbf{p}f(\mathbf{p})$$

$$(\mathbf{x}f)(\mathbf{p}) = i\hbar\left(\partial\mathbf{p} - \frac{1}{2}\frac{\mathbf{p}}{\mathbf{p}^2 + m^2c^2}\right)f(\mathbf{p})$$
$$= i\hbar(\mathbf{p}^2 + m^2c^2)^{1/4}\,\partial\mathbf{p}(\mathbf{p}^2 + m^2c^2)^{-1/4}.$$

It is well known that the 'imprimitivity system' for $\mathbf{p}$ is satisfied, and also that for $\mathbf{x}$ under the rotations and translations. What is left to be verified is that

$$U^{-1}(\Lambda(\mathbf{u}))i\hbar(\mathbf{p}^2 + m^2c^2)^{1/4}\,\partial\mathbf{p}(\mathbf{p}^2 + m^2c^2)^{-1/4}\,U(\Lambda(\mathbf{u}))$$
$$= i\hbar(\mathbf{p}'^2 + m^2c^2)^{1/4}\sigma(\mathbf{p}, \Lambda(\mathbf{u}))\cdot\partial\mathbf{p}(\mathbf{p}'^2 + m^2c^2)^{-1/4}$$

which is done by a small calculation.

The theory thus constructed is what might be called the Wigner theory for free relativistic particles of spin O. The representation U of $\mathscr{P}$ is the Wigner representation for spin O [3], while $\mathbf{x}$ is what is usually referred to as the Newton–Wigner position operator [4]. The definition of $\mathbf{x}$ given in terms of the above 'systems of imprimitivity' supposes the postulates of localizability for relativistic particles [4, 5], in fact, its representative is uniquely determined by these postulates. However, our definition contains one more condition, that the position transforms in a particular way under a special Lorentz transformation. Though this additional condition is satisfied for the Wigner theory of spin O, one verifies easily that it is *not* satisfied for a Wigner theory of spin $\neq O$. Thus a Wigner theory of spin $\neq O$ does not satisfy the above definition of a relativistic particle.

A reason for this is the following. A realization of (4) is obtained by a double application of the inducing construction to obtain realizations of the 'imprimitivity systems' for $\overline{\mathbf{p}}$ and $\overline{\mathbf{x}}$ and then glue these together by a unitary transformation. Since the action of $\mathscr{P}$ on $\overline{\mathbf{p}}$ and $\overline{\mathbf{x}}$ has as stability groups $S0(3) \times_s \mathbb{R}^4$ and $S0(1, 3)$ respectively, it follows that for this unitary transformation to exist, the representation transforming $\overline{\mathbf{p}}$ and $\overline{\mathbf{x}}$ must be induced from representations of $S0(1, 3) \times_s \mathbb{R}^4$ respectively. The Wigner representations however, are induced from representation of $S0(3) \times_s \mathbb{R}^4$, this being the reason for the non-existence of a position operator satisfying our definition in the Wigner theory, except for the spin O case when the representations from which we induce are trivial.

3. The quantal particle of 'spin' $\neq O$

The inducing construction tells why a Wigner theory of spin $\neq O$ is not a realization of (4). But it does also tell how a notion of 'spin', compatible with the above definition of a relativistic particle, can be introduced; in fact, one should induce from representations of $S0(3, 1) \times_s \mathbb{R}^4$.

Thus let D be an irreducible unitary representation of $S0(3, 1)$ on l^2, then U defined by

$$(U(\Lambda, a^\mu)f)(\mathbf{p}) = e^{(i/\hbar)p^\mu a_\mu}D(\Lambda)f(\Lambda^{-1}\mathbf{p})$$

is a unitary representation of $\mathscr{P}$ on

$$L^2\left(\mathbb{R}^3, \frac{d^3p}{\sqrt{\mathbf{p}^2 + m^2c^2}}\right)\otimes l^2$$

induced from D. If one defines the operators representing the momentum and position as for the spin O case, the theory obtained satisfies the imprimitivity systems. Moreover, the additional degrees of freedom, the 'spins' are described by the six operators $S^{\mu\nu}$ on l^2 forming the representation of the Lie algebra of D. The $S^{\mu\nu}$ thus transform according to

$$U^{-1}(\Lambda, a^\mu)S^{\mu\nu}U(\Lambda, a^\mu) = (\Lambda\Lambda S)^{\mu\nu}.$$

In the case D is a representation in the principal series we can relate this definition of spin to that of Wigner in the following way. Let F be defined by

$$(Ff)(\mathbf{p}) = D^{-1}(L(\mathbf{p}))f(\mathbf{p}) = \psi(\mathbf{p})$$

where $L(\mathbf{p}) = \Lambda(\mathbf{p}/\sqrt{\mathbf{p}^2 + m^2c^2})$. It is a unitary transformation which takes U into $V = FUF^{-1}$,

$$(V(\Lambda, a^\mu)\psi)(\mathbf{p}) = e^{(i/\hbar)p^\mu a_\mu} D(R(\mathbf{p}, \Lambda))\psi(\Lambda^{-1}\mathbf{p})$$

where

$$R(\mathbf{p}, \Lambda) = L^{-1}(\mathbf{p})\Lambda\mathbf{L}(\Lambda^{-1}\mathbf{p})$$

is a Wigner rotation.

This form of the representation has the property that it decomposes into a direct sum of Wigner representations [6, 7]

$$(V(\Lambda, a^\mu)\psi)(\mathbf{p}) = \sum_{s=s', s'+1, \ldots}^{\oplus} e^{(i/\hbar)p^\mu a_\mu} D^{(s)}(R(\mathbf{p}, \Lambda))\psi^{(s)}(\Lambda^{-1}\mathbf{p})$$

where $D^{(s)}$ denotes a unitary representation of $SU(2)$ in $\mathbb{C}^{2s+1}$, and s' is one of the two numbers characterizing the D.

The momentum $\mathbf{p}$ commutes with F and is thus represented by the same operator in the new representation. The position operator $\mathbf{x}$ however, does not commute with F, and moreover, its new representative does not decompose according to the above decomposition.

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