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# Onsager's symmetry in higher order fluid dynamics

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*Abstract.* Example of the higher order fluid dynamics that possesses the properties of the family of dissipative dynamical systems of macroscopic physics, thus in particular the Onsager symmetry, is constructed. The non-equilibrium extension of the thermodynamic entropy depends on gradients of the fluid dynamic state variables.

## I. Introduction

The abstract concept of the family of dissipative dynamical systems of macroscopic physics (hereafter called DDS) that has been developed in References [1], [2], provides a unified mathematical setting for a large class of dynamical systems phenomenologically introduced in macroscopic physics (e.g., kinetic equations of the Boltzmann and Enskog–Vlasov type, Navier–Stokes–Fourier fluid dynamics, non-equilibrium thermodynamics etc. [2]).

In this paper, we construct an example of the higher order fluid dynamics that possesses the properties of DDS. In Section II the Navier–Stokes–Fourier fluid dynamics and the concept of DDS are reviewed. The higher order fluid dynamics generalizing the Navier–Stokes–Fourier fluid dynamics and satisfying the properties of DDS is constructed in Section III. In Section IV a brief review of other approaches to higher order fluid dynamics is presented.

The state of a one component physical system is completely described in fluid mechanics (completely with respect to the fluid mechanics observations and measurements) by the triple $(E, N, \mathbf{U}) \equiv f$, where $E: \Omega \to \mathbb{R}_+$, $(\mathbf{r} \mapsto E(\mathbf{r}))$; $N: \Omega \to \mathbb{R}_+$, $(\mathbf{r} \mapsto N(\mathbf{r}))$; $\mathbf{U}: \Omega \to \mathbb{R}^3 \cdot (\mathbf{r} \mapsto \mathbf{U}(\mathbf{r}))$; the physical system is confined in $\Omega \subset \mathbb{R}^3$, we shall assume that volume of $\Omega$ equals to one. $\mathbb{R}_+$ denotes the positive real line. Let $\mathscr{H}$ be the set of all admissible (again with respect to the fluid mechanic observations and measurements) states. We shall assume that $\mathscr{H}$ is a smooth (infinite dimensional) manifold. We define in $\mathscr{H}$ the involution $J: \mathscr{H} \to \mathscr{H}$; $(E, N, \mathbf{U}) \overset{J}{\mapsto} (E, N, -\mathbf{U})$. The subset of $\mathscr{H}$ that consists of elements $f$ of $\mathscr{H}$ that are invariant with respect to $J$ (i.e., the elements of the type $(E, N, 0)$) is denoted $\mathscr{H}^{(+)}$, its elements $f^{(+)}$. We can also say that $\mathscr{H}$ is a bundle $\mathscr{H} \equiv (\mathscr{H}, \pi^{(+)}, \mathscr{H}^{(+)})$, where $\mathscr{H}^{(+)}$ is the base space, $\pi^{(+)}$ is the projection $\mathscr{H} \to \mathscr{H}^{(+)}$ and $\mathscr{H}$ is the total space. In terms of fluid mechanics observations and measurements $N(\mathbf{r})$ is the local density of mass $E(\mathbf{r})$ is the local density of inner energy and $\mathbf{U}(\mathbf{r})$ is the local density of velocity of the fluid considered.

The time evolution of $f$ is governed by the following equations:

$$\frac{\partial f}{\partial t} = Rf,$$

in components

$$\begin{aligned}
\frac{\partial E}{\partial t} &= -U_\alpha \nabla_\alpha(E) - \frac{1}{N}(P_{\alpha\beta}\nabla_\alpha U_\beta + \nabla_\alpha Q_\alpha) \\
\frac{\partial N}{\partial t} &= -\nabla_\alpha(NU_\alpha) \qquad (1) \\
\frac{\partial U_\alpha}{\partial t} &= -U_\beta \nabla_\beta(U_\alpha) - \frac{1}{N}(\nabla_\beta P_{\alpha\beta}).
\end{aligned}$$

The symbol $\nabla_\alpha \equiv \partial/\partial r_\alpha$, the summation convention is used. The tensor $P_{\alpha\beta}$ is symmetric $P_{\alpha\beta} = P_{\beta\alpha}$. If the quantities $P_{\alpha\beta}$ and $Q_\alpha$ are specified as functions of $(E, N, \mathbf{U})$, or by using the terminology used in fluid mechanics, if the constitutive laws are specified, then (1) represents a self-contained system of the time evolution equations. The parameters introduced in the constitutive laws are the phenomenological quantities entering fluid dynamics (through these phenomenological quantities the individuality of the particular physical system is expressed) and (1) is a family of the time evolution equations parametrized by the phenomenological quantities. Boundary conditions for $(E, N, \mathbf{U})$ have to enter into the specification of $\mathscr{H}$. The class of the boundary conditions used in this paper (BC1)–(BC5) in Section II and (GBC1)–(GBC5) in Section III) reflects our interest in the thermodynamic equilibrium states and the time approach to these states.

Let $\mathscr{L}$ be the set of all constitutive laws. We shall assume that the physical system considered is isotropic and thus we require that the functional dependence of $P_{\alpha\beta}$, $Q_\alpha$ on $f$ is invariant with respect to the orthogonal transformations in $\Omega$. This requirement determines the subset $\mathscr{L}_0$ of $\mathscr{L}$. (See, e.g., Reference [3] pp. 135.) Physical considerations of constitutive laws are based on the physical interpretation of $P$ and $Q$ ($P_{\alpha\beta}$ is the pressure tensor, $Q_\alpha$ is the heat flow). From the mathematical point of view, one can look for $\mathscr{L}_{\text{math}}$ such that (1) will be mathematically well defined (e.g., the flow generated by (1) will exist etc.). It has been shown in Reference [1] that the Navier–Stokes–Fourier constitutive laws transform (2) into an example of DDS. According to the general theory of DDS it means in particular that (i) the study of the solutions of $R^{(+)}f = 0$ and $R^{(-)}f\,|_{R^{(+)}f=0} = 0$ gives rise to thermodynamics ($R^{(\pm)} \equiv \frac{1}{2}(R \pm JRJ)$); (ii) the flow exists, at least locally, in the vicinity of the solutions of the problem (i) that moreover correspond to the thermodynamically stable equilibrium states. In this paper we shall find a generalization of the Navier–Stokes–Fourier constitutive laws that transform (1) again into an example of DDS and the resulting system of partial differential equations is of order higher than two.

For later use, we shall write (1) also in the following form:

$$\frac{\partial f}{\partial t} = R^{(-)}(f) + R^{(+)}(f), \qquad (1.1)$$

in components

$$\frac{\partial E}{\partial t} = R^{(-)}_{(E)}(f) + R^{(+)}_{(E)}(f)$$

$$\begin{aligned}
\frac{\partial N}{\partial t} &= R^{(-)}_{(N)}(f)\\
\frac{\partial U_\alpha}{\partial t} &= R^{(-)}_{(U)\alpha}(f) + R^{(+)}_{(U)\alpha}(f),
\end{aligned}$$

$R^{(\pm)} \equiv \frac{1}{2}(R \pm JRJ)$, thus $R^{(+)}(Jf) = R^{(+)}(f)$; $R^{(-)}(Jf) = -R^{(-)}(f)$. If we write

$$\begin{aligned}
P_{\alpha\beta} &= P^{(+)}_{\alpha\beta} + P^{(-)}_{\alpha\beta}\\
Q_\alpha &= Q^{(+)}_\alpha + Q^{(-)}_\alpha
\end{aligned} \tag{2}$$

($P^{(\pm)}_{\alpha\beta}(Jf) = \pm P^{(\pm)}_{\alpha\beta}(F)$, similarly $Q_\alpha$), then

$$\begin{aligned}
R^{(-)}_{(E)}(f) &= -U_\alpha\nabla_\alpha E - \frac{1}{N}P^{(+)}_{\alpha\beta}\nabla_\alpha U_\beta - \frac{1}{N}\nabla_\alpha Q^{(-)}_\alpha\\
R^{(+)}_{(E)}(f) &= -P^{(-)}_{\alpha\beta}\nabla_\alpha U_\beta - \nabla_\alpha Q^{(+)}\\
R^{(-)}_{(N)}(f) &= -\nabla_\alpha(NU_\alpha)\\
R^{(-)}_{(U)\alpha}(f) &= -U_\beta\nabla_\beta U_\alpha - \frac{1}{N}\nabla_\beta P^{(+)}_{\alpha\beta}\\
R^{(+)}_{(U)\alpha}(f) &= -\frac{1}{N}\nabla_\beta P^{(-)}_{\alpha\beta}.
\end{aligned} \tag{3}$$

## II. Navier–Stokes–Fourier fluid dynamics

The Navier–Stokes–Fourier constitutive laws are the following:

$$\begin{aligned}
P^{(+)}_{\alpha\beta} &= N^*\,\delta_{\alpha\beta}\\
P^{(-)}_{\alpha\beta} &= p^{(-)}\,\delta_{\alpha\beta} + p^{(-)}_{\alpha\beta}\\
Q^{(+)}_\alpha &= -\lambda\nabla_\alpha E^*\\
Q^{(-)}_\alpha &\equiv 0\\
p^{(-)} &= -\eta_v\nabla_\alpha U^*_\alpha\\
p^{(-)}_{\alpha\beta} &= -2\eta(\overline{\overline{\nabla_\beta U^*_\alpha}}).
\end{aligned} \tag{4}$$

If $T_{\alpha\beta}$ is a second order tensor, then $\overline{\overline{T}}_{\alpha\beta} = \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) - \frac{1}{3}T_{\gamma\gamma}\,\delta_{\alpha\beta}$. Thus $p^{(-)}_{\alpha\beta}$ is a symmetric tensor with zero trace. $E^*$, $N^*$ and $\mathbf{U}^*$ are functions $(E, N, \mathbf{U})(\mathbf{r}) \mapsto E^*(\mathbf{r})$, $(E, N, \mathbf{U})(\mathbf{r}) \mapsto N^*(\mathbf{r})$, $(E, N, \mathbf{U}) \mapsto \mathbf{U}^*(\mathbf{r})$ of class $C^2$. Thus, the values of $E^*$, $N^*$, $\mathbf{U}^*$ at $\mathbf{r} \in \Omega$ depend only on the values of $E$, $N$, $\mathbf{U}$ at the same $\mathbf{r} \in \Omega$. $\eta_v$, $\eta$ and $\lambda$ are functions $(E, N)(\mathbf{r}) \mapsto \zeta_v(\mathbf{r})$ etc. The values of $\eta_v$, $\eta$, $\lambda$ at $\mathbf{r} \in \Omega$ depend also only on the values of $E$, $N$ at the same $\mathbf{r} \in \Omega$. We shall assume moreover that $E^*(Jf) = E^*(f)$, $N^*(Jf) = N^*(f)$ and $\mathbf{U}^*(Jf) = -\mathbf{U}^*(f)$. The phenomenological quantities $Q_{\text{NSF}}$ introduced by the Navier–Stokes–Fourier constitutive laws are thus $Q_{\text{NSF}} = \{E^*, N^*, \mathbf{U}^*, \eta_v, \eta, \lambda$ and the relations (4.1), (4.2), (4.3) introduced later in this Section$\}$.

We shall review the general definition of DDS.

(DDS1) The simultaneous solutions of $R^{(+)}f = 0$ and $R^{(-)}f\,|_{R^{(+)}f=0} = 0$, denoted $F$, are physically interpreted as thermodynamically equilibrium states.

(DDS2) The states $F$ found in (DDS1) are identical with the extremal points of a smooth function $\mathbb{V}: \mathscr{H} \times \Xi \to \mathbb{R}$, where $\Xi$ is a two dimensional (in the case of one-component system) space, its elements are physically interpreted as thermodynamical fields. In the process of the solution of (DDS1), the components $\sigma_E$, $\sigma_N$ of $\sigma \in \Xi$ appear as integration constants. The requirement that $\mathbb{V}$ is invariant with respect to the orthogonal transformations in $\Omega$ (the systems considered are isotropic) implies $\mathbb{V}(Jf) = \mathbb{V}(f)$. The function $\mathbb{V}$ evaluated at its extremal points (extremal with respect to $f \in \mathscr{H}$), denoted $\gamma$, is in general a multivalued function $\Xi \in \mathbb{R}$. For $\sigma \in \Xi$ for which $\gamma$ is single-valued the function $\gamma$ is identified with the thermodynamic potential.

(DDS3) Let $F$ be a solution of (DDS1). Thermodynamic stability of $F$ is equivalent to the local dynamic stability of $F$. This point is explained later in this Section.

One particular consequence of (DDS1), (DDS2) and (DDS3) is Onsager's symmetry of the vector field $R$ linearized at $F$.

In the case of the time evolution equations (1) and (4) the equation $R^{(+)}f = 0$ takes the form

$$\nabla_\alpha(\eta_v \nabla_\beta U^*_\beta) + 2\nabla_\beta(\overline{\overline{\eta \nabla_\beta U^*_\alpha}}) = 0 \tag{5}$$

$$3\eta_v(\nabla_\beta U^*_\beta)(\nabla_\alpha U_\alpha) + 2\eta(\overline{\overline{\nabla_\beta U^*_\alpha}})(\nabla_\alpha U_\beta) + \nabla_\alpha(\lambda \nabla_\alpha E^*) = 0. \tag{6}$$

If we multiply (5) by $U^{(*)}_\alpha$ sum over $\alpha$, integrate over $\Omega$ and integrate per parts, we have

$$\int_\Omega d^3\mathbf{r}[(\nabla_\alpha U^*_\alpha)\eta_v(\nabla_\beta U^*_\beta) + 2(\overline{\overline{\nabla_\beta U^*_\alpha}})\eta(\overline{\overline{\nabla_\beta U^*_\alpha}})] + s_U = 0. \tag{5.1}$$

We shall consider the boundary conditions that will guarantee that the surface integrals $s_U$ equal to zero

$$s_U = 0. \tag{BC1}$$

We shall moreover assume that

$$\begin{aligned} \eta &> 0 \\ \eta_v &> 0 \end{aligned} \tag{4.1}$$

for all $E$, $N$ considered.

As a consequence of (5.1), (BC1) and (4.1), we have

$$\begin{aligned} \nabla_\beta U^*_\alpha &= 0 \\ \nabla_\alpha U^*_\alpha &= 0. \end{aligned} \tag{7}$$

If we assume moreover that $U^*_\alpha$ on the boundary $\partial\Omega$ of $\Omega$ equals to zero

$$U^*_\alpha \mid_{\partial\Omega} = 0, \tag{BC2}$$

then (7) implies

$$U^*_\alpha = 0. \tag{8}$$

From (8) we then have

$$U_\alpha = 0 \tag{8.1}$$

since $U_\alpha \to U_\alpha^*$ preserves the bundle structure $(\mathscr{H}, \pi^{(+)}, \mathscr{H}^{(+)})$. By inserting (7) into (6), one obtains

$$\nabla_\alpha(\lambda \nabla_\alpha E^*) = 0. \tag{6.1}$$

If (6.1) is multiplied by $E^*$, integrated over $\Omega$ and integrated per parts, one obtains

$$\int_\Omega d^3\mathbf{r}(\nabla_\alpha E^*)\lambda(\nabla_\alpha E^*) + s_E = 0. \tag{6.2}$$

The boundary conditions that we assume will guarantee that the surface integral $s_E$ equals to zero

$$s_E = 0. \tag{BC3}$$

Moreover, we assume that

$$\lambda > 0. \tag{4.2}$$

Then (6.2) implies

$$\nabla_\alpha E^* = 0 \tag{9}$$

and thus

$$E^* = \sigma_E, \tag{10}$$

where $\sigma_E$ is a constant.

The equation $R^{(-)}f = 0$ restricted to the solutions of $R^{(+)}f = 0$ takes the form

$$\nabla_\alpha N^* \,\big|_{E^* = \sigma_E} = 0 \tag{11}$$

and thus

$$N^* = \tilde{\sigma}_N, \tag{12}$$

where $\tilde{\sigma}_N$ is a function of $E^*$. By $\sigma_N$ we shall denote $\tilde{\sigma}_N$ restricted to $E^* = \sigma_E$.

Following the general theory of DDS, we look for a function $\mathbb{V}: \mathscr{H} \times \Xi \to \mathbb{R}$ (the elements of $\Xi$ are $\sigma_E$ and $\sigma_N$ introduced in (10) and (12)) such that the solutions of (8), (10), (12) will be identical with extremal points of $\mathbb{V}$. There are, of course, many such functions. First of all we restrict ourselves to isotropic systems and thus $\mathbb{V}(Jf) = \mathbb{V}(f)$. The property (DDS3) defining the families of the dynamical systems DDS will restrict considerably the class of the functions $\mathbb{V}$ that may be considered.

Let $\sigma_0 \equiv (\sigma_{E0}, \sigma_{N0}) \in \Xi$ be such that the equations (8), (10), (12) have a unique solution (denoted $F_0$). We assume that if $\sigma \in U_0$, where $U_0$ is a neighbourhood of $\sigma_0$ in $\Xi$, then the corresponding unique solutions of (8), (10), (12) form a submanifold $\mathscr{F}_0$ in $\mathscr{H}$. The equilibrium state $F_0$ is called thermodynamically stable if the thermodynamic potential $\gamma$ is a convex function of $\sigma \in (\sigma_E, \sigma_N)$ at $\sigma_0$. According to (DDS2) $F_0$ is an extremal point of the function $\mathbb{V}$ and the thermodynamic potential $\gamma$ defined on $U_0$ is identical with the function $\mathbb{V}$ evaluated at $F_0$. One can show [1] that the function $\gamma$ is convex at $\sigma_0$ if and only if $\mathbb{V}$, as a function of $f \in \mathscr{H}$, is concave at $\sigma_0$. Thus $F_0$ is thermodynamically stable if and only if $A_0 = -D^{(2)}\mathbb{V}\,|_{F_0}$ (the second derivative of $\mathbb{V}$ with respect to $f \in \mathscr{H}$ evaluated at $F_0$) is a positive definite linear operator on $\tilde{H}_0 = T_{F_0}\mathscr{H} \ominus T_{F_0}\mathscr{F}_0$, where $T_{F_0}\mathscr{H}$ resp. $T_{F_0}\mathscr{F}_0$ is the tangent space to $\mathscr{H}$ resp. $\mathscr{F}_0$ at $F_0$.

Let $P_0$ denote the Hessian of the vector field $R$ evaluated at $F_0$. We say that $F_0$ is locally dynamically stable if there exists a unique solution to the initial value problem

$$\frac{\partial f}{\partial t} = P_0 f, \tag{13}$$

$f \in \mathscr{D}(P_0) \subset H_0 \subset \tilde{H}_0$, $f(t = 0) = f_0 \in \mathscr{D}(P_0)$, for all $t > 0$. If $\|f_0\|_0 < \infty$ then also $\|f_t\|_0 < \infty$; $f_t = T_t f_0$, where $\{T_t, t > 0\}$ is a strongly continuous semigroup of contraction operators. $H_0$ is an appropriate Hilbert space, its inner product is denoted $(.,.)_0$, its norm $\| \quad \|_0$.

We recall the Hille–Yoshida–Phillips [4] theorem. A necessary and sufficient condition for an operator $P_0$ to generate a strongly continuous semigroup of contraction operator on $H_0$ to itself is that $P$ is closed and both $P$ and its adjoint $P^\dagger$ are dissipative operators. A linear operator $P_0$ with dense domain $\mathscr{D}(P) \subset H_0$ is said to be dissipative if

$$(P_0 f, f)_0 + (f, P_0 f)_0 \leqslant 0$$

for all $f^K \mathscr{D}(P_0)$.

We want to satisfy the necessary and sufficient condition in the Hille–Yoshida–Phillips theorem by

(i) assuming (4.1), (4.2) already used in the solution of $R^{(+)}f = 0$ and $R^{(-)}f \mid_{R^{(+)}f=0} = 0$,

(ii) choosing appropriately the Hilbert space $H_0$.

Let $\tilde{H}_0$ be a fixed Hilbert space, say $L_2$ space, its inner product is denoted by $(\cdot,\cdot)$. We shall find, by using (4.1), (4.2), an operator $A$ such that the necessary and sufficient condition in the Hille–Yoshida–Phillips theorem will be satisfied for $AP_0$ in $\tilde{H}_0$. The boundary condition that will guarantee that $AP_0$ is closed will be formally denoted (BC4) (see, e.g., Reference [5]).

$$(AP_0 \text{ is closed}). \tag{BC4}$$

Let us suppose that such an operator $A$ has been found. The operator $A$ will be identified with $A_0 = -D^{(2)}\mathbb{V} \mid_{F_0}$. The assumption of thermodynamic stability of $F_0$ becomes equivalent to the assumption that $A$ defines an inner product $(\cdot, A.) \equiv (\cdot,\cdot)_0$ in $\mathscr{D}(A) \subset H_0$ and subsequently in the Hilbert space $H_0 \subset \tilde{H}_0$. We shall see that the identification of $A$ with $A_0 = -D^{(2)}\mathbb{V} \mid_{F_0}$ that is equivalent to (DDS3), restricts considerably the class of the functions $\mathbb{V}$ that can be considered in the context of DDS.

Now, we shall find the operator $A$. Since $\mathbb{V}(Jf) = \mathbb{V}(f)$, we look for $A$ in the form

$$A = \begin{pmatrix} a_1 & a_{12} & 0 & 0 & 0 \\ a_{12} & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & a_3 \end{pmatrix}, \tag{14}$$

where $a_1, a_2, a_3, a_{12}$ are real numbers (in the next Section (see (25) they will be differential operators). Thus, in this Section, $H_0$ and $\tilde{H}_0$ are identical ($\mathscr{D}(A) \equiv \tilde{H}_0$) except the difference in the definition of the inner product. The inequalities (4.1) and (4.2) enter only in $P^{(+)}$ ($P^{(+)}$ is the Hessian of $R^{(+)}$ evaluated at $F_0$). Thus, we look for $A$ such that $(AP_0)^{(+)}$ is selfadjoint (more generally, essentially selfadjoint) and dissipative and $(AP_0)^{(-)}$ is skewadjoint (more generally essentially skewadjoint). We assume the boundary conditions, denoted (BC5) that will guarantee the identity of the domains of $AP$ and its adjoint (or a weaker requirement in the case of essential selfadjointness—see Reference [5]). By using the terminology introduced in [1], [2], we look for $A$ such that $AP_0$ is an Onsager operator in $\tilde{H}_0$ or in other words $P_0$ is an Onsager operator in $H_0$. [The Hilbert space $H_0$ is equipped with the inner product $(.,.)_A$ and with the fundamental decomposition $H_0 = H_0^{(+)} \oplus H_0^{(-)}$ that reflects locally the bundle structure of $\mathscr{H}$. If $\Pi^{(+)}$ resp. $\Pi^{(-)}$ are projection operators to the complete subspaces $H_0^{(+)}$ resp. $H_0^{(-)}$, then the operator $J = \Pi^{(+)} - \Pi^{(-)}$. One can define in $H_0$ an indefinite inner product $(., J.)_A$ and $H_0$ becomes a Krein space [6]. We say that $P_0$ is an Onsager operator in $H_0$ if $P_0$ is selfadjoint with respect to the indefinite inner product $(., J.)_A$ and $P_0$ is dissipative with respect to the inner product $(.,.)_A$.]

From (1), (3), (4), we have (the notation $E^*_E \equiv \partial E^*/\partial E\,|_{F_0}$ etc. is used);

$$P_0^{(+)} = \begin{pmatrix} \lambda_0\Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & \\ 0 & 0 & & P_{02}^{(+)} & \\ 0 & 0 & & & \end{pmatrix} \cdot \begin{pmatrix} E^*_E & E^*_N & 0 & 0 & 0 \\ a_{12} & a_2 & 0 & 0 & 0 \\ 0 & 0 & U^*_U & 0 & 0 \\ 0 & 0 & 0 & U^*_U & 0 \\ 0 & 0 & 0 & 0 & U^*_U \end{pmatrix}, \tag{15}$$

where

$$P_{02}^{(+)} = \begin{pmatrix} (P_{02}^{(+)})_{11} & (P_{02}^{(+)})_{12} & (P_{02}^{(+)})_{13} \\ (P_{02}^{(+)})_{12} & (P_{02}^{(+)})_{22} & (P_{02}^{(+)})_{23} \\ (P_{02}^{(+)})_{13} & (P_{02}^{(+)})_{23} & (P_{02}^{(+)})_{33} \end{pmatrix}$$

$$(P_{02}^{(+)})_{ij} = \eta_0\, \delta_{ij}\Delta + (\eta_{v0} + \tfrac{1}{3}\eta_0)\nabla_i\nabla_j,$$
$$\Delta = \nabla_i\nabla_i, \qquad \lambda_0 = \lambda\,|_{F_0}, \qquad \eta_{v0} = \eta_v\,|_{F_0}, \qquad \eta_0 = \eta\,|_{F_0}.$$

Thus, if we make the identifications

$$\begin{aligned} a_1 &= -E^*_E, \qquad a_{12} = -E^*_N \\ a_3 &= -U^*_{U_1} = -U^*_{U_2} = -U^*_{U_3} = -U^*_U, \end{aligned} \tag{16}$$

then $(AP_0)^{(+)}$ (using (4, 1), (4, 2) and the boundary conditions (BC5)) is selfadjoint and dissipative.

From (1), (3), (4), we obtain also

$$P_0^{(-)} = \begin{pmatrix} 0 & 0 & -\frac{1}{N_0} N_0^* \nabla_1 & -\frac{1}{N_0} N_0^* \nabla_2 & -\frac{1}{N_0} N_0^* \nabla_3 \\ 0 & 0 & -N_0 \nabla_1 & -N_0 \nabla_2 & -N_0 \nabla_3 \\ -\frac{1}{N_0} N_E^* \nabla_1 & -\frac{1}{N_0} N_N^* \nabla_1 & 0 & 0 & 0 \\ -\frac{1}{N_0} N_E^* \nabla_1 & -\frac{1}{N_0} N_N^* \nabla_2 & 0 & 0 & 0 \\ -\frac{1}{N_0} N_E^* \nabla_1 & -\frac{1}{N_0} N_N^* \nabla_2 & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

where

$$N_0 = N \,|_{F_0}, \; N_0^* = N^* \,|_{F_0}, \; N_E^* = \left.\frac{\partial N^*}{\partial E}\right|_{F_0}, \; N_N^* = \left.\frac{\partial N^*}{\partial N}\right|_{F_0}.$$

By multiplying (17) from the left by $A$, taking into account (16), one obtains the following conditions which have to be satisfied for $AP_0^{(-)}$ to be skewadjoint:

$$\begin{aligned} \frac{1}{N_0} U_U^* N_E^* &= \frac{1}{N_0} E_E^* N_0^* + N_0 E_N^* \\ \frac{1}{N_0} U_U^* N_N^* &= \frac{1}{N_0} E_N^* N_0^* - N_0 a_2 \end{aligned} \tag{18}$$

If we choose the function $\mathbb{V}$ such that the following relations

$$\begin{aligned} \frac{\partial \mathbb{V}}{\partial E} &= E^* - \sigma_E \\ \frac{\partial \mathbb{V}}{\partial N} &= -\frac{1}{N_2} (N^* E^* - \sigma_N) \\ \frac{\partial \mathbb{V}}{\partial U_\alpha} &= -E^* U_\alpha \end{aligned} \tag{19}$$

are satisfied then (18) is satisfied and also the solutions of (8), (10), (12) are identical with the extremal points of $\mathbb{V}$. The equations (19) imply that the functions $E^*$, $N^*$ entering the Navier–Stokes–Fourier constitutive laws are not independent. The relation between them (the generalized Maxwell's relation) is implied by the symmetry of $D^{(2)}\mathbb{V}$

$$(D^{(2)}\mathbb{V} \text{ is symmetric}) \tag{4.3}$$

The general solution of (19) is

$$\mathbb{V}(E, N, \mathbf{U}; \sigma_E, \sigma_N) = \int_\Omega d^3\mathbf{r} \left[ S(E(\mathbf{r}), N(\mathbf{r}), \mathbf{U}(\mathbf{r})) - \sigma_E E - \frac{1}{N} \sigma_N \right], \tag{20}$$

where

$$\frac{\partial S}{\partial E} = E^*,\ \frac{\partial S}{\partial(1/N)} = E^*N^* \text{ and } \frac{\partial S}{\partial U_\alpha} = -E^*U_\alpha.$$

This function $\mathbb{V}$ is identical with the non-equilibrium extension of the thermodynamic potential postulated in non-equilibrium thermodynamics [7]. We have obtained this function $\mathbb{V}$ as the consequence of the postulated Navier–Stokes–Fourier constitutive laws and the requirement that the resulting family of the time evolution equations possesses the formal structure of DDS.

## III. Higher order Navier–Stokes–Fourier fluid dynamics

We shall consider the following generalization of the Navier–Stokes–Fourier constitutive laws:

$$\begin{aligned}
P^{(+)}_{\alpha\beta} &= \mathscr{N}^*\delta_{\alpha\beta} + p^{(+)}_{\alpha\beta}\\
P^{(-)}_{\alpha\beta} &= p^{(-)}\delta_{\alpha\beta} + p^{(-)}_{\alpha\beta}\\
Q^{(+)}_{\alpha} &= -\lambda\nabla_\alpha\mathscr{E}^*\\
Q^{(-)}_{\alpha} &= \mu_v\nabla_\alpha(\nabla_\beta U^*_\beta) + \mu\Delta U^*_\alpha\\
p^{(+)}_{\alpha\beta} &= \overline{\overline{\tfrac{3}{2}\nabla_\alpha(\Lambda\nabla_\beta E^*)}}\\
p^{(-)} &= {}^{-}\eta_v\nabla_\alpha U^*_\alpha\\
p^{(-)}_{\alpha\beta} &= \overline{\overline{-2\eta\nabla_\beta U^*_\alpha}}.
\end{aligned} \tag{21}$$

Let $j(E, N) \equiv (\Delta E, \Delta N, \nabla_\alpha E\nabla_\alpha E, \nabla_\alpha N\nabla_\alpha N, \nabla_\alpha E\nabla_\alpha N)$. The functions $\mathscr{E}^*$ and $\mathscr{N}^*$ are defined as follows: $\mathscr{E}^*\colon (E, N, j(E, N), \mathbf{U})(\mathbf{r}) \mapsto \mathscr{E}^*(\mathbf{r})$ and $\mathscr{E}^*(\mathbf{r})\,|_{j(E,N)=0} = E^*(\mathbf{r})$ introduced in (4). Similarly $\mathscr{N}^*\colon (E, N, j(E, N), \mathbf{U})(\mathbf{r}) \mapsto \mathscr{N}^*(\mathbf{r})$ and $\mathscr{N}^*(\mathbf{r})\,|_{j(E,N)=0} = N^*(\mathbf{r})$ introduced in (4). We assume moreover that $\mathscr{E}^*(Jf) = \mathscr{E}^*(f)$ and $\mathscr{N}^*(Jf) = \mathscr{N}^*(f)$. $\mathbf{U}^*$ is assumed to be the same as in (4). $\Lambda$, $\mu_v$, $\mu$, $\lambda$, $\eta_v$, $\eta$ are real valued functions of $(E, N, j(E, N))$ (e.g., $\Lambda\colon (E, N, j(E, N))(\mathbf{r}) \to \Lambda(\mathbf{r})$ etc.). The phenomenological quantities $Q_{\text{GNSF}}$ introduced by the Generalized Navier–Stokes–Fourier constitutive laws are thus $Q_{\text{GNSF}} \equiv \{\mathscr{E}^*, \mathscr{N}^*, U^*, \Lambda, \mu_v, \mu, \lambda, \eta_v, \eta$ and the relations (21.1), (21.2), (21.3) introduced later in this Section$\}$.

If (21) is inserted into (1) then clearly the system of partial differential equations of the order higher than two is obtained. In the constitutive laws (21) the assumption, used in (4), that $P_{\alpha\beta}$ and $Q_\alpha$ at $\mathbf{r}$ depend only on the values of $E$, $N$, $\mathbf{U}$ at the same $\mathbf{r}$ has been relaxed. The non-locality of the dependence of $P_{\alpha\beta}$ and $Q_\alpha$ on $(E, N, \mathbf{U})$ is expressed in (21) by letting the quantities entering (4) to depend also on the gradients of $E$ and $N$.

The discussion of (1), (21) follows closely the discussion of (1), (4) in Section II. The equation $R^{(+)}f = 0$ remains the same as in (5)–(10). We assume again that

$$\eta > 0,\ \eta_v > 0 \tag{21.1}$$

$$\lambda > 0 \tag{21.2}$$

for all $(E, N, j(E, N))$ considered and the boundary conditions (GBC1)–(GBC3) corresponding to (BC1)–(BC3) in Section II. The equation (10) reads now

$$\mathscr{E}^* = \sigma_E \tag{22}$$

The equation $R^{(-)}f = 0$ restricted to the solutions of $R^{(+)}f = 0$ takes the form

$$\mathscr{N}^* \,|_{\mathscr{E}^* = \sigma_E} = 0 \tag{23}$$

and thus

$$\mathscr{N}^* = \tilde{\sigma}_N \tag{24}$$

where $\tilde{\sigma}_N$ is a function of $\mathscr{E}^*$. By $\sigma_N$ we shall denote again $\tilde{\sigma}_N$ restricted to $\mathscr{E}^* = \sigma_E$. The problem is now to find the function $\mathscr{V}$: $(E, N, \mathbf{U}, j(E, N); \sigma_E, \sigma_N) \rightarrow \mathscr{V}(E, N, \mathbf{U}, j(E, N); \sigma_E, \sigma_N)$ such that (i) the solutions of (8), (22), (24) will be identical with the extremal points of $\mathscr{V}$, (ii) thermodynamic stability of a thermodynamically equilibrium state will be equivalent to its local dynamic stability, (iii) if $j(E, N) \equiv 0$ then $\mathscr{V}$ reduces to $\mathbb{V}$ that has been introduced in (20). We shall indeed find an example of such a function $\mathscr{V}$. It will become evident however that the requirement (DDS3) – the equivalence of thermodynamic and local dynamic stability – is not strong enough to single out the function $\mathscr{V}$. A systematic study of the higher order fluid dynamic requires a generalization of (DDS3). The local dynamic stability has to be replaced by a type of global dynamic stability (the requirement that $\mathscr{V}$ is a Liapunov function).

Let $F_0$ be a solution of (8), (22), (24) (thus also an extremal point of the function $\mathscr{V}$) and let moreover $F_0$ does not depend on $\mathbf{r}$. Similarly as in Section II, we shall find the operator $A$ such that $AP$ is an Onsager operator. The operator $A$ will be then identified with $-D^{(2)}\mathscr{V} \,|_{F_0}$. Since $F_0$ is independent of $\mathbf{r}$ and $\mathscr{V}$ can depend only on $(E, N, \mathbf{U}, j(E, N))$ the operator $A$ has to have the form

$$A = A^{(0)} + A^{(1)}\Delta, \tag{25}$$

where $A^{(0)}$ is a $5 \times 5$ matrix of real numbers identical with (14) and

$$A^{(1)} = \begin{pmatrix} b_1 & b_{12} & 0 & 0 & 0 \\ b_{12} & b_2 & 0 & 0 & 0 \\ 0 & 0 & b^3 & 0 & 0 \\ 0 & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 0 & b_3 \end{pmatrix} \tag{26}$$

where $b_1$, $b_2$, $b_{12}$, $b_3$ are real numbers. Since the operator $A$ is a differential operator, i.e., $A$ is unbounded operator in $\tilde{H}_0$, $(\mathscr{D}(A) \not\equiv H_0)$ the Hilbert space $H_0 \not\equiv \tilde{H}_0$. Construction of $H_0 \subset \tilde{H}_0$ in which the given operator $A$ defined in $\mathscr{D}(A) \subset \tilde{H}_0$ is bounded and can thus be used, if moreover $A$ is positive definite, to define an inner product in $H_0$ is explained for example in Reference [8].

From (1), (3), (21), we have

$$P_0^{(+)} = \begin{pmatrix} \lambda_0\Delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & \\ 0 & 0 & & P_{02}^{(+)} & & \\ 0 & 0 & & & & \end{pmatrix} \left[ \begin{pmatrix} E_E^* & E_N^* & 0 & 0 & 0 \\ a_{12} & a_2 & 0 & 0 & 0 \\ 0 & 0 & U_U^* & 0 & 0 \\ 0 & 0 & 0 & U_U^* & 0 \\ 0 & 0 & 0 & 0 & U_U^* \end{pmatrix} + \right.$$

$$\left.\begin{pmatrix} \mathcal{E}^*_{2E} & \mathcal{E}^*_{2N} & 0 & 0 & 0 \\ b_{12} & b_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Delta \right] \tag{27}$$

where $P_{02}^{(+)}$ is the same as in (15), $\lambda_0 = \lambda \,|_{F_0}$, $E_E^*$, $E_N^*$, $a_{12}$, $a_2$ are the same as in (15) and $\mathcal{E}^*_{2E}$ is the part of the $\partial \mathcal{E}^* / \partial E \,|_{F_0}$ that is proportional to $\Delta$. Thus if $a_1$, $a_2$, $a_{12}$ are the same as in (16),

$$\begin{aligned} b_1 &= -\mathcal{E}^*_{2E} \\ b_{12} &= -\mathcal{E}^*_{2E}, \\ b_3 &= 0. \end{aligned} \tag{28}$$

and appropriate boundary conditions (GBC4), (GBC5) that correspond to (BC4), (BC5) are assumed, then $(AP_0)^{(+)}$ is selfadjoint and dissipative.

From (1), (3), (21), we obtain also

$$P_0^{(-)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{N_0} N_0^* \nabla_1 & -\frac{1}{N_0} N_0^* \nabla_2 & -\frac{1}{N_0} N_0^* \nabla_3 \\ 0 & 0 & 0 & -N_0 \nabla_1 & -N_0 \nabla_2 & -N_0 \nabla_3 \\ -\frac{1}{N_0} N_E^* \nabla_1 & -\frac{1}{N_0} N_N^* \nabla_1 & 0 & 0 & 0 \\ -\frac{1}{N_0} N_E^* \nabla_2 & -\frac{1}{N_0} N_N^* \nabla_2 & 0 & 0 & 0 \\ -\frac{1}{N_0} N_E^* \nabla_3 & -\frac{1}{N_0} N_N^* \nabla_3 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & p_1 \nabla_1 & p_2 \nabla_2 & p_3 \nabla_3 \\ 0 & 0 & 0 & 0 & 0 \\ p_2 \nabla_1 & p_3 \nabla_1 & 0 & 0 & 0 \\ p_2 \nabla_2 & p_3 \nabla_2 & 0 & 0 & 0 \\ p_2 \nabla_3 & p_3 \nabla_3 & 0 & 0 & 0 \end{pmatrix} \Delta, \tag{29}$$

where

$$p_1 = -\frac{1}{N_0} (\mu_{v0} + \mu_0), \qquad p_2 = -\frac{1}{N_0} (\mathcal{N}^*_{2E} + \Lambda E_E^*)$$
$$p_3 = -\frac{1}{N_0} (\mathcal{N}^*_{2N} + \Lambda E_N^*). \tag{30}$$

The first term on the right hand side of (29) is identical with (17) since, according to our assumption, $F_0$ is independent of $\mathbf{r}$ and thus $j(E, N) \,|_{F_0} \equiv 0$. By multiplying (27) from the left by $A$ (see (25), (26)), taking into account the boundary conditions

(GBC4), (GBC5), and considering only the derivatives up to and including the fourth order, one obtains the following conditions that have to be satisfied for $(AP_0)^{(-)}$ to be skewadjoint:

$$
\begin{aligned}
&a_1 p_1 + b_1\left(-\frac{1}{N_0} N_0^*\right) + b_{12}(-N_0) = a_3 p_2 \\
&a_{12} p_1 + b_{12}\left(-\frac{1}{N_0} N_0^*\right) + b_2(-N_0) = a_3 p_3 .
\end{aligned}
\tag{31}
$$

By inserting (28), (16) and (30) into (31), we have

$$
\begin{aligned}
&\frac{1}{N_0} E_E^*(\mu_{v_0} + \mu_0) + \frac{1}{N_0} \mathscr{E}_{2E}^* N_0^* + N_0 \mathscr{E}_{2N}^* = +\frac{1}{N_0} E_0^*(\mathscr{N}_{2E}^* + \Lambda E_E^*) \\
&\frac{1}{N_0} E_N^*(\mu_{v_0} + \mu_0) + \frac{1}{N_0} \mathscr{E}_{2N}^* N_0^* + N_0 b_2 = -\frac{1}{N_0} E_0^*(\mathscr{N}_{2N}^* + \Lambda E_N^*).
\end{aligned}
\tag{32}
$$

If the function $\mathscr{V}$ satisfies the following relations:

$$
\begin{aligned}
&\frac{\delta_v \mathscr{V}}{\delta_v E} = \mathscr{E}^* - \sigma_E \\
&\frac{\delta_v \mathscr{V}}{\delta_v N} = -\frac{1}{N^2}(\mathscr{E}^* \mathscr{N}^* + \nabla_\alpha(\Lambda \nabla_\alpha E^*) + (\mu_v + \mu)\Delta E^* - \sigma_N) \\
&\frac{\delta_v \mathscr{V}}{\delta_v U_\alpha} = -\mathscr{E}^* U_\alpha ,
\end{aligned}
\tag{33}
$$

then (32) is satisfied, the extremal points of $\mathscr{V}$ are identical with the solutions of (8), (22), (24) and $\mathscr{V}$ reduces to $\mathbb{V}$ introduced in (20) if $j(E, N) \equiv 0$. The differentiation

$$
\frac{\delta_v \mathscr{V}}{\delta_v \phi} \equiv \frac{\delta \mathscr{V}}{\delta \phi} - \nabla_\alpha\left(\frac{\delta \mathscr{V}}{\delta \nabla_\alpha \phi}\right) + \Delta\left(\frac{\delta \mathscr{V}}{\delta \Delta \phi}\right)
$$

is the variational differentiation, $\phi \equiv E$ or $\phi \equiv N$. The quantities $\mathscr{E}^*$, $\mathscr{N}^*$, $\mu_v$, $\mu$, $\Lambda$ are not independent. The relation among them is implied by the generalized Maxwell's relations

$$
(D^{(2)}\mathscr{V} \text{ is symmetric}). \tag{21.3}
$$

A function $\mathscr{V}$ satisfying (33) is

$$
\mathscr{V}(E, N, \mathbf{U}, j(E, N); \sigma_E, \sigma_N) = \int_\Omega d^3\mathbf{r}\left[\mathscr{S}(E(\mathbf{r}), N(\mathbf{r}), \mathbf{U}(\mathbf{r}), j(E, N)(\mathbf{r})) - \sigma_E E - \frac{1}{N}\sigma_N\right], \tag{34}
$$

$$
\begin{aligned}
&\mathscr{S}\,\big|_{j(E,N)\equiv 0} = S \text{ introduced in (20)}, \\
&\frac{\delta_v \mathscr{S}}{\delta_v E} = \mathscr{E}^*, \\
&\frac{\delta_v \mathscr{S}}{\delta_v\left(\dfrac{1}{N}\right)} = \mathscr{E}^* \mathscr{N}^* + \nabla_\alpha(\Lambda \nabla_\alpha E^*) + (\mu_v + \mu)\Delta E^*, \\
&\frac{\delta_v \mathscr{S}}{\delta_v U_\alpha} = -\mathscr{E}^* U_\alpha .
\end{aligned}
$$

The thermodynamic potential $\gamma_{\mathscr{V}}$ constructed from $\mathscr{V}$ is in general different from the thermodynamic potential $\gamma_{\mathbb{V}}$ constructed from $\mathbb{V}$. In particular, a thermodynamic equilibrium state $F_0$ can be thermodynamically stable with respect to $\gamma_{\mathscr{V}}$ but thermodynamically unstable with respect to $\gamma_{\mathbb{V}}$. There is a possibility of relating these different notions of thermodynamic stability to the appearance of the metastable states known from experience with fluids. This problem will be hopefully discussed in detail in a subsequent paper.

We have found that if the constitutive laws (21) are postulated and the non-equilibrium extension of the thermodynamic potential is given by (34) then the linearized infinitesimal generator of the time evolution possesses the Onsager symmetry. The non-equilibrium thermodynamic potential $\mathscr{V}$ is not however determined uniquely by studying the equivalence of the thermodynamic and local dynamic stability. We cannot decide for example if in (34) should be the term $\nabla_\alpha(\Delta\nabla_\alpha E^*)$ or the term $\Lambda\Delta E^*$. A systematic study of higher order fluid dynamics will require the study of a type of global dynamic stability where the function $\mathscr{V}$ will play the role of a Liapunov function.

## IV. Discussion

An example of the higher order fluid dynamics possessing the structure of the family of dissipative dynamical systems of macroscopic physics [1], [2] has been constructed.

The non-equilibrium extension of the thermodynamic potential cannot be obtained in higher order fluid dynamics by using the method used in non-equilibrium thermodynamics [7] (i.e., $S(E, N)$ is extended to $\int_\Omega d^3\mathbf{r} S(E(\mathbf{r}), N(\mathbf{r}))$, where $E$, $N$ are the thermodynamical conjugates of the thermodynamic fields $\sigma_E$ and $\sigma_N$, $E(\mathbf{r})$ and $N(\mathbf{r})$ are the fluid mechanics state variables, called the assumption of the local thermodynamic equilibrium). The situation in higher order fluid mechanics is similar, as for example in the Boltzmann kinetic theory. The non-equilibrium extension of the thermodynamic potential, so called Boltzmann's H-function, cannot be also obtained by using the method used in non-equilibrium thermodynamics. We have seen that the method associated with the theory of DDS applies in the Navier–Stokes–Fourier fluid dynamics (Section II) kinetic theory [1] and higher order fluid dynamics (Section III). The higher order fluid dynamics indicates however the necessity of generalizing the point (DDS3) in the definition of the formal structure of DDS. A type of global–Liapunov–dynamic stability has to replace the local, Hille–Yoshida–Phillips–dynamic stability.

Several different approaches to higher order fluid dynamics have been developed in fluid mechanics. We mention three of them.

In the point of view advocated by Coleman [3] the inequality $\partial S/\partial t + J \geqslant 0$ is added to the system of equations (1). $J$ is a known function of $P_{\alpha\beta}$, $Q_\alpha$; $S$ is another, in addition to $P_{\alpha\beta}$ and $Q_\alpha$, unknown function of fluid mechanics state variables. The higher order fluid dynamics in the framework of Coleman's point of view has been discussed recently by de Sobrino [9].

Reduction of the Boltzmann kinetic theory to fluid dynamics inspired the Burnett [10] [11] and the Waldmann [12] [13] approach to higher order fluid dynamics. The Burnett approach has been inspired by the iterations of the Chapman–Enskog method, the Waldmann approach by Grad's thirteen moment method of solution of Boltzmann's equation. The linearized Burnett equations together with the non-equilibrium thermodynamic potential used in non-equilibrium thermodynamics do not possess the Onsager symmetry [11]. Waldmann has suggested the generalization of the entropy production and the entropy flux that are appropriate for the constitutive laws he used (e.g., the entropy production is evidently positive definite) but the generalization of the non-equilibrium entropy itself, that has to depend on gradients, has not been discussed.

## REFERENCES

[1] M. GRMELA, Helv. Phys. Acta *47*, 667 (1974); J. Math. Phys. *16*, 2441 (1975).
[2] M. GRMELA and I. ISCOE, Ann. Inst. H. Poincaré, Sect. A. (submitted).
[3] C. TRUESDELL, *Rational Thermodynamics* (McGraw-Hill, New York 1969).
[4] R. S. PHILLIPS, Trans. Amer. Math. Soc. *90*, 193 (1959).
[5] G. HELLWIG, *Differential Operators of Mathematical Physics* (Addison-Wesley, Reading, Mass. 1967).
[6] J. BOGNÁR, *Indefinite Inner Product Spaces* (Springer-Verlag, Heidelburg, New York 1974).
[7] S. R. DE GROOT and P. MAZUR, *Non-Equilibrium Thermodynamics* (North-Holland, Amsterdam 1962).
[8] J.-P. AUBIN, *Applied Functional Analysis*, Wiley-Interscience (to appear).
[9] L. DE SOBRINO, Can. J. Phys. *54*, 105 (1976).
[10] D. BURNETT, Proc. London Math. Soc. *40*, 382 (1935).
S. CHAPMAN and T. G. COWLING, *The Mathematical Theory of Non-Uniform Gases* (Cambridge U.P., Cambridge, England 1953).
J. A. MCLENNAN, Phys. Rev. A*8*, 1479 (1973).
[11] J. A. MCLENNAN, Phys. Rev. A*10*, 1272 (1974).
[12] L. WALDMANN, Z. Naturforsch. *22a*, 1269 (1967).
[13] L. WALDMANN and H. VESTNER, Physica *80A*, 523 (1975).